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Abstract

We investigate the spectrum of a differential operator of fourth order with bounded operator coefficients and find a formula for the trace of this operator.

Key Words: Hilbert Space, Self-adjoint operator, Kernel operator, Spectrum, Essential spectrum, Resolvent.

1. Introduction

Let H be a separable Hilbert space of infinite dimension. Consider the operators L_0 and L in the space $H_1 = L_2(H; [0.\pi])$ which are formed by differential expressions

$$l_0(y) = y^{\text{IV}}(x), \qquad l(y) = y^{\text{IV}}(x) + Q(x)y(x)$$

with the same boundary conditions $y'(o) = y'(\pi) = 0$ and $y'''(0) = y'''(\pi) = 0$, respectively. Suppose that the operator function $Q(x)$ in the expression $l(y)$ satisfies the following conditions:

1. For every $x \in [0, \pi]$, $Q(x) : H \to H$ is a self adjoint kernel operator. Moreover, $Q(x)$ has weak derivative of second order in this interval and for $x \in [0, \pi]$, $Q^{(i)}(x) : H \to H$ are self-adjoint operators $(i = 1, 2)$.
2. $$||Q||_{H_1} < \frac{1}{2}.$$
3. There is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of the space H such that
$$\sum_{n=1}^{\infty} ||Q(x)\varphi_n||_{H_1} < \infty.$$

4. The functions $||Q^{(i)}(x)||_{\sigma_1(H)}$ are bounded and measurable functions in $[0,\pi]$, $(i = 0,1,2)$.

Here $\sigma_1(H)$ is the space of kernel operators from H to H as in [16]. Moreover, we denote the norms by $||\cdot||$ and $||\cdot||_{H_1}$ and the inner products by $(\cdot,\cdot)$ and $(\cdot,\cdot)_{H_1}$in H and H_1, respectively and also denote the sum of eigenvalues of a kernel operator A by $\mathrm{tr}A = \mathrm{trace}A$.

The spectrum of operator L_0 is the set $\{m^4\}_{m=0}^{\infty}$. Every point of this set is an eigenvalue of L_0which has infinite multiplicity. The orthonormal eigenfunctions corresponding to eigenvalue m^4 are in the form

$$\psi_{mn}^{0}(x) = d_m \cos mx \cdot \varphi_n \quad (n = 1,2,...) \tag{1}$$

where

$$d_m = \left\{ \begin{array}{cc} \frac{1}{\sqrt{\pi}} & m = 0 \\ \sqrt{\frac{2}{\pi}} & m = 1,2,... \end{array} \right\} \tag{2}$$

In this work , we will firstly investigate the spectrum of operator L and find a formula for the sum of the series

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn} - m^4\right) - \frac{1}{\pi}\int_0^{\pi}\mathrm{tr}Q(x)dx\right], \tag{3}$$

where $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of operator L which belong to the interval $\left[m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}\right]$ $(m = 0,1,2,...)$

Trace formulas for the scalar differential operators have been found by Gelfand and Levitan [1], Dikiy [2], Halberg and Kramer [3], Levitan [4], Lidskiy and Sadovniçiy [5], Guseynov and Levitan [6] and many others. A list of the works on this subject is presented by Levitan and Sargsyan [7] and Fulton and Pruess [8]. On the other hand, trace formulas for differential operators with operator coefficients has been investigated by Adıgözelov [9], Chalilova [10], Maksudov, Bayramoglu and Adıgözelov [11], Adıgüzelov, Avci and Gül [12], Albayrak, Baykal and Gül [13] and Maksudov, Bairamoglu and Adigezalov [17]. A trace formula for higher order, including fourth order, differential operators with operator coefficients has been given in [17]. It is this latter problem we study in the present work, but with differential operators and boundary conditions different from those in [17].

2. The Spectrum of Operator L

Let R_λ^0 and R_λ be resolvents of the operators L_0 and L, respectively.

Lemma 1 *If the operator function $Q(x)$ satisfies condition 3, and $\lambda \notin \{m^4\}_{m=0}^{\infty} = \sigma(L_0)$, then $QR_\lambda^0 : H_1 \to H_1$ is a kernel operator, i.e. $QR_\lambda^0 \in \sigma_1(H_1)$.*

Proof. System (1) of the eigenfunctions of L_0 is an orthonormal basis of space H_1. As known in [16], to show that QR_λ^0 is a kernel operator, it is enough to see that the series

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} ||QR_\lambda^0 \psi_{mn}^0||_{H_1}$$

is convergent. From (1) and (2), we find

$$\begin{aligned}
\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} ||QR_\lambda^0 \psi_{mn}^0||_{H_1} &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} |m^4-\lambda|^{-1} \cdot ||Q\psi_{mn}^0||_{H_1} \\
&= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} |m^4-\lambda|^{-1} [\int_0^\pi (Q(x) d_m \cos mx \cdot \varphi_n, \\
&\quad Q(x) d_m \cos mx \cdot \varphi_n)]^{\frac{1}{2}} \\
&= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} |m^4-\lambda|^{-1} \\
&\quad \cdot \left[\int_0^\pi d_m^2 \cos^2 mx ||Q(x)\varphi_n||^2 dx\right]^{\frac{1}{2}} \\
&\leq \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} |m^4-\lambda|^{-1} \left[\int_0^\pi ||Q(x)\varphi_n||^2 dx\right]^{\frac{1}{2}} \\
&= \sum_{m=0}^{\infty} |m^4-\lambda|^{-1} \sum_{n=1}^{\infty} ||Q(x)\varphi_n||_{H_1}. \qquad (4)
\end{aligned}$$

In view of (4) and condition 3 we conclude that

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} ||QR_\lambda^0 \psi_{mn}^0||_{H_1} < \infty.$$

