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PINAR KİREZLİ

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Analyzes of algebraic classification of higher dimensional Kundt geometries with large D method

Pınar KİREZLİ*
Department of Physics, Faculty of Arts and Sciences, Namık Kemal University, Tekirdağ, Turkiye

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Abstract: In this paper, the classification of higher dimensional Kundt geometry is revisited as the dimension of the spacetime $D \to \infty$. In addition to previous studies, in order to Kundt geometry becomes algebraically special spacetime obligatory conditions are determined. Additionally, Type II, Type III, Type N, Type O, and Type D Kundt geometries are explicitly analyzed. Classification of several metrics such as pp-waves, nongyratonic Kundt metric, and VSI spacetime, which are well-known subclasses of Kundt geometry are studied.

Keywords: Kundt, large D method, algebraic classification

1. Introduction

One of the most important classes of exact spacetimes in general relativity is Kundt spacetime which is introduced by Wolfgang Kundt [1, 2]. It admits a null geodesic congruence which is nonexpanding, twist-free, and shear-free and it contains many different vacuum and pure radiation solutions in $D = 4$ dimensions such as pp-waves, plane waves, Bertotti-Robinson, (anti)-Nariai, Plesańki-Hacyan. Also, Kundt spacetime includes various Petrov types. Kundt waves which are Type N spacetimes representing plane-fronted gravitational waves are commonly investigated [3–6]. Further, the geodesic motion of Kundt Type III spacetime is analyzed and it is shown that chaotic motion appears under certain conditions [7]. Additionally, Kundt solutions were widely studied with alternative theories of general relativity to understand structure of this spacetime, such as Brans-Dicke theory [8], Eddington-inspired Born-Infeld gravity [9], Modified Gravity [10], Einstein-Gauss-Bonnet theory [11]. While, explicit extension to a higher dimension can be obtained for Kundt geometry [12, 13], such spacetimes which are subclasses of Kundt spacetime in $D = 4$ dimension, extensions of higher dimensions should be investigated and analyzed.

Higher dimensional solutions in general relativity gained a new perspective with large D expansion method that pioneers are Emparan and et al. [14–19]. Recently, a review [20] is prepared about general aspects of black holes and effective membrane theories in the large D limit and some physical problems are discussed in this limit. According to this limitation method, one expects the simplification of Einstein field equations and novel reformulation of the dynamics. In this paper, we will show how the equations have become simpler for the classification of the Kundt spacetime which

*Correspondence: pkirezli@nku.edu.tr

was studied by [21] and its subclasses which are novel classification of the Kundt geometry, as the spacetime dimension $D \to \infty$.

Petrov classification [22] enables a better comprehension of several aspects of General Relativity in 4-dimensions [23, 24] and it was extended to any $D > 4$ dimensions by [25]. The explicit and complete classification of the algebraic types and its subtypes of the Kundt geometry based on the Weyl aligned null direction (WAND) for any arbitrary dimension $D > 4$ is introduced by [26]. Although general aspects of the classification of the Kundt spacetime with large D limitation method found by [21], the obligatory conditions are not explicitly studied for Type II, Type III, Type N, Type O, and Type D.

In Section 2 we introduce higher dimensional Kundt spacetime and obtain Weyl scalars with large D limitation method. In Section 3, although the Kundt spacetime is not algebraically special, the obligatory conditions are given in order for spacetime to become algebraically special. In addition, the necessary conditions for subtypes of Type II are analyzed. Type III, Type N, and Type O Kundt geometries for primary WAND $\mathbf{k}$ are discussed in Section 4 and in Section 5 Type D geometries for secondary WAND ℓ are studied. Finally, subclasses of the Kundt geometry are explicitly analyzed and the classification of these subclasses is determined in Section 6.

2. Higher dimensional Kundt geometries

Nontwisting, shearfree, nonexpanding spacetime is named Kundt spacetime and the higher dimensional metric of this spacetime can be written in the form;

$$ds^2 = g_{pq}(u,x)\, dx^p dx^q + 2g_{up}(u,r,x)\, du dx^p - 2du dr + g_{uu}(u,r,x)\, du^2 \tag{2.1}$$

where the indeces $p, q..$ count to 2- $(D-1)$ and $x = x^2, x^3, ...x^{(D-1)}$ shorthand of the $(D-2)$ spatial coordinates on the traverse space. Other coordinate r is an affine parameter along the optically privileged null congruence ($k = \partial_r$) and the $u =$const. is a null hypersurface. Although the spatial metric coefficient g_{pq} is independent of the parameter r, other metric components g_{up}, g_{uu} are functions of (r, u, x). Some of the relations between covariant and contravariant metric components can be written as;

$$g^{ur} = -1, \quad g^{rp} = g^{pq} g_{uq}, \quad g^{rr} = -g_{uu} + g^{pq} g_{up} g_{uq}, \quad g_{up} = g^{rq} g_{pq}.$$

To find out Weyl scalar components of the metric (2.1), which are necessary to determine the algebraic classification of spacetime, we introduce most natural null frames $\mathbf{k}, \ell, \mathbf{m_i}$, whose vectors satisfy the normalization conditions $\mathbf{k}.\ell = -1$, $\mathbf{m_i}.\mathbf{m_j} = \delta_{ij}$ and $\mathbf{k}.\mathbf{k} = \ell.\ell = 0$, $\mathbf{k}.\mathbf{m_i} = \ell.\mathbf{m_i} = 0$. Boosts are defined, according to rescaling of these null frames such as; $\mathbf{k} = \lambda\mathbf{k}$, $\ell = \lambda^{-1}\ell$, $\mathbf{m_i} = \mathbf{m_i}$, and boost weight which is used to determine the classification of the spacetimes in higher dimensions, becomes $+1, -1, 0$, respectively. If the Weyl scalar which has $+2$ boost weight, vanishes the null direction of $\mathbf{k}$ becomes a primary Weyl-aligned null direction (WAND). In addition to this, if all Weyl scalars which have $+1$ boost weight, equal zero, the spacetime becomes algebraically special. The secondary WAND can be defined as the null direction of ℓ for fixed $\mathbf{k}$ and the spacetime can be classified more explicitly by secondary WAND.

