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## On Graded Primary Ideals

*Mashhoor Refai, Khaldoun Al-Zoubi*

### Abstract

Let  $G$  be a group and  $R$  be a  $G$ -graded commutative ring, i.e.,  $R = \bigoplus_{g \in G} R_g$  and  $R_g R_h \subseteq R_{gh}$  for all  $g, h \in G$ . In this paper, we study the graded primary ideals and graded primary  $G$ -decomposition of a graded ideal.

**Key words and phrases:** Graded rings, Graded primary ideals, Decomposition of graded ideals

### 0. Introduction

Let  $G$  be a group with identity  $e$  and  $R$  be a commutative ring. Then  $R$  is a  $G$ -graded ring if there exist additive subgroups  $R_g$  of  $R$  indexed by the elements  $g \in G$  such that  $R = \bigoplus_{g \in G} R_g$  and  $R_g R_h \subseteq R_{gh}$  for all  $g, h \in G$ . We denote this by  $(R, G)$ , and we consider  $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$ . The elements of  $R_g$  are called homogeneous of degree  $g$ . If  $x \in R$ , then  $x$  can be written uniquely as  $\sum_{g \in G} x_g$ , where  $x_g$  is the component of  $x$  in  $R_g$ . Also, we write  $h(R) = \bigcup_{g \in G} R_g$ .

Let  $I$  be an ideal of  $R$ . Then  $I$  is a graded ideal of  $(R, G)$  if  $I = \bigoplus_{g \in G} (I \cap R_g)$ . Clearly  $\bigoplus_{g \in G} (I \cap R_g) \subseteq I$  and hence  $I$  is graded ideal of  $(R, G)$  if  $I \subseteq \bigoplus_{g \in G} (I \cap R_g)$ .

In this paper we introduce the concepts of graded primary ideals and graded primary  $G$ -decomposition. We study these concepts in analogous way to that done for graded

ideals in [1] . However, if  $G$  is a finitely generated abelian group then  $G$  is isomorphic to the direct sums of some copies of  $Z_m$  and  $Z^n$ , and for this case the results are well known. Throughout this paper  $G$  is a non-finitely generated abelian group. Our work is a new direction in the study of graded ideals as well as an integration study of that done for the graded ideals. We believe that this work will lead to constructive ideals which introduce good tools for solving open problems of primary ideals and primary decomposition by turning them over into graded prime ideals and graded primary  $G$ -decomposition. This paper will be the primary ground to initiate more useful studies concerning the graded primary ideal and graded  $G$ -decomposition.

In section 1, we give some basic definitions and facts concerning graded primary ideals. In section 2, we define the graded primary  $G$ -decomposition of a graded ideal and study the uniqueness of this decomposition.

## 1. Graded Primary Ideals

In this section, we define the graded primary ideals and give some of their basic properties.

**Definition 1.1** [6]. *Let  $I$  be a graded ideal of  $(R, G)$ . Then:*

1.  *$I$  is a graded prime ideal (in abbreviation, “ $G$ -prime ideal”) if  $I \neq R$ ; and whenever  $rs \in I$ , we have  $r \in I$  or  $s \in I$ , where  $r, s \in h(R)$ .*

2.  *$I$  is a graded maximal ideal (in abbreviation, “ $G$ -maximal ideal”) if  $I \neq R$  and there is no graded ideal  $J$  of  $(R, G)$  such that  $I \subsetneq J \subsetneq R$ .*

3. *The graded radical of  $I$  (in abbreviation “ $Gr(I)$ ”) is the set of all  $x \in R$  such that for each  $g \in G$  there exists  $n_g > 0$  with  $x_g^{n_g} \in I$ . Note that, if  $r$  is a homogeneous element of  $(R, G)$ , then  $r \in Gr(I)$  iff  $r^n \in I$  for some  $n \in \mathbf{N}$ .*

**Notation** If  $M \subseteq R$  then let  $V(M)$  denote the set of all  $G$ -prime ideals of  $(R, G)$  that contains  $M$ . Also, let  $GX$  denote the set of all  $G$ -prime ideals of  $(R, G)$ .

**Proposition 1.2** [6]. *Let  $I, J$  be graded ideals of  $(R, G)$ . Then*

1.  $I \subseteq Gr(I)$ .
2.  $Gr(Gr(I)) = Gr(I)$ .
3.  $Gr(I) = R$  iff  $I = R$ .

