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A New Characteristic of Möbius Transformations by Use of Apollonius Points of Pentagons

Serap Bulut, Nihal Yılmaz Özgür

Abstract

In this paper, we give a new characterization of Möbius transformations. To this end, a new concept of “Apollonius points of pentagons” is used.

Key Words: Möbius transformations; Apollonius points of pentagons

1. Introduction

Throughout the paper, unless otherwise stated, let $w = f(z)$ be a nonconstant meromorphic function on the complex plane \mathbb{C} . Let us consider the following Property 1:

Property 1. $w = f(z)$ maps circles in the z -plane onto circles in the w -plane, including straight lines among circles.

The well known principle of circle transformation (see [1], [3]) reads as follows:

Theorem 1.1 $w = f(z)$ satisfies Property 1 iff $w = f(z)$ is a Möbius transformation.

In [2], Haruki and Rassias introduced the definition of the Apollonius point of a triangle, afterwards in [5], Piyapong Niamsup extended this definition to the (k, l) -Apollonius point of a triangle. Then, by means of these definitions, two new invariant characteristic properties of Möbius transformations were obtained. We recall that the following two definitions from [2] and [5], respectively.

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Definition 1.2 [2] Let $\triangle ABC$ be an arbitrary triangle and L be a point on the complex plane. We denote by $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = by = cz$ holds, then L is said to be an Apollonius point of $\triangle ABC$.

Definition 1.3 [5] Let $\triangle ABC$ be an arbitrary triangle and L be a point on \mathbb{C} . We denote by $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = k(by) = l(cz)$ holds, where $k, l > 0$, then L is said to be a (k, l) -Apollonius point of $\triangle ABC$.

The purpose of this paper is to give a new characterization of Möbius transformations. To do this, we introduce the notions of an Apollonius point and of a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point of a pentagon in Section 2 where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}^+$. Then we give the following new property:

Property 2. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z -plane. Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary pentagon contained in R and let its $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point L be a point of R . If we set $Z'_i = f(Z_i)$ for $1 \leq i \leq 5$, $L' = f(L)$ and if the five different points Z'_i ($1 \leq i \leq 5$) form a pentagon (i.e., any triple of Z'_i ($1 \leq i \leq 5$) are not collinear), then the point L' is also a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point of $Z' = Z'_1Z'_2Z'_3Z'_4Z'_5$.

Finally we prove the following theorem as a main theorem of this paper in Section 3.

Main Theorem. The following propositions are equivalent:

(i) $w = f(z)$ is a Möbius transformation.

(ii) Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z -plane. For every quadruple $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, if L is a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point of the pentagon $Z = Z_1Z_2Z_3Z_4Z_5$ contained in R , then $f(L)$ is a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point of the pentagon $Z' = Z'_1Z'_2Z'_3Z'_4Z'_5$ where $Z'_i = f(Z_i)$, $1 \leq i \leq 5$.

2. $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius Points of a Pentagon

Definition 2.1 Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary pentagon (not necessarily simple) and L be a point on \mathbb{C} . If the following equality holds for $2 \leq k \leq 5$, then L is said to be a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point of Z :

$$|L - Z_1| \cdot |Z_2 - Z_3| \cdot |Z_4 - Z_5| = \lambda_{k-1} |L - Z_k| \cdot |Z_{k+1} - Z_{k+2}| \cdot |Z_{k+3} - Z_{k+4}|,$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}^+$. In the right side of the above equation, if the values depend on k are different from 5, then we consider these values in $\text{mod}(5)$.

Remark 2.2 For $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$, this definition gives the definition of Apollonius point of an arbitrary pentagon.

Theorem 2.3 Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary pentagon on the complex plane \mathbb{C} and let the positive real numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be fixed. Then the number of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius points of Z is at most 2.

Proof. The proof follows from the Theorem of Apollonius, [2], and from the fact that if two circles meet, including straight lines among circles, then there are at most two points of intersection. \square

Example 2.4 Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary regular pentagon. Then, the center of circumscribed circle of Z is its only Apollonius point.

For the proof of the Theorem 2.7 we need the following definition and theorem from [6].

Definition 2.5 A hexagon $ABCDEF$ (not necessarily simple) on the complex plane for which $\overline{AB} \cdot \overline{CD} \cdot \overline{EF} = \lambda \overline{BC} \cdot \overline{DE} \cdot \overline{FA}$ holds (where the bar denotes the length of the segment) is an λ -Apollonius hexagon where $\lambda > 0$.