Weyl scalars of the metric (2.1) is obtained very uncomplicated as the dimension of the spacetime $D \to \infty$ (Christoffel symbols, Riemann tensor, Ricci tensor, Ricci scalar, and Weyl tensor are calculated in the Appendix 7);

$$
\begin{aligned}
\Psi_{0^{ij}} &= C_{abcd}k^a m_i^b k^c m_j^d = 0, && (2.2)\\
\Psi_{1T^i} &= C_{abcd}k^a \ell^b k^c m_i^d = m_i^p\left(-\frac{1}{2}g_{up,rr}\right), && (2.3)\\
\Psi_{1^{ijk}} &= C_{abcd}k^a m_i^b m_j^c m_k^d = 0, && (2.4)\\
\Psi_{2S} &= C_{abcd}k^a \ell^b \ell^c k^d = \frac{1}{2}g_{uu,rr} - \frac{1}{4}g^{pq}g_{up,r}g_{uq,r}, && (2.5)\\
\Psi_{2T^{ij}} &= C_{abcd}k^a m_i^b \ell^c m_j^d \\
&= m_i^p m_j^q\bigg(\frac{1}{2}g_{up}g_{uq,rr} + \frac{1}{4}g_{up,r}g_{uq,r} + \frac{1}{2}g_{pn}g^{ms}g_{us,r}\ {}^s\Gamma^n{}_{mq} + \frac{1}{2}g_{pn}\left(g^{nm}g_{um,r}\right)_{,q}\bigg), && (2.6)\\
\Psi_{2^{ijkl}} &= C_{abcd}m_i^a m_j^b m_k^c m_l^d = m_i^p m_j^q m_k^n m_l^m C_{pqnm}, && (2.7)\\
\Psi_{2^{ij}} &= C_{abcd}k^a \ell^b m_i^c m_j^d = m_i^p m_j^q\left(g_{u[p,q]r} + g_{u[p}g_{q]u,rr}\right), && (2.8)\\
\Psi_{3T^i} &= C_{abcd}\ell^a k^b \ell^c m_i^d = m_i^p\left(\frac{1}{4}g_{uu}g_{up,rr} - g_{u[u,p]r} + \frac{1}{2}g^{mn}g_{um,r}E_{np} - \frac{1}{2}g_{up}g_{uu,rr}\right), && (2.9)\\
\Psi_{3^{ijk}} &= C_{abcd}\ell^a m_i^b m_j^c m_k^d \\
&= m_i^p m_j^q m_k^m\bigg(-2g_{up}g_{u[q,m]r} + g_{up}g_{u[q}g_{m]u,rr} + g_{up}g_{u[q}g_{m]u,r} \\
&\quad + g^{\ell s}g_{us,r}\ {}^s\Gamma^n{}_{\ell p}g_{u[q}g_{m]n} + \frac{1}{2}g_{up,r}g_{u[q}g_{m]u,r} + g_{u[q}g_{m]n}\left(g^{n\ell}g_{u\ell,r}\right)_{,p} - 2g_{pn}g^{sk}E_{k[q}\ {}^s\Gamma^n{}_{m]s} \\
&\quad - g_{pn}\left(g^{rn}g_{u[m,r}\right)_{,q]} - E_{p[m}g_{q]u,r} - 2g_{pn}\left(g^{ns}E_{s[q}\right)_{,m]} - g_{pn}g^{rs}\ {}^s\Gamma^n{}_{s[q}g_{m]u,r}\bigg), && (2.10)\\
\Psi_{4^{ij}} &= C_{abcd}\ell^a m_i^b \ell^c m_j^d = m_i^p m_j^q\bigg(-g_{uq}g_{u[u,p]r} + \frac{g_{uu}}{2}\left(g^{ms}g_{us,r}\ {}^s\Gamma^n{}_{m(p}g_{q)n} + \left(g^{nm}g_{um,r}\right)_{,(p}g_{q)n}\right) \\
&\quad + g^{mn}g_{um,r}E_{n(p}g_{q)u} - \frac{g^{rm}}{2}g_{um,r}g_{u(p}g_{q)u,r} + \frac{1}{2}g^{pq}g_{up,r}g_{uq,r}g_{up}g_{uq} \\
&\quad - g_{pn}\left(-\frac{{}^s\Gamma^n{}_{sq}}{2}\left(g^{rs}g_{uu,r} + 2g^{sm}E_{um}\right) + \left(g^{rn}g_{u[u,r}\right)_{,q]} - 2\left(g^{nm}E_{m[q}\right)_{,u]}\right)\bigg). && (2.11)
\end{aligned}
$$

which were written in order by their boost weight $2, 1, 0, -1, -2$. These scalars and classification of the Kundt geometry for any arbitrary dimension $D > 4$ are discussed in [26]. The nonzero Weyl scalars do not change with large D method, only the equations become easier.

Further, there are some relations between the above Weyl scalars such as;

$$
\begin{aligned}
\Psi_{1T^i} &= \Psi_{1^k{}_{k}{}_{i}}, && (2.12)\\
\Psi_{2S} &= \Psi_{2T^k{}_k}, && (2.13)\\
\Psi_{3T^i} &= \Psi_{3^k{}_{k}{}_{i}}. && (2.14)
\end{aligned}
$$

Additionally, the symmetric and antisymmetric part of the $\Psi_{2T^{ij}}$ becomes;

$$\begin{aligned} \Psi_{2T^{(ij)}} &= m_i^p m_j^q \left(\frac{1}{4} g_{up,r} g_{uq,r} + \frac{1}{2} g_{pn} g^{ms} g_{us,r} \ {}^s\Gamma^n{}_{mq} + \frac{1}{2} g_{u(p} g_{q)u,rr} + \frac{1}{2} g_{pn} g_{um,r} g^{nm}{}_{,q} + \frac{1}{2} g_{u(p,q)r} \right) \\ \Psi_{2T^{[ij]}} &= m_i^p m_j^q \left(\frac{1}{2} g_{u[p} g_{q]u,rr} + \frac{1}{2} g_{u[p,q]r} \right) \end{aligned} \tag{2.15}$$

which satisfies the relation $\Psi_{2^{ij}} = 2\Psi_{2T^{[ij]}}$. The irreducible components of Weyl scalars are given [27];

$$\tilde{\Psi}_{1^{ijk}} \equiv \Psi_{1^{ijk}} - \frac{1}{D-3} \left(\delta_{ij} \Psi_{1T^k} - \delta_{ik} \Psi_{1T^j} \right), \tag{2.16}$$

$$\tilde{\Psi}_{2T^{(ij)}} \equiv \Psi_{2T^{(ij)}} - \frac{1}{D-2} \delta_{ij} \Psi_{2S}, \tag{2.17}$$