This proves the lemma □

Theorem 2 *If $Q(x)$ satisfies conditions 2, and 3, then the spectrum of operator L is a subset of the union of pairwise disjoint intervals $F_m = \left[m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}\right]$ $(m = 0, 1, 2, ...)$; and the following conditions are satisfied:*

1. *Each point of spectrum of L which is different from m^4 in F_m is an isolated eigenvalue which has finite multiplicity.*
2. *m^4 can be an eigenvalue of L which has finite or infinite multiplicity.*
3. *$\lim\limits_{n\to\infty} \lambda_{mn} = m^4$ such that $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of L in F_m.*

Proof. If

$$\lambda \in R \backslash \cup_{m=0}^{\infty} [m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}],$$

then we get

$$|\lambda - m^4| > ||Q||_{H_1} \qquad (m = 0, 1, 2, ...). \tag{5}$$

For the self adjoint operator $R_\lambda^0 = (L_0 - \lambda I)^{-1}$, since $||\, R_\lambda^0||_{H_1} = \max\limits_m |\lambda - m^4|^{-1}$, then from (5) we can write

$$||R_\lambda^0||_{H_1} < ||Q||_{H_1}^{-1}.$$

Because of this, we have

$$||QR_\lambda^0||_{H_1} \leq ||Q||_{H_1} \cdot ||R_\lambda^0||_{H_1} < 1.$$

By considering this inequality, we conclude that

$$A(B) = R_\lambda^0 - BQR_\lambda^0$$

is a contraction operator from $L(H_1, H_1)$ to $L(H_1, H_1)$, where $B \in L(H_1, H_1)$. In this case, it is known that there exists an unique solution $B = B_0$ which belongs to the space $L(H_1, H_1)$ of the equation $R_\lambda^0 - BQR_\lambda^0 = B$. Moreover, since $R_\lambda^0 - R_\lambda QR_\lambda^0 = R_\lambda$ we have $R_\lambda = B_0 \in L(H_1, H_1)$ and so $\lambda \in \rho(L)$ (resolvent set of L). Hence, the spectrum of L is a subset of the union of the pairwise disjoint intervals $\left[m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}\right]$,

$(m=0,1,2,...)$, i.e. $\sigma(L)\subset\ \cup_{m=0}^{\infty}[m^4-||Q||_{H_1},m^4+||Q||_{H_1}]$. From Lemma 1 and the equation $R_\lambda=R_\lambda^0-R_\lambda QR_\lambda^0$, for every $\lambda\in\rho(L_0)\cap\rho(L)$ we see that $R_\lambda-R_\lambda^0$ is a kernel operator from $H_1$to H_1. This means that, as known from [14], the essential spectra of L and L_0 coincide. According to this, and since L_0 has only the essential spectrum, the essential spectrum of L will be the set $\{m^4\}_{m=0}^{\infty}$ and this shows that conditions 1, 2, 3, in the hypothesis of theorem 2 are satisfied. □

3. A Formula for the Trace of L

In this section, we obtain a formula for the sum of series (3). The sum of this series is called the regularized trace of operator L.

Lemma 3 *If $Q(x)$ satisfies conditions* 2, *and* 3,*, then operator function $R_\lambda-R_\lambda^0$ is analytic in the region $\rho(L)$ with respect to the norm in $\sigma_1(H_1)$.*

Proof. Since $R_\lambda-R_\lambda^0=-R_\lambda QR_\lambda^0$, to prove this lemma we need to show that the operator function $R_\lambda QR_\lambda^0$ is analytic in the region $\rho(L)$. First, from Theorem 2, it follows that $\rho(L)\subset\rho(L_0)$. Moreover, by using the relation $R_\lambda-R_\mu=(\lambda-\mu)R_\lambda R_\mu$ we have

$$\begin{aligned}
D(\lambda,\Delta\lambda) &= \frac{R_{\lambda+\Delta\lambda}QR^0_{\lambda+\Delta\lambda}-R_\lambda QR^0_\lambda}{\Delta\lambda}-R^2_\lambda QR^0_\lambda-R_\lambda Q(R^0_\lambda)^2\\
&= \frac{1}{\Delta\lambda}[(R_{\lambda+\Delta\lambda}QR^0_{\lambda+\Delta\lambda}-R_{\lambda+\Delta\lambda}QR^0_\lambda)+(R_{\lambda+\Delta\lambda}QR^0_\lambda\\
&\quad -R_\lambda QR^0_\lambda)]-R^2_\lambda QR^0_\lambda-R_\lambda Q(R^0_\lambda)^2\\
&= \frac{1}{\Delta\lambda}R_{\lambda+\Delta\lambda}Q(R^0_{\lambda+\Delta\lambda}-R^0_\lambda)+\frac{1}{\Delta\lambda}(R_{\lambda+\Delta\lambda}-R_\lambda)QR^0_\lambda\\
&\quad -R^2_\lambda QR^0_\lambda-R_\lambda Q(R^0_\lambda)^2\\
&= R_{\lambda+\Delta\lambda}QR^0_\lambda R^0_{\lambda+\Delta\lambda}+R_{\lambda+\Delta\lambda}R_\lambda QR^0_\lambda-R^2_\lambda QR^0_\lambda-R_\lambda Q(R^0_\lambda)^2\\
&= [R_{\lambda+\Delta\lambda}QR^0_\lambda R^0_{\lambda+\Delta\lambda}-R_{\lambda+\Delta\lambda}Q(R^0_\lambda)^2]+[R_{\lambda+\Delta\lambda}(R^0_\lambda)^2\\
&\quad -R_\lambda Q(R^0_\lambda)^2]+(R_{\lambda+\Delta\lambda}R_\lambda QR^0_\lambda-R^2_\lambda QR^0_\lambda)\\
&= R_{\lambda+\Delta\lambda}QR^0_\lambda(R^0_{\lambda+\Delta\lambda}-R^0_\lambda)+(R_{\lambda+\Delta\lambda}-R_\lambda)Q(R^0_\lambda)^2\\
&\quad +(R_{\lambda+\Delta\lambda}-R_\lambda)R_\lambda QR^0_\lambda. \qquad (6)
\end{aligned}$$