4.  $Gr(IJ) = Gr(I \cap J) = Gr(I) \cap Gr(J)$ .
5. If  $P$  is a  $G$ -prime ideal of  $(R, G)$ , then  $Gr(P^n) = P$  for all  $n > 0$ .

**Definition 1.3** [7]. Let  $I, J$  be ideals of  $R$ . We define the ideal quotient  $(I:J)$  by  $(I:J) = \{a \in R : aJ \subseteq I\}$ .

**Proposition 1.4** [3]. Let  $I_1, \dots, I_n$  be graded ideals of  $(R, G)$ . Let  $P$  be a  $G$ -prime ideal such that  $\bigcap_{i=1}^n I_i \subseteq P$ . Then  $I_i \subseteq P$  for some  $1 \leq i \leq n$ . Also, if  $P = \bigcap_{i=1}^n I_i$  then  $P = I_i$  for some  $1 \leq i \leq n$ .

**Definition 1.5** Let  $I$  be a graded ideal of  $(R, G)$ . Then we say that  $I$  is a graded primary ideal of  $(R, G)$  (in abbreviation, “ $G$ -primary ideal”) if  $I \neq R$ ; and whenever  $a, b \in h(R)$  with  $ab \in I$  then  $a \in I$  or  $b \in Gr(I)$ .

**Example 1.6** Let  $R = \mathbf{Z}[i]$  (The Gaussian integers) and let  $G = \mathbf{Z}_2$ . Then  $R$  is a  $G$ -graded ring with  $R_0 = \mathbf{Z}$ ,  $R_1 = i\mathbf{Z}$ . Let  $I = 2R$  be a graded prime ideal. Then  $I$  is a graded primary ideal. But  $I$  is not a primary ideal because  $2$  is not irreducible element of  $R = \mathbf{Z}[i]$ .

**Proposition 1.7** Let  $I$  be a graded ideal of  $(R, G)$ . Then  $I$  is a graded primary ideal of  $(R, G)$  iff  $R/I$  is not trivial and has the property that every homogeneous zero divisor in  $R/I$  is a nilpotent.

**Proof.** Suppose that  $I$  is a graded primary ideal. Since  $I \neq R$ , we deduce that  $R/I$  is not trivial. Let  $r \in h(R) - I$  be such that  $r + I \in R/I$  is a zero divisor. Then there exists  $a \in R - I$  such that  $a + I \neq I$  and  $(a + I)(r + I) = I$ , which implies  $ar + I = I$  which implies  $ar \in I$ . Since  $I$  is a graded primary ideal and  $a \notin I$ ,  $r \in Gr(I)$ . So, there exists  $n \in \mathbf{N}$  such that  $r^n \in I$ . Hence  $(r + I)^n \subseteq r^n + I = I$ . Therefore,  $r + I$  is nilpotent in  $R/I$ .

Conversely,  $I \neq R$  since  $R/I$  is not trivial. Let  $r, s \in h(R)$  and  $rs \in I$  with  $r \notin I$ . Then,  $rs + I = I$  and hence  $(r + I)(s + I) = I$ . If  $s \notin I$  then,  $s + I$  a zero divisor in  $R/I$  and hence  $s + I$  is nilpotent in  $R/I$ . So, there exists  $n \in \mathbf{N}$  such that  $(s + I)^n = I$  which implies  $(s + I)^n \subseteq s^n + I = I$ . Hence  $s^n \in I$  and then  $s \in Gr(I)$ .  $\square$

**Lemma 1.8** Let  $Q$  be a graded primary ideal of  $(R, G)$ . Then  $P = Gr(Q)$  is a graded prime ideal of  $(R, G)$ , and we say that  $Q$  is a graded  $G$ - $P$ -primary.