Property 3. Suppose that f is analytic and univalent on a nonempty open region Δ on the complex plane. Let $ABCDEF$ be a λ -Apollonius hexagon in Δ . If we set $Z' = f(Z)$ ($Z = ABCDEF$), then $A'B'C'D'E'F'$ is also a λ -Apollonius hexagon.

Theorem 2.6 $w = f(z)$ satisfies Property 3 iff $w = f(z)$ is a Möbius transformation.

Now we can give the following theorem.

Theorem 2.7 Property 1 implies Property 2.

Proof. Let $w = f(z)$ satisfies Property 1. Suppose that $w = f(z)$ is analytic in a nonempty domain R on the z -plane. Then by Theorem 1.1 $w = f(z)$ is a Möbius transformation and so univalent in R . Let $Z = Z_1Z_2Z_3Z_4Z_5$ be an arbitrary pentagon contained in R and let its $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point L be a point of R . If we set $Z'_i = f(Z_i)$ for $1 \leq i \leq 5$, then by the univalence of $w = f(z)$, the five points Z'_i ($1 \leq i \leq 5$) are different.

We now prove that if any triple of Z'_i ($1 \leq i \leq 5$) are not collinear, then the point $L' = f(L)$ is also a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point of $Z' = Z'_1 Z'_2 Z'_3 Z'_4 Z'_5$. Since L is a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point of Z , by the Definition 2.1, for $k = 5$, we have

$$|L - Z_1| \cdot |Z_2 - Z_3| \cdot |Z_4 - Z_5| = \lambda_4 |L - Z_5| \cdot |Z_1 - Z_2| \cdot |Z_3 - Z_4|.$$

Therefore by the Definition 2.5, $LZ_1Z_2Z_3Z_4Z_5$ is a λ_4 -Apollonius hexagon. By the Theorem 2.6 and [6], $L'Z'_1Z'_2Z'_3Z'_4Z'_5$ is a λ_4 -Apollonius hexagon. Hence we obtain

$$|L' - Z'_1| \cdot |Z'_2 - Z'_3| \cdot |Z'_4 - Z'_5| = \lambda_4 |L' - Z'_5| \cdot |Z'_1 - Z'_2| \cdot |Z'_3 - Z'_4|. \quad (1)$$

Similarly, we have

$$\lambda_4 |L' - Z'_5| \cdot |Z'_1 - Z'_2| \cdot |Z'_3 - Z'_4| = \lambda_3 |L' - Z'_4| \cdot |Z'_5 - Z'_1| \cdot |Z'_2 - Z'_3|, \quad (2)$$

$$\lambda_3 |L' - Z'_4| \cdot |Z'_5 - Z'_1| \cdot |Z'_2 - Z'_3| = \lambda_2 |L' - Z'_3| \cdot |Z'_4 - Z'_5| \cdot |Z'_1 - Z'_2|, \quad (3)$$

$$\lambda_2 |L' - Z'_3| \cdot |Z'_4 - Z'_5| \cdot |Z'_1 - Z'_2| = \lambda_1 |L' - Z'_2| \cdot |Z'_3 - Z'_4| \cdot |Z'_5 - Z'_1|. \quad (4)$$

By (1) – (4), we obtain that the following products is equal for every $2 \leq k \leq 5$:

$$|L' - Z'_1| \cdot |Z'_2 - Z'_3| \cdot |Z'_4 - Z'_5| = \lambda_{k-1} |L' - Z'_k| \cdot |Z'_{k+1} - Z'_{k+2}| \cdot |Z'_{k+3} - Z'_{k+4}|.$$

By the Definition 2.1, we obtain that $L' = f(L)$ is also a $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ -Apollonius point of Z' . Consequently, $w = f(z)$ satisfies Property 2. \square

3. Proof of the Main Theorem

Proof of the Main Theorem. Let $w = f(z)$ be a Möbius transformation. Then by Theorem 1.1, $w = f(z)$ satisfies Property 1. Thus by Theorem 2.7, $w = f(z)$ satisfies Property 2. This proves (ii).

Now assume that the function $w = f(z)$ satisfies (ii). Since $w = f(z)$ is analytic and univalent in the domain R , by a well known lemma

$$f'(z) \neq 0 \quad (5)$$

holds in R .