From this, we obtain

$$\begin{aligned}||D(\lambda,\Delta\lambda)||_{\sigma_1(H_1)} &\leq ||R_{\lambda+\Delta\lambda}QR_\lambda^0||_{\sigma_1(H_1)}||R^0_{\lambda+\Delta\lambda}-R^0_\lambda||_{H_1}\\ &\quad +||R_{\lambda+\Delta\lambda}-R_\lambda||_{H_1}[||Q(R^0_\lambda)^2||_{\sigma_1(H_1)}\\ &\quad +||R_\lambda QR^0_\lambda||_{\sigma_1(H_1)}]\\ &\leq ||R_{\lambda+\Delta\lambda}||_{H_1}||QR^0_\lambda||_{\sigma_1(H_1)}||R^0_{\lambda+\Delta\lambda}-R^0_\lambda||_{H_1}\\ &\quad +||R_{\lambda+\Delta\lambda}-R_\lambda||_{H_1}||QR^0_\lambda||_{\sigma_1(H_1)}[||R^0_\lambda||_{H_1}\\ &\quad +||R_\lambda||_{H_1}]. \qquad (7)\end{aligned}$$

Since

$$\lim_{\Delta\lambda\to\infty}||R_{\lambda+\Delta\lambda}-R_\lambda||_{H_1}=\lim_{\Delta\lambda\to\infty}||R^0_{\lambda+\Delta\lambda}-R^0_\lambda||_{H_1}=0,$$

and from (6) and (7), we find

$$\lim_{\Delta\lambda\to\infty}||\frac{R_{\lambda+\Delta\lambda}QR^0_{\lambda+\Delta\lambda}-R_\lambda QR^0_\lambda}{\Delta\lambda}-R^2_\lambda QR^0_\lambda-R_\lambda Q(R^0_\lambda)^2||_{\sigma_1(H_1)}=0.$$

This shows that the operator function $R_\lambda - R^0_\lambda = -R_\lambda QR^0_\lambda$ is analytic in the region $\rho(L)$ with respect to the norm in $\sigma_1(H_1)$, as desired. □

Let $\{\psi_{mn}(x)\}_{m,n=1}^\infty$ be orthonormal eigenfunctions corresponding to eigenvalues $\{\lambda_{mn}\}_{m,n=1}^\infty$ of L and let

$$\Gamma_p=\{\lambda:|\lambda-p^4|=\frac{1}{2}\}\ ;\quad B^0_{mn}=(\cdot,\psi^0_{mn})_{H_1}\psi^0_{mn};\quad B_{mn}=(\cdot,\psi_{mn})_{H_1}\psi_{mn};$$

and

$$L^{(r)}_{0m}=\sum_{n=1}^\infty m^{4r}B^0_{mn};\qquad L^{(r)}_m=\sum_{n=1}^\infty \lambda^r_{mn}B_{mn},\quad (r=-1,1).$$

The spectra of operators L and L_0 only consist of eigenvalues and its limit points. Hence, from [15], we know that

$$R^0_\lambda=\sum_{m=0}^\infty\sum_{n=1}^\infty\frac{B^0_{mn}}{m^4-\lambda}\ ;\quad R_\lambda=\sum_{m=0}^\infty\sum_{n=1}^\infty\frac{B_{mn}}{\lambda_{mn}-\lambda}. \qquad (8)$$

Theorem 4 *If $Q(x)$ satisfies conditions 2 and 3, then the series*

$$\sum_{n=1}^{\infty}(\lambda_{pn} - p^4) \qquad (p = 0, 1, ...)$$

are absolutely convergent.

Proof. Since $\{\lambda_{mn}\}_{n=1}^{\infty} \subset \left[m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}\right]$, and from the assumption $||Q||_{H_1} < \frac{1}{2}$, for $m < p$, we have

$$\lambda_{mn} \leq m^4 + ||Q||_{H_1} < m^4 + \frac{1}{2} \leq (m+1)^4 - \frac{1}{2} \leq p^4 - \frac{1}{2} \qquad (n = 1, 2, ...).$$

Thus we find

$$\lambda_{mn} < p^4 - \frac{1}{2} \quad or \quad |\lambda_{mn} - p^4| > \frac{1}{2} \qquad (m < p; n = 1, 2, ...). \tag{9}$$

For $p < m$,

$$p^4 + \frac{1}{2} \leq (p+1)^4 - \frac{1}{2} \leq m^4 - \frac{1}{2} < m^4 - ||Q||_{H_1} \leq \lambda_{mn} \quad (n = 1, 2, ...).$$

and so we obtain

$$\lambda_{mn} > p^4 + \frac{1}{2} \quad or \quad |\lambda_{mn} - p^4| > \frac{1}{2} \qquad (m > p; n = 1, 2, ...). \tag{10}$$

By using (8), (9) and (10), we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Gamma_p} \lambda(R_\lambda - R_\lambda^0)d\lambda &= \frac{1}{2\pi i}\int_{\Gamma_p} \lambda[\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{B_{mn}}{\lambda_{mn} - \lambda} \\
&\quad - \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{B_{mn}^0}{m^4 - \lambda}d\lambda] \\
&= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}[B_{mn}\frac{1}{2\pi i}\int_{\Gamma_p}\frac{\lambda}{\lambda_{mn} - \lambda}d\lambda \\
&\quad - B_{mn}^0\frac{1}{2\pi i}\int_{\Gamma_p}\frac{\lambda}{m^4 - \lambda}d\lambda]
\end{aligned}$$

$$= \sum_{n=1}^{\infty}[B_{pn}^{0}\frac{1}{2\pi i}\int_{\Gamma_p}\frac{\lambda}{\lambda-p^4}d\lambda - B_{pn}\frac{1}{2\pi i}\int_{\Gamma_p}\frac{\lambda}{\lambda-\lambda_{pn}}d\lambda]$$

$$= \sum_{n=1}^{\infty}\left(p^4B_{pn}^{0}-\lambda_{pn}B_{pn}\right) = L_{0p}^{(1)} - L_{p}^{(1)} \qquad p=0,1,...$$