**Proof.** Since  $1 \notin Q$  we have  $1 \notin \text{Gr}(Q)$  and hence  $\text{Gr}(Q) \neq R$ . So,  $P$  is proper. Suppose we have  $a, b \in h(R)$  such that  $ab \in \text{Gr}(Q)$  and  $a \notin \text{Gr}(Q)$ . Then there exists  $n \in \mathbb{N}$  such that  $(ab)^n = a^n b^n \in Q$ . Since  $a \notin \text{Gr}(Q)$ ,  $a^n \notin \text{Gr}(Q)$  which implies  $b^n \in Q$ . Hence  $b \in \text{Gr}(Q)$ . Therefore,  $\text{Gr}(Q)$  is a graded prime ideal of  $(R, G)$ .  $\square$

**Corollary 1.9** *If  $Q$  is a graded  $G$ - $P$ -primary, then  $P$  is the smallest graded prime ideal of  $(R, G)$  that contains  $Q$ .*

**Proof.** Let  $P' \in \text{GX}$  with  $Q \subseteq P'$ . Then  $P = \text{Gr}(Q) \subseteq \text{Gr}(P')$  and since  $P'$  is a graded prime ideal,  $\text{Gr}(P') = P'$ . Hence  $\text{Gr}(Q) = P \subseteq P'$ . Therefore,  $P$  is the unique minimal graded prime ideal of  $Q$ .  $\square$

**Lemma 1.10** *Let  $I, J$  be graded ideals of  $(R, G)$ . If  $\text{Gr}(I) + \text{Gr}(J) = R$ , then  $I + J = R$ .*

**Proof.** Direct.  $\square$

Let  $Q$  be a graded ideal such that  $P = \text{Gr}(Q)$  is a graded prime ideal of  $(R, G)$ . Then  $Q$  is not necessary  $G$ - $P$ -primary. In fact, let  $K$  be a field and  $R = K[X, Y]$  with  $\deg X = \deg Y = 1$ .  $R$  is  $\mathbb{Z}$ -graded ring. Put  $Q = (X^2, XY)$ . Then  $Q$  is graded ideal with  $\text{Gr}(Q) = (X)$ . Since  $X \notin Q$  and  $Y \notin \text{Gr}(Q)$ ,  $Q$  is not  $G$ - $P$ -primary.

However, if  $\text{Gr}(Q)$  is a graded maximal, the following result holds:

**Proposition 1.11** *Let  $Q$  be a graded ideal of  $(R, G)$  such that  $\text{Gr}(Q) = M$  is a graded maximal ideal of  $(R, G)$ . Then  $Q$  is a graded  $G$ - $M$ -primary.*

**Proof.** Let  $Q$  be a graded ideal of  $(R, G)$  such that  $\text{Gr}(Q) = M$  is a graded maximal ideal of  $(R, G)$ . Since  $Q \subseteq \text{Gr}(Q) = M \subseteq R$ ,  $Q$  is proper. Let  $a, b \in h(R)$  such that  $ab \in Q$  with  $b \notin \text{Gr}(Q)$ . Then  $\text{Gr}(Q) \subsetneq \text{Gr}(Q) + Rb$  and hence  $M + Rb = R$ , as  $M$  is maximal.

So,  $\text{Gr}(Q) + \text{Gr}(Rb) = R$ . Hence by Lemma 1.10,  $Q + Rb = R$  and then there exist  $q \in Q$ ,  $r \in R$  such that  $q + rb = 1$ . Hence  $a = a1 = a(q + rb) = aq + r(ab) \in Q$  since  $q, ab \in Q$ . Therefore,  $Q$  is graded  $G$ - $M$ -primary.  $\square$

**Corollary 1.12** *Let  $M$  be a graded maximal ideal of  $(R, G)$ . Then for  $n \in \mathbf{N}$ ,  $M^n$  is a graded  $G$ - $M$ -primary.*

**Proof.** Let  $M$  be a graded maximal ideal of  $(R, G)$ . By Proposition 1.11,  $M^n$  is graded  $G$ - $M$ -primary because  $\text{Gr}(M^n) = M$ .  $\square$