$$\begin{aligned} \tilde{\Psi}_{2^{ijk\ell}} \equiv\ & \Psi_{2^{ijk\ell}} - \frac{2}{D-4} \left(\delta_{ik} \tilde{\Psi}_{2T^{(j\ell)}} + \delta_{j\ell} \tilde{\Psi}_{2T^{(ik)}} - \delta_{i\ell} \tilde{\Psi}_{2T^{(jk)}} - \delta_{jk} \tilde{\Psi}_{2T^{(i\ell)}} \right) \\ & - \frac{2}{(D-2)(D-3)} \left(\delta_{ik} \delta_{j\ell} - \delta_{i\ell} \delta_{jk} \right) \Psi_{2S}, \end{aligned} \tag{2.18}$$

$$\tilde{\Psi}_{3^{ijk}} \equiv \Psi_{3^{ijk}} - \frac{1}{D-3} \left(\delta_{ij} \Psi_{3T^k} - \delta_{ik} \Psi_{3T^j} \right). \tag{2.19}$$

where $\delta_{ij} = g_{pq} m_i{}^p m_j{}^q$. As the dimension of the spacetime $D \to \infty$, after the first term of the right-hand side vanishes and they can be written;

$$\tilde{\Psi}_{1^{ijk}} \equiv \Psi_{1^{ijk}}, \qquad \tilde{\Psi}_{2T^{(ij)}} \equiv \Psi_{2T^{(ij)}}, \qquad \tilde{\Psi}_{2^{ijk\ell}} \equiv \Psi_{2^{ijk\ell}}, \qquad \tilde{\Psi}_{3^{ijk}} \equiv \Psi_{3^{ijk}}. \tag{2.20}$$

Boost weight $+2$ which is corresponding to $\Psi_{0^{ij}}$, vanishes, and as the dimension of the spacetime $D \to \infty$ the Kundt geometry becomes Type I similar to any arbitrary dimensions $D > 4$. Even though one of the Weyl scalar ($\Psi_{1^{ijk}}$) which is corresponding to the $+1$ component of boost weight vanishes, the Kundt geometry is not algebraically special spacetime because of nonzero Weyl scalar (Ψ_{1T^i}) which is corresponding $+1$ component of boost weight, too. Rest of the paper, first we will analyze algebraically special Kundt geometries without solving field equations and then examine several subclasses of Kundt spacetime. In addition, vanishing Weyl scalar and corresponding (sub)types are summarized in Table 1 for primary and secondary WANDs **k** and ℓ which is a road map of classification in higher dimensional spacetimes.

3. Algebraically special Kundt geometries

The spacetime is always Type I(b) because the Weyl scalars $\Psi_{0^{ij}}$ and $\Psi_{1^{ijk}}$ vanish. However, it will be algebraically special when all boost weight $+1$ of Weyl scalars becomes zero. The obligatory condition for the Kundt spacetime to become algebraically special is vanishing the Weyl scalar Ψ_{1T^i}. This condition allows us to obtain metric coefficient g_{up} as;

$$g_{up} = f_p(u,x) r + e_p(u,x). \tag{3.1}$$

and the general Kundt metric becomes;

$$ds^2 = g_{pq}(u,x)\, dx^p dx^q + 2 \left(f_p(u,x) r + e_p(u,x) \right) du dx^p - 2du dr + g_{uu}(u,r,x)\, du^2 \tag{3.2}$$

Table 1. Algebraic classification of the Kundt geometry for the primary and secondary WANDs $\mathbf{k}, \ell$ [27]

Types	**Vanishing Weyl Scalar**
I	$\Psi_{0^{ij}}$
I(a)	$\Psi_{0^{ij}}, \Psi_{1T^i}$
I(b)	$\Psi_{0^{ij}}, \Psi_{1^{ijk}}$
II	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}$
II(a)	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2S}$
II(b)	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2T^{(ij)}}$
II(c)	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2^{ijk\ell}}$
II(d)	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2^{ij}}$
III	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2S}, \Psi_{2T^{(ij)}}, \Psi_{2^{ijk\ell}}, \Psi_{2^{ij}}$
III(a)	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2S}, \Psi_{2T^{(ij)}}, \Psi_{2^{ijk\ell}}, \Psi_{2^{ij}}, \Psi_{3T^i}$
III(b)	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2S}, \Psi_{2T^{(ij)}}, \Psi_{2^{ijk\ell}}, \Psi_{2^{ij}}, \Psi_{3^{ijk}}$
N	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2S}, \Psi_{2T^{(ij)}}, \Psi_{2^{ijk\ell}}, \Psi_{2^{ij}}, \Psi_{3T^i}, \Psi_{3^{ijk}}$
O	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2S}, \Psi_{2T^{(ij)}}, \Psi_{2^{ijk\ell}}, \Psi_{2^{ij}}, \Psi_{3T^i}, \Psi_{3^{ijk}}, \Psi_{4^{ij}}$
I_i	$\Psi_{0^{ij}}, \Psi_{4^{ij}}$
II_i	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{4^{ij}}$
III_i	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{2S}, \Psi_{2T^{(ij)}}, \Psi_{2^{ijk\ell}}, \Psi_{2^{ij}},, \Psi_{4^{ij}}$
D	$\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{3T^i}, \Psi_{3^{ijk}}, \Psi_{4^{ij}}$

.

where the only metric component of g_{uu} is dependent of the parameter r. Additionally, R_{rurp} and R_{rp} vanish without any other conditions. The Weyl scalars become;

$$\Psi_{2S} = \frac{1}{2}g_{uu,rr} - \frac{1}{4}f^p f_p, \tag{3.3}$$

$$\Psi_{2T^{(ij)}} = m_i^p m_j^q\left(\frac{1}{4}f_p f_q + \frac{1}{2}f^m g_{pn}\ {}^s\Gamma^n_{mq} + \frac{1}{2}g_{pn}f_m g^{nm}{}_{,q} + \frac{1}{2}f_{(p,q)}\right), \tag{3.4}$$

$$\Psi_{2^{ijkl}} = m_i^p m_j^q m_k^n m_l^m\ {}^sR_{pqnm}, \tag{3.5}$$

$$\Psi_{2^{ij}} = m_i^p m_j^q f_{[p,q]}, \tag{3.6}$$

$$\Psi_{3T^i} = \frac{m_i^p}{2}\left(f_{p,u} - g_{uu,pr} + f^n E_{np} - (f_p r + e_p)\, g_{uu,rr}\right), \tag{3.7}$$