From this last relation and Lemma 3 we obtain

$$L_{0p}^{(1)} - L_{p}^{(1)} \in \sigma_1(H_1) \quad (p=0,1,2,...). \tag{11}$$

This time, let us show that $L_p^{(-1)} - L_{0p}^{(-1)} \in \sigma_1(H_1)$. Again if we use (8), (9) and (10) we find

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_p}\lambda^{-1}(R_\lambda - R_\lambda^0)d\lambda &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}[B_{mn}\frac{1}{2\pi i}\int_{\Gamma_p}\frac{d\lambda}{\lambda(\lambda_{mn}-\lambda)}\\ &\quad -B_{pn}^{0}\frac{1}{2\pi i}\int_{\Gamma_p}\frac{d\lambda}{\lambda(m^4-\lambda)}]\\ &= \sum_{n=1}^{\infty}[B_{pn}^{0}\frac{1}{2\pi i}\int_{\Gamma_p}\frac{d\lambda}{\lambda(\lambda-p^4)}\\ &\quad -B_{pn}\frac{1}{2\pi i}\int_{\Gamma_p}\frac{d\lambda}{\lambda(\lambda-\lambda_{pn})}]\\ &= \sum_{n=1}^{\infty}\left(p^{-4}B_{mn}^{0}-\lambda_{pn}^{-1}B_{pn}\right)\\ &= L_{0p}^{(-1)} - L_{p}^{(-1)} \quad (p=1,2,...). \end{aligned} \tag{12}$$

According to Lemma 3, the operator function $\lambda^{-1}(R_\lambda - R_\lambda^0)$ is analytic in the region $\rho(L)$ with respect to the norm in $\sigma_1(H_1)$. Hence, from (12)

$$L_{0p}^{(-1)} - L_{p}^{(-1)} \in \sigma_1(H_1) \quad (p=1,2,...). \tag{13}$$

Now, we can show that the series $\sum_{n=1}^{\infty}(\lambda_{pn}-p^4)$ $(p=0,1,...)$ are convergent. The spectrum of the operator $L_{0p}^{(r)}$ only consist of the points 0 and p^{4r}. In this case, from [15], we write

$$p^{4r} \geq (L_{0p}^{(r)}\psi_{pn},\psi_{pn})_{H_1} \qquad (p=1,2,...).$$

On the other hand,

$$\lambda_{pn}^r = (L_p^{(r)}\psi_{pn}, \psi_{pn})_{H_1}.$$

By using last two relation, we find

$$\begin{aligned}\sum_n(\lambda_{pn}^r - p^{4r})_{\lambda_{pn}^r > p^{4r}} &\leq \sum_n((L_p^{(r)} - L_{0p}^{(r)})\psi_{pn}, \psi_{pn})_{H_1} \\ &\leq \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left|((L_p^{(r)} - L_{0p}^{(r)})\psi_{mn}, \psi_{mn})_{H_1}\right| \end{aligned} \quad (14)$$

From (11) and (13) we have $L_p^{(r)} - L_{0p}^{(r)} \in \sigma_1(H_1)$ $(r = 1, -1;\ p = 1, 2, ...)$. Hence, from [16] we write

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left|((L_p^{(r)} - L_{0p}^{(r)})\psi_{mn}, \psi_{mn})_{H_1}\right| \leq \left\|L_{0p}^{(r)} - L_p^{(r)}\right\|_{\sigma_1(H_1)}. \quad (15)$$

From (14) and (15) we find

$$\sum_n(\lambda_{pn}^r - p^{4r})_{\lambda_{pn}^r > p^{4r}} \leq \left\|L_{0p}^{(r)} - L_p^{(r)}\right\|_{\sigma_1(H_1)} \qquad (p \geq 1) \quad (16)$$

$$\sum_n(\lambda_{pn} - p^4)_{\lambda_{pn} > p^4} < \infty \qquad (p \geq 1)$$

and

$$\begin{aligned}\sum_n(p^4 - \lambda_{pn})_{\lambda_{pn} < p^4} &\leq const \cdot \sum_n(p^4 - \lambda_{pn})_{\lambda_{pn} < p^4} p^{-4}\lambda_{pn}^{-1} \quad (p \geq 1) \\ &= const \cdot \sum_n(\lambda_{pn}^{-1} - p^{-4})_{\lambda_{pn}^{-1} > p^{-4}} < \infty \quad (p \geq 1). \end{aligned} \quad (17)$$

From (16) and (17) we have

$$\sum_{n=1}^{\infty}|\lambda_{pn} - p^4| < \infty \qquad (p \geq 1).$$

Moreover, by considering $L_{00}^{(1)} = 0$ and (11) we obtain

$$\sum_{n=1}^{\infty} |\lambda_{0n}| < \infty.$$

This proves Theorem 4 □

For every $\lambda \in \rho(L)$, since $R_\lambda - R_\lambda^0 \in \sigma_1(H_1)$ and from (8) and Theorem 4 we find

$$\operatorname{tr}(R_\lambda - R_\lambda^0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn}-\lambda} - \frac{1}{m^4-\lambda}\right).$$

Let us multiply this equation by $\frac{\lambda}{2\pi i}$ and integrate on the circle $|\lambda| = b_p = p^4 + 2p^3$, $(p \geq 1)$:

