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MOHAMED I. RIFFI

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## Moments Equalities for Nonnegative Integer-Valued Random Variables

*Mohamed I. Riffi*

### Abstract

We present and prove two theorems about equalities for the  $n$ th moment of nonnegative integer-valued random variables. These equalities generalize the well known equality for the first moment of a nonnegative integer-valued random variable  $X$  in terms of its cumulative distribution function, or in terms of its tail distribution.

**Key words and phrases:** Expectation, Moments, Equalities.

### 1. Introduction

There is a well-known equality for the  $n$ th moment of a nonnegative random variable  $Y$  as an integral of a function of its tail distribution. A similar equality for the first moment of a nonnegative integer-valued random variable as a sum over  $x$  of a function of its tail distribution is also well known and used a lot in the literature (See [1, p. 43], for example). What we prove in this paper is a generalization of this sum equality when the random variable is integer-valued.

In the next section we will prove a generalization of the well-known equality in the discrete case. Our equality gives a neat formula of the  $n$ th moment, when it exists, for nonnegative integer-valued variables.

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## 2. Main Theorems

In this section we prove two identities that each will be used to prove one of our main theorems. The first identity is used to express a product of terms of the form  $(X - i)$  as a finite sum of products of similar terms when the sum ranges from 1 to a nonnegative integer  $x$ . The second identity is used to express  $x^n$  as a finite sum that ranges from 1 to a nonnegative integer  $x$ .

Before we proceed to the main theorems, we need the following lemma.

**Lemma 2.1** *Let  $x$  and  $n$  be nonnegative integers such that  $x > n$ . Then*

$$\binom{x}{n+1} = \sum_{j=1}^{x-n} \binom{x-j}{n}.$$

**Proof.** Apply Pascal's identity, namely

$$\binom{x}{n+1} = \binom{x-1}{n} + \binom{x-1}{n+1},$$

to each last term on the right-hand side of the resulting equation. Continue this procedure  $x - n - 1$  times to get

$$\begin{aligned} \binom{x}{n+1} &= \sum_{j=1}^{x-n-1} \binom{x-j}{n} + \binom{n+1}{n+1} \\ &= \sum_{j=1}^{x-n} \binom{x-j}{n}. \end{aligned}$$

□

**Lemma 2.2** *Let  $g_n(x) := \sum_{i=1}^x \prod_{j=1}^n (i - j)$  and  $f_n(x) := \prod_{j=0}^n (x - j)$ , where  $x$  and  $n$  are nonnegative integers such that  $x > n$ . Then  $(n+1)g_n(x) = f_n(x) = (x)_{n+1}$ , where  $(z)_m = z(z-1)\dots(z-m+1)$ .*

**Proof.** Notice that

$$f_n(x) = x(x-1)\cdots(x-n) = (x)_{n+1} = (n+1)! \binom{x}{n+1}.$$

We also notice that  $g_n(x)$  is a finite sum of terms of the form  $(x-j)(x-j-1)(x-j-2)\cdots$  of length  $n$ . Such a general term can be expressed as  $(x-j)_n$  for  $j = 1, 2, \dots, n$ . The number of these terms is  $x - n$ . Therefore,

$$g_n(x) = \sum_{j=1}^{x-n} (x-j)_n. \quad (2.1)$$

Since  $(x-j)_n = n! \binom{x-j}{n}$ , (2.1) can be written as

$$g_n(x) = n! \sum_{j=1}^{x-n} \binom{x-j}{n}.$$

By Lemma 2.1, we have

$$g_n(x) = n! \binom{x}{n+1}.$$

This implies that  $(n+1)g_n(x) = f_n(x)$ . □

**Remark 2.1** Lemma 2.2 simply says that, for  $x > n \geq 2$ ,

$$\prod_{i=0}^{n-1} (x-i) = n \sum_{y=1}^x \prod_{i=1}^{n-1} (y-i). \quad (2.2)$$

**Theorem 2.1** Let  $X$  be a nonnegative integer-valued random variable and  $n \geq 2$ . Then

$$E \left( \prod_{i=0}^{n-1} (X-i) \right) = \begin{cases} n \sum_{x=n-1}^{\infty} \left( \prod_{i=0}^{n-2} (x-i) P(X > x) \right) & \text{if } X > n \\ 0 & \text{if } X \leq n, \end{cases} \quad (2.3)$$

provided that the sum on the right-hand side of (2.3) exists.

**Proof.** Note that when  $X < n$ , the left-hand side of (2.3) is 0. The proof of this

theorem when  $X > n$  mainly depends on (2.2) which is proved in Lemma 2.2.

$$\begin{aligned}
\mathbb{E} \left( \prod_{i=0}^{n-1} (X - i) \right) &= \sum_{x \geq 0} \prod_{i=0}^{n-1} (x - i) P(X = x) \\
&= n \sum_{x \geq 0} P(X = x) \sum_{y=1}^x \prod_{i=1}^{n-1} (y - i) \quad (\text{by (2.2)}) \\
&= n \sum_{y=1}^{\infty} \sum_{x=y}^{\infty} \prod_{i=1}^{n-1} (y - i) P(X = x) \quad (\text{Fubini's Theorem}) \\
&= n \sum_{y=1}^{\infty} \prod_{i=1}^{n-1} (y - i) P(X \geq y) \\
&= n \sum_{x=n}^{\infty} \prod_{i=1}^{n-1} (x - i) P(X \geq x) \\
&= n \sum_{x=n-1}^{\infty} \prod_{i=0}^{n-2} (x - i) P(X > x),
\end{aligned} \tag{2.4}$$

where we have changed the dummy variable  $y$  to  $x$ . □

**Remark 2.2** *We may start the sum on the right-hand side of (2.3) at  $x = 0$ , since the first  $n - 1$  terms of the product  $n \prod_{i=0}^{n-2} (x - i)$  are equal to 0.*