$$\begin{aligned}
\Psi_{3^{ijk}} &= m_i^p m_j^q m_k^m \Big(-2\left(f_p r + e_p\right) f_{[q,m]} + \left(f_p r + e_p\right) e_{[q} f_{m]} - g_{pn} g^{rs}\, {}^s\Gamma^n{}_{s[q} f_{m]} \\
&+ f^\ell\, {}^s\Gamma^n{}_{\ell p} \left(f_{[q} g_{m]n} r + e_{[q} g_{m]n}\right) + \frac{1}{2} f_p f_{[m} e_{q]} + \left(f_{[q} g_{m]n} r + e_{[q} g_{m]n}\right) f^n{}_{,p} \\
&- 2 g_{pn} g^{sk} E_{k[q}\, {}^s\Gamma^n{}_{m]s} - g_{pn} r \left(f^n f_{[m}\right)_{,q]} - g_{pn}\left(e^n f_{[m}\right)_{,q]} - E_{p[m} f_{q]} - 2 g_{pn}\left(g^{ns} E_{s[q}\right)_{,m]} \Big), \quad (3.8) \\
\Psi_{4^{ij}} &= m_i^p m_j^q \Big(-\frac{f_q r + e_q}{2}\left(g_{uu,pr} - f_{p,u}\right) + \frac{g_{uu}}{2}\left(f^m\, {}^s\Gamma^n{}_{m(p} g_{q)n} + f^n{}_{,(p} g_{q)n}\right) \\
&+ f^n\left(E_{n(p} f_{q)} r + E_{n(p} e_{q)}\right) - \frac{g^{rm} f_m}{2}\left(f_{(p} f_{q)} r + e_{(p} f_{q)}\right) + \frac{f^p f_p}{2}\left(f_p r + e_p\right)\left(f_q r + e_q\right) \\
&- g_{pn}\left[-\frac{{}^s\Gamma^n{}_{sq}}{2}\left(g^{rs} g_{uu,r} + 2 g^{sm} E_{um}\right) - 2\left(g^{nm} E_{m[q}\right)_{,u]} + \frac{1}{2}\left(g^{rn} g_{uu,r}\right)_{,q} - \frac{1}{2}\left(g^{rn} f_q\right)_{,u}\right]\Big). \quad (3.9)
\end{aligned}$$

where $f^p = g^{pq} f_q$.

- The Kundt spacetime becomes algebraically special Type II(a) if the Weyl scalar $\Psi_{2S} = 0$. When the condition is applied, we get;

$$g_{uu} = \frac{r^2}{4} f^p f_p + b(u,x) r + c(u,x). \tag{3.10}$$

  This is the same as the result of any arbitrary dimension $D > 4$ in [26].

- The Kundt spacetime becomes algebraically special Type II(b) if the Weyl scalar $\Psi_{2T^{(ij)}} = 0$ which reads;

$$\frac{1}{2} f_p f_q + f_{(p,q)} = -g_{pn}\left(f^m\, {}^s\Gamma^n{}_{mq} + f_m g^{mn}{}_{,q}\right). \tag{3.11}$$

  In this case, it is not possible to obtain an exact result for the metric functions.

- The Kundt spacetime becomes algebraically special Type II(c) if the Weyl scalar $\Psi_{2^{ijkl}} = 0$. According to this condition, the Riemann tensor of the (D-2) dimensional traverse space becomes;

$${}^s R_{pqmn} = 0. \tag{3.12}$$

- The Kundt spacetime becomes algebraically special Type II(d) if the Weyl scalar $\Psi_{2^{ij}} = 0$ which satisfies;

$$f_{p,q} - f_{q,p} = 0. \tag{3.13}$$

4. Type III, Type N, and Type O Kundt geometries

Kundt spacetime becomes Type III or more special when the above conditions (3.10-3.13) are satisfied at the same time. We have to analyze the Weyl scalars of Ψ_{3T^i} and $\Psi_{3^{ijk}}$ for more special spacetimes with respect to primary WAND $\mathbf{k}$. Further, the classification of Kundt spacetime is summarized in Table 2.

Table 2. Algebraic classification of the higher dimensional Kundt geometries for the primary WAND **k** as the dimension of the spacetime $D \to \infty$.

Types	**Obligatory conditions**
I	-
I(a)	$g_{up} = f_p r + e_p$
I(b)	-
II	$g_{up} = f_p r + e_p$
II(a)	$g_{uu} = \frac{r^2}{4} f^p f_p + br + c$
II(b)	$\frac{1}{2} f_p f_q + f_{(p,q)} = -g_{pn} \left(f^m \, {}^s\Gamma^n{}_{mq} + f_m g^{mn}{}_{,q} \right)$
II(c)	${}^sR_{pqmn} = 0$
II(d)	$f_{[p,q]} = 0$
III	$g_{uu} = \frac{r^2}{4} f^p f_p + br + c$, $\frac{1}{2} f_p f_q + f_{(p,q)} = -g_{pn} \left(f^m \, {}^s\Gamma^n{}_{mq} + f_m g^{mn}{}_{,q} \right)$, ${}^sR_{pqmn} = 0$, $f_{[p,q]} = 0$
III(a)	$(f^n f_n)_{,p} = f^n f_n f_p$, $b_{,p} = f_{p,u} + f^n e_{[n,p]} + \frac{f^n}{2} (g_{np,u} + f_n e_p)$
III(b)	Equation (4.4)
N	$(f^n f_n)_{,p} = f^n f_n f_p$, $b_{,p} = f_{p,u} + f^n e_{[n,p]} + \frac{f^n}{2} (g_{np,u} + f_n e_p)$, Equation (4.4)
O	$Ar^2 + Br + C = 0$

- The Type III Kundt geometry is subtype Type III(a) when the Weyl scalar $\Psi_{3T^i} = 0$ which yields;

$$f_{p,u} - b_p + \frac{f^n}{2} \left(2e_{[n,p]} + g_{np,u} \right) + \frac{1}{2} f^n f_n e_p = \frac{r}{2} \left[-\left(f^n f_n \right)_{,p} + f^n f_n f_p \right]. \tag{4.1}$$