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda \cdot \operatorname{tr}(R_\lambda - R_\lambda^0) \;=\; & \frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda \\ & \cdot \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn}-\lambda} - \frac{1}{m^4-\lambda}\right) d\lambda \\ & + \frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda \\ & \cdot \sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn}-\lambda} - \frac{1}{m^4-\lambda}\right) d\lambda. \end{aligned} \tag{18}$$

On the other hand, for $m \leq p$ and $p \geq 1$ we have

$$m^4 - ||Q||_{H_1} \leq \lambda_{mn} \leq m^4 + ||Q||_{H_1} \leq p^4 + ||Q||_{H_1} < p^4 + 2p^3 = b_p$$

and so

$$|\lambda_{mn}| < b_p, \qquad m \leq p;\ p \geq 1;\ n = 1, 2, ... \tag{19}$$

and for $m > p$ we have

$$\lambda_{mn} \geq m^4 - ||Q||_{H_1} \geq (p+1)^4 - ||Q||_{H_1} > p^4 + 2p^3 = b_p$$

or

$$|\lambda_{mn}| > b_p, \qquad m > p;\ p \geq 1;\ n = 1, 2, ... \tag{20}$$

Hence, from (18), (19) and (20) we obtain

$$\begin{aligned}
\frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda \cdot \mathrm{tr}(R_\lambda - R_\lambda^0)d\lambda &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}[\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda - m^4}d\lambda \\
&\quad -\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-\lambda_{mn}}d\lambda] \\
&\quad +\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}[\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda - m^4}d\lambda \\
&\quad -\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-\lambda_{mn}}d\lambda] \\
&= \sum_{m=0}^{p}\sum_{n=1}^{\infty}(m^4 - \lambda_{mn}).
\end{aligned} \tag{21}$$

By using the formula $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$, we find

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{2}(-1)^j R_\lambda^0 (QR_\lambda^0)^j - R_\lambda (QR_\lambda^0)^3.$$

If we put this expression in equation (21), we have

$$\begin{aligned}
\sum_{m=0}^{p}\sum_{n=1}^{\infty}(m^4 - \lambda_{mn}) &= \sum_{j=1}^{2}\frac{(-1)^j}{2\pi i}\int_{|\lambda|=b_p}\lambda \cdot \mathrm{tr}[R_\lambda^0(QR_\lambda^0)^j]d\lambda \\
&\quad -\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda \cdot \mathrm{tr}[R_\lambda(QR_\lambda^0)^3]d\lambda.
\end{aligned} \tag{22}$$

Now, let

$$M_{pj} = \frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_p}\lambda \cdot \mathrm{tr}[R_\lambda^0(QR_\lambda^0)^j]d\lambda \ \ (j = 1, 2) \tag{23}$$

and

$$M_p = \frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda \cdot \operatorname{tr}[R_\lambda (QR_\lambda^0)^3] d\lambda. \tag{24}$$

Then we can write equation (22) in the form

$$\sum_{m=0}^{p} \sum_{n=1}^{\infty} (\lambda_{mn} - m^4) = M_{p1} + M_{p2} + M_p \tag{25}$$

In a similar way to the proof of Lemma 3, we can prove that the operator function QR_λ^0 is analytic with respect to the norm in $\sigma_1(H_1)$ at every point $\lambda \neq m^4 (m = 0, 1, 2, ...)$; and so we can show that the formulas

$$M_{pj} = \frac{(-1)^j}{2\pi i j} \int_{|\lambda|=b_p} \operatorname{tr}[(QR_\lambda^0)^j] d\lambda \quad (j = 1, 2) \tag{26}$$

are satisfied.

Lemma 5 *If $Q(x)$ satisfies condition 3 and the function $||Q(x)||_{\sigma_1(H_1)}$ is integrable in the interval $[0, \pi]$, then the formula*

$$M_{p1} = \frac{2p+1}{2\pi} \int_0^\pi trQ(x)dx + \frac{1}{2} \sum_{m=0}^{p} \sum_{n=1}^{\infty} d_m^2 \int_0^\pi (Q(x)\varphi_n, \varphi_n) \cos 2mx dx$$

holds.

Proof. Since the system (1) of eigenfunctions ψ_{mn}^0 $(m = 0, 1, 2, ...; n = 1, 2, ...)$ corresponding to eigenvalue m^4 of operator L_0 is an orthonormal basis of space H_1, and by using the formula (26), we have

$$\begin{aligned} M_{p1} &= -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \operatorname{tr}(QR_\lambda^0) d\lambda \\ &= -\frac{1}{2\pi i} \int_{|\lambda|=b_p} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (QR_\lambda^0 \psi_{mn}^0, \psi_{mn}^0)_{H_1} d\lambda. \end{aligned} \tag{27}$$

Taking advantage of (1) we can estimate the expression $|(QR_\lambda^0\psi_{mn}^0, \psi_{mn}^0)_{H_1}|$:

$$
\begin{aligned}
|(QR_\lambda^0\psi_{mn}^0, \psi_{mn}^0)_{H_1}| &= \left|\int_0^\pi (QR_\lambda^0\psi_{mn}^0(x), \psi_{mn}^0(x))dx\right| \\
&= |m^4-\lambda|^{-1}\left|\int_0^\pi (Q(x)\psi_{mn}^0(x), \psi_{mn}^0(x))dx\right| \\
&= |m^4-\lambda|^{-1} \\
&\quad \cdot\left|\int_0^\pi (Q(x)d_m\cos mx\cdot\varphi_n, d_m\cos mx\cdot\varphi_n)dx\right| \\
&\leq |m^4-\lambda|^{-1}\left|\int_0^\pi (Q(x)\varphi_n, \varphi_n)dx\right| \\
&\leq |m^4-\lambda|^{-1}\int_0^\pi ||Q(x)\varphi_n||dx \\
&\leq \sqrt{\pi}|m^4-\lambda|^{-1}\left(\int_0^\pi ||Q(x)\varphi_n||^2dx\right)^{\frac{1}{2}} \\
&= \sqrt{\pi}|m^4-\lambda|^{-1}||Q(x)\varphi_n||_{H_1}.
\end{aligned}
$$