**Proposition 1.13** *Let  $I, J$  be graded ideals of  $(R, G)$ . Then  $(I:J)$  is a graded ideal of  $(R, G)$ .*

**Proof.** Suppose  $I, J$  are graded ideals of  $(R, G)$ . Let  $r = \sum_{g \in G} r_g \in (I:J)$ . Our goal is to show that  $r_g \in (I:J)$  for all  $g \in G$ , i.e.,  $r_g J \subseteq I$  for all  $g \in G$ . Without loss of generality assume  $r = \sum_{i=1}^n r_{g_i}$  where  $r_{g_i} \neq 0$  for all  $i = 1, \dots, n$  and  $r_g = 0$  for all  $g \notin \{g_1, \dots, g_n\}$ . Since  $r \in (I:J)$ ,  $rJ \subseteq I$ , which implies  $\sum_{g \in G} r_g J \subseteq I$ . Thus  $\sum_{i=1}^n r_{g_i} J \subseteq I$ . We put  $a = r_{g_i}$ . Then, it suffices to show that  $ac \in I$  for any  $c \in J$ . Since  $J$  is a graded ideal, we may show  $ac \in I$  for any  $c \in J \cap h(R)$ . Now, let  $r = \sum_{i=1}^n r_{g_i}$  and  $c = c_h$  for some  $h \in G$ . Since  $rc \in I$ ,  $\sum_{i=1}^n r_{g_i} c \in I$ , if  $g_i \neq g_j$ , we have  $g_i h \neq g_j h$  and hence  $r_{g_i} c_h \in I \cap R_{g_i h} \subseteq I$  as  $I$  is a graded ideal. Thus  $a \in (I:c)$  and then  $r_{g_i} J \subseteq I$  for all  $i = 1, \dots, n$ . Hence  $r_{g_i} \in (I:J)$  for all  $i = 1, \dots, n$ . Therefore, for all  $g \in G$ ,  $r_g \in (I:J)$  and hence  $(I:J)$  is a graded ideal of  $(R, G)$ .  $\square$

**Proposition 1.14** *Let  $P$  be a graded prime ideal of  $(R, G)$  and for  $n \geq 1$ , let  $Q_1, \dots, Q_n$  be graded  $G$ - $P$ -primary ideals of  $(R, G)$ . Then  $\bigcap_{i=1}^n Q_i$  is also graded  $G$ - $P$ -primary.*

**Proof.** Let  $Q_1, \dots, Q_n$  be graded primary ideals of  $(R, G)$  with  $\text{Gr}(Q_i) = P$  for  $i = 1, \dots, n$ . By Proposition 1.2,  $\text{Gr}(Q_1 \cap \dots \cap Q_n) = \text{Gr}(Q_1) \cap \dots \cap \text{Gr}(Q_n) = P \subseteq R$ . Suppose  $a, b \in h(R)$  such that  $ab \in \bigcap_{i=1}^n Q_i$  with  $b \notin \bigcap_{i=1}^n Q_i$ . Then there exists an integer  $j$  with  $1 \leq j \leq n$  such that  $b \notin Q_j$ . Since  $ab \in Q_j$  and  $Q_j$  is graded  $G$ - $P$ -primary, we have  $a \in \text{Gr}(Q_j) = P = \text{Gr}(\bigcap_{i=1}^n Q_i)$ . Hence  $\bigcap_{i=1}^n Q_i$  is graded  $G$ - $P$ -primary.  $\square$

**Proposition 1.15** *Let  $Q$  be a graded  $G$ - $P$ -primary ideal of  $(R, G)$  and  $a \in h(R)$ .*

- (i) *If  $a \in Q$ , then  $(Q:a) = R$ .*
- (ii) *If  $a \notin Q$ , then  $(Q:a)$  is graded  $G$ - $P$ -primary, in particular  $Gr((Q:a)) = P$ .*
- (iii) *If  $a \notin P$ , then  $(Q:a) = Q$ .*

**Proof.** (i) Since  $1.a = a \in Q$ ,  $1 \in (Q:a)$ . Thus  $(Q:a) = R$ .

(ii) Let  $Q$  be a graded  $G$ - $P$ -primary and let  $a \in h(R)$  with  $a \notin Q$ .