Despite the right-hand side of the above equation depending on the coordinate r, the left-hand side is independent of the r parameter. As a result;

$$\left(f^n f_n \right)_{,p} = f^n f_n f_p \tag{4.2}$$

$$b_{,p} = f_{p,u} + f^n e_{[n,p]} + \frac{f^n}{2} \left(g_{np,u} + f_n e_p \right), . \tag{4.3}$$

- The Type III Kundt geometry is subtype Type III(b) when the Weyl scalar $\Psi_{3^{ijk}} = 0$;

$$r \left[f_p e_{[q} f_{m]} - g_{pn} f^s \, {}^s\Gamma^n{}_{s[q} f_{m]} + \left(f^\ell \, {}^s\Gamma^n{}_{\ell p} + f^n{}_{,p} \right) f_{[q} g_{m]n} - g_{pn} f^n{}_{,[q} f_{m]} \right]$$

$$+ e_p e_{[q} f_{m]} + \frac{1}{2} f_p f_{[m} e_{q]} - g_{pn} \left(e^s \, {}^s\Gamma^n{}_{s[q} f_{m]} + e^n{}_{,[q} f_{m]} \right) + \left(f^\ell \, {}^s\Gamma^n{}_{\ell p} + f^n{}_{,p} \right) e_{[q} g_{m]n}$$

$$-2g_{pn}\left[g^{sk}\tilde{E}_{k[q}\,{}^{s}\Gamma^{n}{}_{m]s}+\tilde{E}_{s[q}g^{ns}{}_{,m]}+g^{ns}\tilde{E}_{s[q,m]}\right]-\tilde{E}_{p[m}f_{q]}=0 \tag{4.4}$$

where $\tilde{E}_{pq}=e_{[p,q]}+\frac{1}{2}g_{pq,u}$. The first line of the equation contains only r parameter and the parenthesis of the first line should be zero for the equation vanishes. This result contains only f_p, e_p and g_{pq} coefficients and the relation between them can be written. Further, the sum of the second and third lines of the equation should be zero which will give some relation between the same coefficients.

- The Kundt geometry is type N when the equations (4.2) and (4.3) and $\Psi_{3ijk}=0$ are satisfied simultaneously.
- The Kundt spacetime becomes Type O when all Weyl scalars vanish. We will not investigate $\Psi_{4ij}=0$ condition which is not easy to analyze but it is written in quadratic form of $A(u,x)r^2+B(u,x)r+C(u,x)$ where the A,B,C contain f_q, e_q, b, c, and g_{pq}.

5. Type D Kundt geometry

Kundt spacetime becomes Type D when the Weyl scalars of $\Psi_{0^{ij}}, \Psi_{1T^i}, \Psi_{1^{ijk}}, \Psi_{3T^i}, \Psi_{3^{ijk}}, \Psi_{4^{ij}}$ are vanishing. First, in order to necessary condition $\Psi_{1T^i}=0$, the metric coefficient of g_{up} is obtained in equation (3.1). By using this metric coefficient, other vanishing Weyl scalars are explicitly given in equations (3.7)-(3.9) which become the dimension of the spacetime $D\to\infty$;

$$f_{p,u}+f^nE_{np}-(f_pr+e_p)\,g_{uu,rr}=g_{uu,pr}, \tag{5.1}$$

$$\begin{aligned}&-2\left(f_pr+e_p\right)f_{[q,m]}+\left(f_pr+e_p\right)e_{[q}f_{m]}-g_{pn}g^{rs}\,{}^{s}\Gamma^{n}{}_{s[q}f_{m]}\\&+f^{\ell}\,{}^{s}\Gamma^{n}{}_{\ell p}\left(f_{[q}g_{m]n}r+e_{[q}g_{m]n}\right)+\frac{1}{2}f_pf_{[m}e_{q]}+\left(f_{[q}g_{m]n}r+e_{[q}g_{m]n}\right)f^{n}{}_{,p}\\&-g_{pn}r\left(f^nf_{[m}\right)_{,q]}-g_{pn}\left(e^nf_{[m}\right)_{,q]}-E_{p[m}f_{q]}=2g_{pn}\left[g^{sk}E_{k[q}\,{}^{s}\Gamma^{n}{}_{m]s}+\left(g^{ns}E_{s[q}\right)_{,m]}\right],\end{aligned} \tag{5.2}$$

$$\begin{aligned}&g_{pn}\left[-\frac{{}^{s}\Gamma^{n}{}_{sq}}{2}\left(g^{rs}g_{uu,r}+2g^{sm}E_{um}\right)-2\left(g^{nm}E_{m[q}\right)_{,u]}+\frac{1}{2}\left(g^{rn}g_{uu,r}\right)_{,q}\right]\\&=\frac{\left(f_pr+e_p\right)\left(f_qr+e_q\right)}{2}g_{uu,rr}+\frac{1}{2}f^nE_{nq}\left(f_pr+e_p\right)+\frac{g_{uu}}{2}\left(f^m\,{}^{s}\Gamma^{n}{}_{m(p}g_{q)n}+f^{n}{}_{,(p}g_{q)n}\right)\\&-\frac{g^{rm}f_m}{2}\left(f_{(p}f_{q)}r+e_{(p}f_{q)}\right)+\frac{f^pf_p}{2}\left(f_pr+e_p\right)\left(f_qr+e_q\right).\end{aligned} \tag{5.3}$$

6. Examples

We will investigate the classification of some spacetime which are subclasses of Kundt spacetime.

1. $ds^2=g_{pq}dx^pdx^q+2e_pdudx^p-2dudr+cdu^2$

   This metric corresponds to pp-waves which are a subclass of Kundt spacetime with all metric functions independent of the parameter r. They naturally become Type II(abd). pp-waves become Type III(a) when the Riemann tensor of traverse space is zero (${}^{s}R_{pqmn}=0$). This means Kundt spacetime Type III(a) or more special when the traverse space is flat as the dimension of the spacetime $D\to\infty$.