Since $Q(x)$ satisfies condition (3), and from this last estimation we conclude that the series

$$\alpha_m(\lambda) = \sum_{n=1}^{\infty}(QR_\lambda^0\psi_{mn}^0, \psi_{mn}^0)_{H_1}(m=0,1,2,...);\ \sum_{m=0}^{\infty}\alpha_m(\lambda)$$

are absolutely and uniformly convergent with respect to λ on the circle $|\lambda| = b_p$. And so, from (27) we find

$$
\begin{aligned}
M_{p1} &= -\frac{1}{2\pi i}\int_{|\lambda|=b_p}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(QR_\lambda^0\psi_{mn}^0, \psi_{mn}^0)_{H_1}d\lambda \\
&= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(Q\psi_{mn}^0, \psi_{mn}^0)_{H_1}\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{\lambda-m^4}.
\end{aligned}
$$

By using (1), (19), (20) and this last relation we obtain

$$\begin{aligned} M_{p1} &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}(Q\psi_{mn}^{0}, \psi_{mn}^{0})_{H_1} \\ &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}(Q(x)d_m\cos mx\cdot\varphi_n, d_m\cos mx\cdot\varphi_n)\,dx \\ &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}d_m^2\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos^2 mx dx \\ &= \frac{1}{2}\sum_{m=0}^{p}\sum_{n=1}^{\infty}d_m^2\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)(1+\cos 2mx)dx. \qquad (28) \end{aligned}$$

Moreover, since

$$\left|\sum_{n=1}^{q}(Q(x)\varphi_n,\varphi_n)\right| \le \sum_{n=1}^{\infty}|(Q(x)\varphi_n,\varphi_n)| \le ||Q(x)||_{\sigma_1(H)} \ (q=1,2,...),$$

and by assumption since

$$\int_0^{\pi}||Q(x)||_{\sigma_1(H)} < \infty,$$

and also by applying the Lebesgue theorem, we find

$$\sum_{n=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)dx = \int_0^{\pi}[\sum_{n=1}^{\infty}(Q(x)\varphi_n,\varphi_n)]dx = \int_0^{\pi} trQ(x)dx. \qquad (29)$$

From (2), (28) and (29) we obtain

$$M_{p1} = \frac{2p+1}{2\pi}\int_0^{\pi}\mathrm{tr}Q(x)dx + \frac{1}{2}\sum_{m=0}^{p}\sum_{n=1}^{\infty}d_m^2\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos 2mx dx.$$

This proves lemma. □

Now, we want to show that

$$\lim_{p\to\infty} M_{p2} = 0. \qquad (30)$$

From (26), we find

$$\begin{aligned} M_{p2} &= \frac{1}{4\pi i}\int_{|\lambda|=b_p} \mathrm{tr}[(QR_\lambda^0)^2]d\lambda \\ &= \frac{1}{4\pi i}\int_{|\lambda|=b_p} [\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}[(QR_\lambda^0)^2\psi_{mn}^0, \psi_{mn}^0)_{H_1}]d\lambda. \end{aligned} \tag{31}$$

Moreover, we have

$$\begin{aligned} QR_\lambda^0\psi_{mn}^0 &= \frac{Q\psi_{mn}^0}{m^4-\lambda} \\ (QR_\lambda^0)^2\psi_{mn}^0 &= (m^4-\lambda)^{-1}QR_\lambda^0 Q\psi_{mn}^0 \\ &= (m^4-\lambda)^{-1}QR_\lambda^0\left\{\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}\psi_{rq}^0\right\} \\ &= (m^4-\lambda)^{-1}\left\{\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}(r^4-\lambda)^{-1}(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}Q\psi_{rq}^0\right\}. \end{aligned}$$

If we put this expression in (31), we obtain

$$M_{p2} = \frac{1}{4\pi i}\int_{|\lambda|=b_p}\left[\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left(\frac{(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}(Q\psi_{rq}^0,\psi_{mn}^0)_{H_1}}{(\lambda-m^4)(\lambda-r^4)}\right)\right]d\lambda. \tag{32}$$

On the other hand

$$\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-m^4)(\lambda-r^4)} = 0\ ;\ m,r\le p \tag{33}$$

In fact, if $m=r$ then

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-m^4)} = 0.$$

If $m\neq r$ then there exists a small number $\varepsilon>0$ such that

$$\begin{aligned}\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-m^4)(\lambda-r^4)} &= \frac{1}{2\pi i}\int_{|\lambda-m^4|=\varepsilon}\frac{d\lambda}{(\lambda-m^4)(\lambda-r^4)} \\ &\quad +\frac{1}{2\pi i}\int_{|\lambda-r^4|=\varepsilon}\frac{d\lambda}{(\lambda-m^4)(\lambda-r^4)} \\ &= \frac{1}{m^4-r^4}+\frac{1}{r^4-m^4}=0.\end{aligned}$$

So, for $m, r > p$ since

$$\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-m^4)(\lambda-r^4)}=0,$$

and from (32) and (33), we find

$$\begin{aligned}M_{p2} &= \frac{1}{2\pi i}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}(Q\psi_{rq}^0,\psi_{mn}^0)_{H_1} \\ &\quad \cdot\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-m^4)(\lambda-r^4)} \\ &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}(Q\psi_{rq}^0,\psi_{mn}^0)_{H_1} \\ &\quad \cdot\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{1}{m^4-r^4}\left[\frac{1}{(\lambda-m^4)}-\frac{1}{(\lambda-r^4)}\right]d\lambda \\ &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(m^4-r^4)^{-1}\left|(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}\right|^2.\end{aligned}$$