**Claim.**  $Q \subseteq (Q:a) \subseteq P = Gr(Q)$ .

Clearly,  $Q \subseteq (Q:a)$ . Let  $b \in (Q:a)$ . Then  $ba \in Q$ . Since  $Q$  is a graded  $G$ - $P$ -primary and  $a \notin Q$ ,  $b \in Gr(Q) = P$ . Therefore,  $Q \subseteq (Q:a) \subseteq P$ . By the claim, we have  $P = Gr(Q) \subseteq Gr((Q:a)) \subseteq Gr(P) = P$  and hence  $Gr((Q:a)) = P$ . Now, we want to show  $(Q:a)$  is  $G$ - $P$ -primary. Suppose  $c, d \in h(R)$  such that  $cd \in (Q:a)$  and  $d \notin P$ . Then  $cda \in Q$ . Since  $Q$  is graded  $G$ - $P$ -primary and  $d \notin P$ ,  $ca \in Q$  and hence  $c \in (Q:a)$ . Thus  $(Q:a)$  is graded  $G$ - $P$ -primary.

(iii) Let  $b \in (Q:a)$ . Then  $ba \in Q$ . Since  $a \notin P$  and  $Q$  is graded  $G$ - $P$ -primary,  $b \in Q$ . Thus  $(Q:a) \subseteq Q$ . Therefore,  $(Q:a) = Q$ . □

## 2. Graded Primary $G$ -decomposition of a Graded Ideal

In this section, we define the graded primary  $G$ -decomposition of a graded ideal as well as propositions indicating the uniqueness of this decomposition side by side with propositions that study the graded primary  $G$ -decomposition in  $gr$ -Noetherian rings.

**Definition 2.1** [2]. *We say  $(R, G)$  is  $gr$ -Noetherian if it satisfies the ascending chain condition on graded ideals of  $R$ .*

**Proposition 2.2** *Let  $I, J$  be two distinct graded ideals of  $(R, G)$ . Put  $\Omega = \{P' \in GX : J \supseteq P' \supseteq I\}$  and suppose  $\Omega \neq \emptyset$ . Then  $\Omega$  has a minimal member with respect to inclusion.*

**Proof.** (Using Zorn's Lemma). Order  $\Omega$  by reverse inclusion, i.e., for  $P_1, P_2 \in \Omega$ ,  $P_1 \leq P_2$  if  $P_2 \subseteq P_1$ . Clearly,  $(\Omega, \leq)$  is a partially ordered set. Let  $\lambda = \{P_\alpha : \alpha \in \Delta\}$  be any chain of  $\Omega$ . Then  $P_\alpha \subseteq P_\gamma$  or  $P_\gamma \subseteq P_\alpha$  for all  $\alpha, \gamma \in \Delta$ . Let  $P = \bigcap_{\alpha \in \Delta} P_\alpha$ . Then clearly,  $P \in \Omega$  and  $P$  is an upper bound of  $\lambda$ . Therefore, by Zorn's lemma, there exists at least one maximal

element  $M$  in  $\Omega$ . Clearly  $M$  is a graded prime ideal of  $(R, G)$  such that  $J \supseteq M \supseteq I$ . If  $J \supseteq M \supseteq q \supseteq I$ , then  $q \in \Omega$ . But  $M$  is maximal element in  $\Omega$  and  $M \leq q$  which implies  $M = q$ . Thus  $M$  is a minimal graded prime ideal of  $(R, G)$  such that  $J \supseteq M \supseteq I$  with respect to inclusion.  $\square$

**Corollary 2.3** *Let  $I$  be a graded ideal of  $(R, G)$  and  $P \in GX$  such that  $P \supseteq I$ . Then*

(1)  *$V(I)$  has at least one minimal member with respect to inclusion. Such a minimal member is called a minimal graded prime ideal of  $I$ .*

(2)  *$\Lambda = \{ P' \in GX : P \supseteq P' \supseteq I \}$  has a minimal member with respect to inclusion.*

**Proof.** (1) In the proof of Proposition 2.2 take  $J = R$ , then the result is clear.

The set of minimal graded prime ideals of  $I$  is denoted by  $G\text{-Min}(I)$ .

(2) In the proof of Proposition 2.2 take  $J = P$ , then the result is clear.  $\square$

Note that a minimal member of  $\Lambda$  is a minimal graded prime ideal of  $I$ , and so deduce that there exists a minimal graded prime ideal  $P''$  of  $I$  with  $P'' \subseteq P$ .