**Remark 2.3** *The case when  $n = 1$  is well known and can be stated separately for notational convenience, where the proof can be found in [1], for example, and will be part of our next main theorem.*

$$E(X) = \sum_{x=0}^{\infty} P(X > x).$$

Our second main theorem provides an explicit formula for the  $n$ th moment of a nonnegative integer-valued random variable  $X$ . We need the following lemma before we state and prove the second main theorem.

**Lemma 2.3** Let  $A_n(x)$  be defined for  $n = 1, 2, \dots$  as follows:

$$A_n(x) := \sum_{y=1}^n (-1)^{y+1} \binom{n}{y} x^{n-y}, \quad x = 1, 2, \dots$$

Then we have the following:

- (1)  $A_n(x+1) = \sum_{y=1}^n \binom{n}{y} x^{n-y}$
- (2)  $\sum_{i=1}^x A_n(i) = x^n$
- (3)  $\sum_{i=1}^x A_n(i+1) = (1+x)^n - 1$ .

**Proof.** To prove (1), note first that by adding and subtracting the term of the sum when  $y = 0$  we get

$$\begin{aligned} A_n(x) &= \sum_{y=0}^n (-1)^{y+1} \binom{n}{y} x^{n-y} + x^n \\ &= - \sum_{y=0}^n (-1)^y \binom{n}{y} x^{n-y} + x^n \\ &= x^n - (x-1)^n \quad (\text{by the binomial theorem}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} A_n(x+1) &= (1+x)^n - x^n \\ &= \sum_{y=0}^n \binom{n}{y} x^{n-y} - x^n \\ &= \sum_{y=1}^n \binom{n}{y} x^{n-y}. \end{aligned}$$

To prove (2) we see that

$$\begin{aligned} \sum_{i=1}^x A_n(i) &= \sum_{i=1}^x i^n - \sum_{i=1}^x (i-1)^n \\ &= \sum_{i=1}^x i^n - \sum_{j=0}^{x-1} j^n \quad (\text{by setting } j = i-1) \\ &= \sum_{j=0}^x j^n - \sum_{j=0}^{x-1} j^n = x^n. \end{aligned}$$

To prove (3) we see that

$$\begin{aligned}
 \sum_{i=1}^x A_n(i+1) &= \sum_{i=1}^x (1+i)^n - \sum_{i=1}^x i^n \\
 &= \sum_{j=2}^{x+1} j^n - \sum_{i=1}^x i^n \quad (\text{by setting } j = i+1) \\
 &= \sum_{j=1}^x j^n - \sum_{j=1}^x j^n - 1 + (1+x)^n \\
 &= (1+x)^n - 1.
 \end{aligned}$$

□

**Theorem 2.2** *Let  $X$  be a nonnegative integer-valued random variable and  $n \geq 1$ . If the sum  $\sum_{x=0}^{\infty} A_n(x+1)P(X > x)$  exists, then*

$$\begin{aligned}
 E(X^n) &= \sum_{x=0}^{\infty} A_n(x+1)P(X > x) \\
 &= \sum_{x=0}^{\infty} [(1+x)^n - x^n]P(X > x),
 \end{aligned} \tag{2.5}$$

where  $A_n(x+1) := \sum_{y=1}^x \binom{n}{y} x^{n-y}$ .

**Proof.** Note that  $X^n = \sum_{i=1}^X A_n(i)$ . Take the expectation of both sides to get

$$\begin{aligned}
 E(X^n) &= E\left(\sum_{i=1}^X A_n(i)\right) \\
 &= \sum_{x=0}^{\infty} \sum_{i=1}^x A_n(i)P(X = x) \\
 &= \sum_{i=1}^{\infty} \sum_{x=i}^{\infty} A_n(i)P(X = x) \quad (\text{by Fubini's Theorem})
 \end{aligned}$$

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$$\begin{aligned} &= \sum_{i=1}^{\infty} A_n(i)P(X \geq i) \\ &= \sum_{i=0}^{\infty} A_n(i+1)P(X > i) \\ &= \sum_{x=0}^{\infty} A_n(x+1)P(X > x), \end{aligned}$$

by changing the dummy variable  $i$  to  $x$ .

Now the last part of (2.5) follows from the fact that

$$A_n(x+1) = \sum_{y=1}^n \binom{n}{y} x^{n-y} = (1+x)^n - x^n.$$

□

### References

- [1] Kai Lai Chung, *A Course in Probability Theory*, Second Edition, Academic Press, New York, 1974.
- [2] George Casella, Roger L. Berger, *Statistical Inference*, Duxbury Press, California, 1990.

Mohamed I. RIFFI  
Associate Professor of Mathematics  
Department of Mathematics  
Islamic University of Gaza  
Gaza, Palestine  
e-mail: miriffi@hotmail.com

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