And from here we obtain

$$
\begin{aligned}
|M_{p2}| &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(r^4-m^4)^{-1}|(Q\psi^0_{rq},\psi^0_{mn})_{H_1}|^2 \\
&\le \sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(r^4-p^4)^{-1}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}|(Q\psi^0_{rq},\psi^0_{mn})_{H_1}|^2 \\
&= \sum_{r=p+1}^{\infty}(r^4-p^4)^{-1}\sum_{q=1}^{\infty}||Q\psi^0_{rq}||^2_{H_1}. \qquad (34)
\end{aligned}
$$

Since $Q(x)$ satisfies condition (3), and by taking advantage of (1) and (34), we can give an estimation for the sum $\sum_{q=1}^{\infty}||Q\psi^0_{rq}||^2_{H_1}$ as

$$
\begin{aligned}
\sum_{q=1}^{\infty}||Q\psi^0_{rq}||^2_{H_1} &= \sum_{q=1}^{\infty}\int_0^{\pi}||Q(x)d_r\cos rx\cdot\varphi_q||^2dx \\
&\le \sum_{q=1}^{\infty}\int_0^{\pi}||Q(x)\varphi_q||^2dx \\
&= \sum_{q=1}^{\infty}||Q(x)\varphi_q||^2_{H_1} < c, \qquad (35)
\end{aligned}
$$

where c is a positive constant. From (34) and (35) we find

$$|M_{p2}| = c\cdot\sum_{r=p+1}^{\infty}(r^4-p^4)^{-1}.$$

Here we can show that

$$\sum_{r=p+1}^{\infty}(r^4-p^4)^{-1} < p^{-\frac{5}{2}}. \qquad (36)$$

Hence, we obtain

$$|M_{p2}| < c\cdot p^{-\frac{5}{2}}$$

and so

$$\lim_{p\to\infty} M_{p2} = 0.$$

This time, let us show that

$$\lim_{p\to\infty} M_p = 0.$$

To do this, we will first estimate $||QR_\lambda^0||_{\sigma_1(H_1)}$ on the circle $|\lambda| = b_p$. As known from [16],

$$||QR_\lambda^0||_{\sigma_1(H_1)} \leq \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} ||QR_\lambda^0 \psi_{mn}^0||_{H_1}.$$

From (4) and since $Q(x)$ satisfies condition 3, we find

$$||QR_\lambda^0||_{\sigma_1(H_1)} < c \cdot \sum_{m=0}^{\infty} |m^4 - \lambda|^{-1}. \tag{37}$$

Moreover,

$$\begin{aligned}
\sum_{m=0}^{\infty} |m^4 - \lambda|^{-1} &= \sum_{m=0}^{p} |m^4 - \lambda|^{-1} + \sum_{m=p+1}^{\infty} |m^4 - \lambda|^{-1} \\
&\leq \sum_{m=0}^{p} (|\lambda| - m^4)^{-1} + \sum_{m=p+1}^{\infty} (m^4 - |\lambda|)^{-1} \\
&= \sum_{m=0}^{p} (p^4 + 2p^3 - m^4)^{-1} + \sum_{m=p+1}^{\infty} (m^4 - p^4 - 2p^3)^{-1} \\
&< \sum_{m=0}^{p} p^{-1} + \sum_{m=p+1}^{\infty} (m^4 - p^4 - 2p^3)^{-1} \\
&= \frac{p+1}{p} + \sum_{m=p+1}^{\infty} [\frac{1}{2}(m^4 - p^4) + \frac{1}{2}(m^4 - p^4) - 2p^3]^{-1} \\
&< 2 + \sum_{m=p+1}^{\infty} \{\frac{1}{2}(m^4 - p^4) + \frac{1}{2}[(p+1)^4 - p^4] - 2p^3\}^{-1} \\
&< 2 + \sum_{m=p+1}^{\infty} \frac{2}{m^4 - p^4}.
\end{aligned}$$

From (36) we have

$$\sum_{m=0}^{\infty} |m^4 - \lambda|^{-1} < 4,$$

and by using this together with (37) we find

$$||QR_\lambda^0||_{\sigma_1(H_1)} < c_1 \ ; \ |\lambda| = b_p = p^4 + 2p^3, \ c_1 > 0. \tag{38}$$

Now, let us estimate $||R_\lambda^0||_{H_1}$ on the circle $|\lambda| = b_p$. For $m \leq p$ we have

$$|m^4 - \lambda| \geq |\lambda| - m^4 = p^4 + 2p^3 - m^4 \geq 2p^3 > p^3,$$

and for $m \geq p+1$ we have

$$|m^4 - \lambda| \geq m^4 - |\lambda| = m^4 - p^4 - 2p^3 \geq (p+1)^4 - p^4 - 2p^3 > 2p^3 > p^3.$$

On the other hand, since

$$||R_\lambda^0||_{H_1} = \max_m \{|m^4 - \lambda|\},$$

we obtain

$$||R_\lambda^0||_{H_1} \leq p^{-3}. \tag{39}$$

From Theorem 2 we know that $\{\lambda_{mn}\}_{n=1}^{\infty} \subset \left[m^4 - ||Q||_{H_1}, m^4 + ||Q||_{H_1}\right]$, $(m = 0, 1, 2, ...)$. Considering this and the assumption $||Q||_{H_1} < \frac{1}{2}$, we write

$$|\lambda_{mn} - m^4| < \frac{1}{2} \quad (m = 0, 1, 2, ...; n = 1, 2, 3, ...).$$