Additionally, it will be Type N (and also Type III(b)) for primary WAND $\mathbf{k}$, when the metric coefficients satisfy;

$$g^{ns}\left(E_{s[q}\right)_{,m]} = -g^{sk}E_{k[q}\ {}^{s}\Gamma^{n}{}_{m]s} \tag{6.1}$$

and will be Type O with;

$$g^{sm}E_{um}\ {}^{s}\Gamma^{n}{}_{sq} = -2\left(g^{nm}E_{m[q}\right)_{,u]}. \tag{6.2}$$

On the other hand, it will be Type (II)$_{\text{i}}$ for the secondary WAND ℓ, when Weyl scalar $\Psi_{4^{ij}} = 0$ which yields equation (6.2). If the traverse universe is flat and equation (6.2) is satisfied, the spacetime becomes Type (III)$_{\text{i}}$ for the secondary WAND ℓ. Finally, the spacetime becomes Type D for the WAND ℓ, when the equations (6.1) and (6.2) are simultaneously provided. pp-waves classification for primary and secondary WANDs for any dimension $D > 4$ are shown in Table 3 in reference [26] and for the spacetime dimension $D \to \infty$ are discussed in Table 3 in the reference [21].

2. $ds^2 = g_{pq}dx^p dx^q - 2dudr + a(u,x)r^2du^2$

   As we set the metric coefficient $g_{up} = 0$ the metric takes the above form. Physically, this metric is called nongyratonic Kundt geometry because g_{up} carries physical information representing a beam of null radiation with internal spin [28–31]. According to the above metric, the spacetime automatically becomes Type II(b) and Type II(d). It will be Type II(a) and Type II(c) when the conditions $a = 0$ and ${}^{s}R_{pqmn} = 0$ are satisfied, respectively. If these conditions are simultaneously fulfilled, the spacetime becomes Type III(a). It will be Type III(b) and Type N if the case

$$\left(g^{ns}g_{s[q,u}\right)_{,m]} = -g^{sk}g_{k[q,u}\ {}^{s}\Gamma^{n}{}_{m]s} \tag{6.3}$$

   is ensured. With all the above cases, the spacetime becomes Type O when $\left(g^{nm}g_{mq,u}\right)_{,u} = 0$ which gives $g^{nm}g_{mq,u} = c(x)$ where $c(x)$ is a constant.

   Also, taking the classification of the secondary WAND ℓ into account, the spacetime becomes Type I$_{\text{i}}$ and Type II$_{\text{i}}$ while the $\Psi_{4^{ij}}$ vanishes which yields;

$$-r^2\left[g^{sm}a_{,m}\ {}^{s}\Gamma^{n}{}_{sq} + \left(g^{nm}a_{,m}\right)_{,q}\right] + r\left[\left(g^{rn}a\right)_{,q} - g^{rs}a\ {}^{s}\Gamma^{n}{}_{sq}\right] - \frac{1}{2}\left(g^{nm}g_{mq,u}\right)_{,u} = 0. \tag{6.4}$$

   Additionally, the spacetime becomes Type III$_{\text{i}}$ with large D method when the cases $a = 0$, ${}^{s}R_{pqmn} = 0$ and $g^{nm}g_{mq,u} = c(x)$ are provided at the same time.

   Finally, we can say that it will be Type D when

$$ra_{,p} = 0 \leftrightarrow a = a(x), \tag{6.5}$$

$$\left(g^{ns}g_{s[q,u}\right)_{,m]} = -g^{sk}\left({}^{s}\Gamma^{n}{}_{s[m}g_{q]k,u}\right), \tag{6.6}$$

$$-r^2\left[g^{sm}a_{,m}\ {}^{s}\Gamma^{n}{}_{sq} + \left(g^{nm}a_{,m}\right)_{,q}\right] + r\left[\left(g^{rn}a\right)_{,q} - g^{rs}a\ {}^{s}\Gamma^{n}{}_{sq}\right] - \frac{1}{2}\left(g^{nm}g_{mq,u}\right)_{,u} = 0. \tag{6.7}$$

3. $ds^2 = \delta_{pq}dx^p dx^q + 2\left(f_p r + e_p\right)dudx^p - 2dudr + \left(ar^2 + br + c\right)du^2$

   Kundt spacetime can be written as g_{uu} is quadratic in r and the traverse space $g_{pq} = \delta_{pq}$ is flat. Also, it will define as VSI spacetime when their scalar curvature invariants of all orders vanish. Additionally, the coefficients have to satisfy the conditions that;

$$a = \frac{1}{4}f^p f_q, \qquad f_{[p,q]=0}, \qquad f_{(pq)} + \frac{1}{2}f_p f_q + \delta_{pn}f^m \, {}^s\Gamma^n{}_{mq} = 0$$

   which make spacetime Type III or more special. Due to these conditions, the spacetime can be called VSI spacetime.

   VSI spacetime becomes Type III(a) when the conditions;

$$\left(f^n f_n\right)_{,p} = f^n f_n f_p \tag{6.8}$$

$$b_{,p} = f_{p,u}f^n e_{[n,p]} + \frac{f^n f_n e_p}{2} \tag{6.9}$$

   are satisfied as the dimension of the spacetime $D \to \infty$.

   It becomes Type III(b) when the Weyl scalar $\Psi_{3^{ijk}}$ vanishes which yields;

$$\begin{aligned} & r\left[f_p e_{[q}f_{m]} - \delta_{pn}f^s \, {}^s\Gamma^n{}_{s[q}f_{m]} + \left(f^\ell \, {}^s\Gamma^n{}_{\ell p} + f^n_{,p}\right) f_{[q}g_{m]n} - \delta_{pn}f^n{}_{,[q}f_{m]}\right] \\ & + e_p e_{[q}f_{m]} + \frac{1}{2}f_p f_{[m}e_{q]} - \delta_{pn}\left(e^s \, {}^s\Gamma^n{}_{s[q}f_{m]} + e^n{}_{,[q}f_{m]}\right) + \left(f^\ell \, {}^s\Gamma^n{}_{\ell p} + f^n{}_{,p}\right) e_{[q}\delta_{m]n} \\ & -2\delta_{pn}\left[\frac{\delta^{sk}}{4}\left(e_{[k,q]} \, {}^s\Gamma^n{}_{ms} - e_{[k,m]} \, {}^s\Gamma^n{}_{qs}\right) + \frac{\delta^{ns}}{2}e_{[m,q]s}\right] - \frac{1}{2}\left(e_{[p,m]}f_q - e_{[p,q]}f_m\right) = 0 \end{aligned} \tag{6.10}$$

   VSI spacetime is Type N when the equations (6.8)-(6.10) are simultaneously satisfied.