If x is an arbitrary fixed point of R , then by (5), we obtain

$$f'(x) \neq 0. \tag{6}$$

Let L be the point represented by x . Since $L \in R$, there exists a positive real number r such that the r closed circular neighborhood of L is contained in R . We denote this closed circular neighborhood by V .

Throughout the proof let $Z = Z_1Z_2Z_3Z_4Z_5$ denote an arbitrary regular pentagon which is contained in V and whose center is at L . Here the sense of Z_1, Z_2, Z_3, Z_4, Z_5 is counterclockwise. Since $Z = Z_1Z_2Z_3Z_4Z_5$ is a regular pentagon contained in V , we can represent Z_1, Z_2, Z_3, Z_4, Z_5 by complex numbers as

$$x + w_{k+1}y,$$

where $0 < |y| \leq r$ and $w_{k+1} = e^{\frac{i2\pi k}{5}}$, $0 \leq k \leq 4$.

Since $w = f(z)$ is univalent in R , $Z'_1 = f(Z_1), Z'_2 = f(Z_2), Z'_3 = f(Z_3), Z'_4 = f(Z_4), Z'_5 = f(Z_5)$ are different points. By a property of analytic functions (see [4]) and by (6) (any triple of Z_1, Z_2, Z_3, Z_4, Z_5 are not collinear on the z -plane) there exists some sufficiently small positive real number s satisfying

$$s \leq r$$

such that any triple of $Z'_1, Z'_2, Z'_3, Z'_4, Z'_5$ are not collinear on the w -plane for all y satisfying $0 < |y| \leq s$.

Since L is the Apollonius point of the regular pentagon Z ($0 < |y| \leq s$) (see Example 2.4) and any triple of $Z'_1, Z'_2, Z'_3, Z'_4, Z'_5$ are not collinear, by hypothesis $L' = f(L)$ is also an Apollonius point of $Z' = Z'_1Z'_2Z'_3Z'_4Z'_5$. Hence, by definition we obtain

$$|L' - Z'_1| \cdot |Z'_2 - Z'_3| \cdot |Z'_4 - Z'_5| \tag{7}$$

$$= |L' - Z'_2| \cdot |Z'_3 - Z'_4| \cdot |Z'_5 - Z'_1| \tag{8}$$

$$= |L' - Z'_3| \cdot |Z'_4 - Z'_5| \cdot |Z'_1 - Z'_2| \tag{9}$$

$$= |L' - Z'_4| \cdot |Z'_5 - Z'_1| \cdot |Z'_2 - Z'_3| \tag{10}$$

$$= |L' - Z'_5| \cdot |Z'_1 - Z'_2| \cdot |Z'_3 - Z'_4|. \tag{11}$$

Let us consider (7) and (9):

$$|L' - Z'_1| \cdot |Z'_2 - Z'_3| \cdot |Z'_4 - Z'_5| = |L' - Z'_3| \cdot |Z'_4 - Z'_5| \cdot |Z'_1 - Z'_2|.$$

Hence we get

$$|L' - Z'_1| \cdot |Z'_2 - Z'_3| = |L' - Z'_3| \cdot |Z'_1 - Z'_2|.$$

Since $Z'_1, Z'_2, Z'_3, Z'_4, Z'_5, L'$ are represented by

$$f(x + w_{k+1}y), f(x),$$

where $0 \leq k \leq 4$, respectively, from the last equation we obtain

$$\begin{aligned} & |f(x) - f(x + y)| \cdot |f(x + w_2y) - f(x + w_3y)| \\ = & |f(x) - f(x + w_3y)| \cdot |f(x + y) - f(x + w_2y)|, \end{aligned}$$

and so

$$\left| \frac{[f(x) - f(x + y)][f(x + w_2y) - f(x + w_3y)]}{[f(x) - f(x + w_3y)][f(x + y) - f(x + w_2y)]} \right| = 1.$$

By a similar way in [2], after calculations we finally get

$$\frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 = 0$$

holds for all z satisfying $f'(z) \neq 0$.

Hence, the Schwarzian derivative of f vanishes for all z satisfying $f'(z) \neq 0$. Consequently, by a well-known fact f is a Möbius transformation. \square

Corollary 3.1 *This theorem gives a new proof of the only if part of Theorem 1.1.*

Proof. By hypothesis $w = f(z)$ satisfies Property 1. Hence, by the Theorem 2.7, $w = f(z)$ satisfies Property 2. Consequently, by the Main Theorem, $w = f(z)$ is a Möbius transformation. \square

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