In a similar way to the proof of (39), by using this inequality we can prove that

$$||R_\lambda|| < c_3 \cdot p^{-3} \ ; c_3 > 0 \tag{40}$$

on the circle $|\lambda| = b_p$ for the big value of p. From (24), (38), (39) and (40), we find

$$\begin{aligned}
|M_p| &= \frac{1}{2\pi}\left|\int_{|\lambda|=b_p} \lambda \cdot \operatorname{tr}[R_\lambda(QR_\lambda^0)^3]d\lambda\right| \\
&\leq \int_{|\lambda|=b_p} |\lambda| \cdot \left|\operatorname{tr}[R_\lambda(QR_\lambda^0)^3]\right| \cdot |d\lambda| \\
&\leq b_p \int_{|\lambda|=b_p} \left\|R_\lambda(QR_\lambda^0)^3\right\|_{\sigma_1(H_1)} \cdot |d\lambda| \\
&\leq b_p \int_{|\lambda|=b_p} ||R_\lambda||_{H_1} \cdot \left\|(QR_\lambda^0)^3\right\|_{\sigma_1(H_1)} \cdot |d\lambda| \\
&\leq c_3 b_p p^{-3} \int_{|\lambda|=b_p} \left\|QR_\lambda^0\right\|_{H_1}^2 \cdot \left\|QR_\lambda^0\right\|_{\sigma_1(H_1)} \cdot |d\lambda| \\
&\leq c_3 b_p p^{-3} ||Q||^2 p^{-6} c_1 2\pi b_p \leq c_4 p^{-1}
\end{aligned}$$

and so we obtain

$$\lim_{p\to\infty} M_p = 0. \tag{41}$$

Theorem 6 *If* $Q(x)$ *satisfies conditions* 1–4, *then for the regularized trace of the operator* L, *the formula*

$$\sum_{m=0}^{\infty}[\sum_{n=1}^{\infty}\left(\lambda_{mn} - m^4\right) - \frac{1}{\pi}\int_0^\pi \operatorname{tr}Q(x)dx] = \frac{1}{4}[\operatorname{tr}Q(0) + \operatorname{tr}Q(\pi)] - -\frac{1}{2\pi}\int_0^\pi \operatorname{tr}Q(x)dx$$

is satisfied.

Proof. From relations (25) and (41) and Lemma 5 we write

$$\lim_{p\to\infty}\left[\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(\lambda_{mn} - m^4\right) - \frac{2p+1}{2\pi}\int_0^\pi \operatorname{tr}Q(x)dx\right] =$$

$$= \frac{1}{2} \lim_{p\to\infty} \sum_{m=0}^{p} \sum_{n=1}^{\infty} d_m^2 \cdot \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2mx dx$$

or

$$\lim_{p\to\infty} \left[\sum_{m=0}^{p} \sum_{n=1}^{\infty} \left(\lambda_{mn} - m^4\right) - \sum_{m=0}^{p} \frac{1}{\pi} \int_0^{\pi} \mathrm{tr}Q(x)dx \right] =$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} d_m^2 \cdot \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2mx dx - \frac{1}{2\pi} \int_0^{\pi} \mathrm{tr}Q(x)dx$$

or

$$\sum_{m=0}^{\infty} [\sum_{n=1}^{\infty} \left(\lambda_{mn} - m^4\right) - \frac{1}{\pi} \int_0^{\pi} \mathrm{tr}Q(x)dx] =$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} d_m^2 \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2mx dx - \frac{1}{2\pi} \int_0^{\pi} \mathrm{tr}Q(x)dx.$$

Now, let us evaluate the expression

$$I = \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} d_m^2 \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2mx dx$$

in the right side of last equality. Since $Q(x)$ satisfies the conditions (1)-(4), we have

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} d_m^2 \left| \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2mx dx \right| < \infty$$

and so we write

$$
\begin{aligned}
I &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} d_m^2 \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos 2mx dx \\
&= \frac{1}{4}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} [d_m^2 \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos mx dx + \\
&\quad + d_m^2 (-1)^m \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos mx dx] \\
&= \frac{1}{4}\sum_{n=1}^{\infty}\{\sum_{m=0}^{\infty} [d_m^2 \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos mx dx] \cos m0 + \\
&\quad + \sum_{m=0}^{\infty} [d_m^2 \int_0^{\pi} (Q(x)\varphi_n, \varphi_n) \cos mx dx] \cos m\pi\}.
\end{aligned}
$$

Considering d_m as in (2), the sums with respect to m in this last relation are the values at the points 0 and π respectively of Fourier series with respect to the functions $\{\cos mx\}_{m=0}^{\infty}$ in the interval $[0, \pi]$ of the function $(Q(x)\varphi_n, \varphi_n)$ which has the derivative of second order. For this reason, we write

$$
\begin{aligned}
I &= \frac{1}{4}\sum_{n=1}^{\infty} [(Q(0)\varphi_n, \varphi_n) + (Q(\pi)\varphi_n, \varphi_n)] \\
&= \frac{1}{4}[\mathrm{tr}Q(0) + \mathrm{tr}Q(\pi)].
\end{aligned}
$$

And hence we obtain

$$
\begin{aligned}
\sum_{m=0}^{\infty}[\sum_{n=1}^{\infty} \left(\lambda_{mn} - m^4\right) - \frac{1}{\pi}\int_0^{\pi} \mathrm{tr}Q(x)dx] &= \frac{1}{4}[\mathrm{tr}Q(0) + \mathrm{tr}Q(\pi)] \\
&\quad - \frac{1}{2\pi}\int_0^{\pi} \mathrm{tr}Q(x)dx
\end{aligned}
$$

This completes the proof of Theorem 6. □

If $Q(x)$ satisfies the condition

$$
\int_0^{\pi} \mathrm{tr}Q(x)dx = 0
$$

in addition to conditions (1)–(4), then the formula in the above takes the form

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\lambda_{mn}-m^4\right)=\frac{1}{4}[\mathrm{tr}Q(0)+\mathrm{tr}Q(\pi)].$$

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