7. Conclusion

We analyzed the explicit classification of shear-free, twist-free, and nonexpanding geometries as the spacetime dimension $D \to \infty$. Eventually, the algebraic classification of the Kundt geometry with large D method is obtained similar to the classification of the same spacetime for any dimension $D > 4$ but with simpler equations. Christoffel symbols, Riemann tensors, Ricci tensors and scalar, and Weyl tensor of the Kundt spacetime which are used to calculate Weyl scalars, were examined in Appendix A as the dimension of the spacetime $D \to \infty$. Also, coherent with previous studies, this spacetime has been shown to be Type I(b) or more special.

On the other hand, according to our calculations, Kundt spacetime can be algebraically special with respect to primary WAND $\mathbf{k}$, if the metric coefficient g_{up} is a linear function of r which showed previous works for both any dimensions $D > 4$ [26] and as the dimension $D \to \infty$ [21]. In addition, Type III, Type N, Type O, and Type D classifications of the Kundt geometry with large D method were explicitly analyzed, and obligatory conditions were specified, for the first time.

In addition, subclasses of the well-known Kundt spacetime were investigated by analyzing several metrics without solving field equations. We showed that the spacetime is Type II(abd) or more special in Example 1 which corresponds to pp-waves. If the traverse space is flat, it becomes Type III(a) or more special. In Example 2 it is shown that Kundt geometry without gyratonic matter terms is Type II(ac) or more special. VSI spacetime which was studied in Example 3 is Type III or more special. After the detection of gravitational waves was announced [32, 33], the attention of the researchers to them increased incredibly. Our results may help to understand the structure of them as the dimension of the spacetime $D \to \infty$. Recently, pp-waves of Type III in $D = 4$ dimensional spacetime field equations were solved [34]. They will be generalized to higher dimensional Type III spacetime and our results will be useful to analyze them as the dimension of the spacetime $D \to \infty$.

Although the classification of the Kundt spacetime with large D method was studied in our previous work [21], explicit classification of the spacetime was not discussed. Our motivation in preparing this paper was to analyze the classification of Kundt spacetime for both primary and secondary WANDs and to examine the classification of subclasses of Kundt geometry in detail, with the large D method.

In this paper, we obtained the classification of the Kundt geometry without solving Einstein field equations. In future studies, with large D limit, one can discuss the geodesic motion of the Kundt spacetime or can analyze field equations.

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Appendix

The nonzero Christoffel symbols of the general Kundt metric are;

$$\Gamma^u{}_{uu} = \frac{1}{2} g_{uu,r}, \tag{A-1}$$

$$\Gamma^u{}_{up} = \frac{1}{2} g_{up,r}, \tag{A-2}$$

$$\Gamma^r{}_{ur} = \frac{1}{2}\left(g^{rp} g_{up,r} - g_{uu,r}\right), \tag{A-3}$$

$$\Gamma^r{}_{up} = \frac{1}{2}\left(-g^{rr} g_{up,r} - g_{uu,p} + 2g^{rn} E_{np}\right), \tag{A-4}$$

$$\Gamma^r{}_{uu} = \frac{1}{2}\left(-g^{rr} g_{uu,r} - g_{uu,u} + 2g^{rn} E_{un}\right), \tag{A-5}$$

$$\Gamma^r{}_{pq} = +\frac{1}{2} g_{pq,u} - g_{u(p,q)} + g_{un}\ {}^s\Gamma^n{}_{pq}, \tag{A-6}$$

$$\Gamma^r{}_{rp} = \frac{-g_{up,r}}{2}, \tag{A-7}$$

$$\Gamma^p{}_{uu} = \frac{1}{2}\left(-g^{rp} g_{uu,r} + 2g^{pn} E_{un}\right), \tag{A-8}$$

$$\Gamma^p{}_{ur} = \frac{1}{2} g^{pq} g_{uq,r}, \tag{A-9}$$

$$\Gamma^p{}_{uq} = \frac{1}{2}\left(-g^{rp} g_{uq,r} + 2g_{pn} E_{nq}\right), \tag{A-10}$$

$$\Gamma^m{}_{pq} = {}^s\Gamma^m{}_{pq}, \tag{A-11}$$

where ${}^s\Gamma^n{}_{pq}$ is the Chrisstoffel symbol of the spatial metric g_{pq} and,

$$E_{pq} = g_{u[p,q]} + \frac{1}{2} g_{pq,u}, \tag{A-12}$$

$$E_{up} = g_{u[p,u]} + \frac{1}{2} g_{up,u}. \tag{A-13}$$

The Rieman tensors of the general Kundt metric are;

$$R_{prrq} = 0\,, \tag{A-14}$$

$$R_{ruur} = \frac{1}{2} g_{uu,rr} - \frac{1}{4} g^{pq} g_{up,r} g_{uq,r}\,, \tag{A-15}$$

$$R_{ruup} = g_{u[u,p]r} - \frac{1}{2} g^{mn} g_{um,r} E_{np} + \frac{1}{4} g^{rm} g_{um,r} g_{up,r}\,, \tag{A-16}$$

$$
\begin{aligned}
R_{rurp} &= -\frac{1}{2}g_{up,rr} \ , && \text{(A-17)}\\
R_{rupq} &= g_{u[p,q]r} \ , && \text{(A-18)}\\
R_{prmq} &= 0 \ , && \text{(A-19)}\\
R_{pruq} &= -\frac{1}{2}g_{pn}g^{ms}g_{us,r}\ {}^{s}\Gamma^{n}{}_{mq} - \frac{1}{4}g_{up,r}g_{uq,r} - \frac{1}{2}g_{pn}\left(g^{nm}g_{um,r}\right)_{,q} \ , && \text{(A-20)}\\
R_{pumq} &= g_{up}g_{u[q,m]r} + E_{p[m}g_{q]u,r} + g_{pn}g^{rs}\ {}^{s}\Gamma^{n}{}_{s[q}g_{m]u,r} + 2g_{pn}g^{sk}E_{k[q}\ {}^{s}\Gamma^{n}{}_{m]s} \\
&\quad + g_{pn}\left(g^{rn}g_{u[m,r)}\right)_{,q]} + 2g_{pn}\left(g^{ns}E_{s[q)}\right)_{,m]} \ , && \text{(A-21)}\\
R_{puqu} &= g_{up}g_{u[u,q]r} - E_{p[u}g_{q]u,r} + \frac{1}{2}g^{rs}g_{uq,r}E_{ps} - g^{s\ell}E_{ps}E_{\ell q} - g_{pn}\left(g^{rn}g_{u[u,r)}\right)_{,q]} \\
&\quad - \frac{1}{2}g_{pn}\ {}^{s}\Gamma^{n}{}_{sq}\left(g^{rs}g_{uu,r} + 2g^{sm}E_{um}\right) + \frac{1}{4}g_{up,r}\left(g_{uu,q} + g^{rr}g_{uq,r} - 2g^{rs}E_{sq}\right) \\
&\quad - 2g_{pn}\left(g^{nm}E_{m[q)}\right)_{,u]} , && \text{(A-22)}\\
R_{pqmn} &= {}^{s}R_{pqmn}. && \text{(A-23)}
\end{aligned}
$$

Components of the Ricci tensor become;

$$
\begin{aligned}
R_{rr} &= 0, && \text{(A-24)}\\
R_{rp} &= -\frac{1}{2}g_{up,rr} \ , && \text{(A-25)}\\
R_{ru} &= -\frac{1}{2}g_{uu,rr} + \frac{1}{2}g^{rp}g_{up,rr} + \frac{1}{2}g^{ms}g_{us,r}\ {}^{s}\Gamma^{q}{}_{mq} + \frac{1}{2}\left(g^{mq}g_{um,r}\right)_{,q} \ , && \text{(A-26)}\\
R_{uu} &= -\frac{1}{2}g^{rr}g_{uu,rr} - \frac{1}{4}g^{rm}g^{rp}g_{um,r}g_{up,r} - \left(g^{rp}g_{u[u,r)}\right)_{,q]} + \frac{1}{4}g^{pq}g_{up,r}g_{uu,q} - 2\left(g^{mq}E_{m[q)}\right)_{,u]} \\
&\quad + \frac{1}{2}g^{rp}g^{mn}g_{um,r}E_{np} - g^{pq}E_{p[u}g_{q]u,r} - g^{pq}g^{s\ell}E_{ps}E_{\ell q} - \frac{1}{2}\ {}^{s}\Gamma^{q}{}_{sq}\left(g^{rs} + 2g^{sm}E_{um}\right)), && \text{(A-27)}\\
R_{up} &= -\frac{1}{2}g^{rr}g_{up,rr} - g_{u[u,p]r} + \left(g^{rm}g_{u[m,r)}\right)_{,p]} - \frac{1}{2}g^{rm}g_{um,r}g_{up,r} + \frac{1}{2}g^{mq}E_{qm}g_{up,r} \\
&\quad + g^{rs}\ {}^{s}\Gamma^{m}{}_{s[p}g_{m]u,r} - \frac{1}{2}g^{rq}g_{pn}g^{ms}g_{us,r}\ {}^{s}\Gamma^{n}{}_{mq} + 2g^{sk}E_{k[p}\ {}^{s}\Gamma^{m}{}_{m]s} + 2\left(g^{ms}E_{s}[p)\right)_{,m]} \\
&\quad - \frac{1}{2}g^{rq}g_{pn}\left(g^{nm}g_{um,r}\right)_{,q} , && \text{(A-28)}\\
R_{pq} &= {}^{s}R_{pq} - \frac{1}{2}g_{up,r}g_{uq,r} - \left(g^{nm}g_{um,r}\right)_{(,q}\,g_{p)n} + g^{r\ell}\ {}^{s}\Gamma^{n}{}_{\ell[q}g_{n]p,r} - g^{ms}g_{us,r}\ {}^{s}\Gamma^{n}{}_{m(p}g_{q)n} \\
&\quad + g_{pq}g^{rn}g_{un}, && \text{(A-29)}
\end{aligned}
$$

Ricci scalar becomes;

$$
\begin{aligned}
R &= {}^{s}R + g_{uu,rr} - 2g^{rp}g_{up,rr} - g^{ms}g_{us,r}\ {}^{s}\Gamma^{q}{}_{mq} - \left(g^{mq}g_{um,r}\right)_{,q} + (D-2)g^{pq}g_{up}g_{un} \\
&\quad - \frac{g^{pq}}{2}g_{up,r}g_{uq,r}. && \text{(A-30)}
\end{aligned}
$$

We calculate the Weyl tensor of the general Kundt metric as the dimension of the spacetime $D \to \infty$ which simplifies the results as;

$$C_{rprq} = 0, \tag{A-31}$$

$$C_{rpru} = -\frac{1}{2}g_{up,rr}\ , \tag{A-32}$$

$$C_{prmq} = 0, \tag{A-33}$$

$$C_{ruru} = -\frac{1}{2}g_{uu,rr} + \frac{1}{4}g^{pq}g_{up,r}g_{uq,r}, \tag{A-34}$$

$$C_{rpuq} = \frac{1}{2}g_{pn}g^{ms}g_{us,r}\ {}^{s}\Gamma^{n}{}_{mq} + \frac{1}{4}g_{up,r}g_{uq,r} + \frac{1}{2}g_{pn}\left(g^{nm}g_{um,r}\right)_{,q}, \tag{A-35}$$

$$C_{rupq} = g_{u[p,q]r}, \tag{A-36}$$

$$C_{pqmn} = {}^{s}R_{pqmn}, \tag{A-37}$$

$$C_{ruup} = g_{u[u,p]r} - \frac{1}{2}g^{mn}g_{um,r}E_{np} + \frac{1}{4}g^{rm}g_{um,r}g_{up,r}, \tag{A-38}$$

$$\begin{aligned} C_{upmq} = {} & -g_{up}g_{u[q,m]r} - g_{pn}\left(g^{rn}g_{u[m,r)}\right)_{,q]} - E_{p[m}g_{q]u,r} - 2g_{pn}\left(g^{ns}E_{s[q)}\right)_{,m]} \\ & -g_{pn}g^{rs}\ {}^{s}\Gamma^{n}{}_{s[q}g_{m]u,r} - 2g_{pn}g^{sk}E_{k[q}\ {}^{s}\Gamma^{n}{}_{m]s}, \end{aligned} \tag{A-39}$$

$$\begin{aligned} C_{upuq} = {} & g_{up}g_{u[u,q]r} - g_{pn}\left(\left(g^{rn}g_{u[u,r)}\right)_{,q]} - 2\left(g^{nm}E_{m[q)}\right)_{,u]} - \frac{{}^{s}\Gamma^{n}{}_{sq}}{2}\left(g^{rs}g_{uu,r} + 2g^{sm}E_{um}\right)\right) \\ & +\frac{1}{4}g^{rr}g_{up,r}g_{uq,r}. \end{aligned} \tag{A-40}$$