**Definition 2.4** *Let  $I$  be a proper graded ideal of  $(R, G)$ . A graded primary  $G$ -decomposition of  $I$  is an intersection of finitely many graded primary ideals of  $(R, G)$ . Such a graded primary  $G$ -decomposition  $I = Q_1 \cap \dots \cap Q_n$  with  $Gr(Q_i) = P_i$  for  $i = 1, \dots, n$  of  $I$  is said to be a minimal graded primary  $G$ -decomposition of  $I$  precisely when*

(i)  *$p_1, \dots, p_n$  are different graded prime ideals of  $R$ , and*

(ii)  *$Q_j \not\supseteq \bigcap_{\substack{i=1 \\ j \neq i}}^n Q_i$  for all  $j = 1, \dots, n$ .*

*We say  $I$  is  $G$ -decomposable graded ideal of  $(R, G)$  precisely when it has a graded primary  $G$ -decomposition.*

**Remark 2.5** *Every  $G$ -decomposable graded ideal of  $(R, G)$  has a minimal graded primary  $G$ -decomposition .*

**Proposition 2.6** *Let  $I$  be a  $G$ -decomposable graded ideal of  $(R, G)$  such that  $I = Q_1 \cap \dots \cap Q_n$  where  $Gr(Q_i) = P_i$  for  $i = 1, \dots, n$ , is a minimal graded primary  $G$ -decomposition of  $I$ . Let  $P$  be any graded prime ideal of  $(R, G)$ . Then the following statements are equivalent.*



- (1)  $P = P_i$  for some  $i$  with  $1 \leq i \leq n$ .
- (2) There exists  $a \in h(R)$  such that  $(I:a)$  is graded  $G$ - $P$ -primary.
- (3) There exists  $a \in h(R)$  such that  $Gr((I:a)) = P$ .

**Proof. (1) implies (2):** Suppose  $P = P_i$  for some  $i$  with  $1 \leq i \leq n$ . Since the graded primary  $G$ -decomposition  $I = \bigcap_{j=1}^n Q_j$  is minimal, there exists  $a_i \in \bigcap_{\substack{j=1 \\ j \neq i}}^n Q_j \cap h(R) - Q_i$

and hence  $(I:a_i) = (\bigcap_{j=1}^n Q_j : a_i) = \bigcap_{j=1}^n (Q_j : a_i)$ . By Proposition 1.15,  $(Q_j : a_i) = R$  for  $j \neq i$  ( $1 \leq j \leq n$ ). While  $(Q_i : a_i)$  is graded  $G$ - $P_i$ -primary. Since  $P = P_i$ , it follows that  $(I:a_i)$  is graded  $G$ - $P$ -primary.

**(2) implies (3):** Clear.

**(3) implies (1):** Suppose  $P = Gr((I:a))$  for some  $a \in h(R)$ . Then  $(I:a) = (\bigcap_{i=1}^n Q_i : a)$   
 $= \bigcap_{i=1}^n (Q_i : a)$ . By Proposition 1.15,  $(Q_i : a) = R$  if  $a \in Q_i$ , while  $(Q_i : a)$  is graded  $G$ - $P_i$ -primary if  $a \notin Q_i$ . Hence,  $P = Gr((I:a)) = \bigcap_{i=1}^n Gr(Q_i : a) = \bigcap_{\substack{i=1 \\ a \notin Q_i}}^n P_i$  and by Proposition 1.4,  $P = P_i$  for some  $i$ . □

The next Corollary shows that the number of terms appearing in a minimal graded primary  $G$ -decomposition of  $I$  is independent of the choice of minimal graded primary  $G$ -decomposition.

**Corollary 2.7** *Let  $I$  be a  $G$ -decomposable graded ideal of  $(R, G)$ . Suppose  $I = Q_1 \cap \dots \cap Q_n$ , where  $Gr(Q_i) = P_i$  for  $i = 1, \dots, n$  and  $I = Q'_1 \cap \dots \cap Q'_{n'}$ , where  $Gr(Q'_i) = P'_i$  for  $i = 1, \dots, n'$ , are two minimal graded primary  $G$ -decompositions of  $I$ . Then  $n = n'$  and  $\{P_1, \dots, P_n\} = \{P'_1, \dots, P'_{n'}\}$ .*

**Proof.** Since the graded  $G$ -decomposition is minimal,  $\bigcap_{\substack{j=1 \\ j \neq i}}^n Q_j \not\subseteq Q_i$  for all  $i$  with

$1 \leq i \leq n$ . So there exists  $a \in \bigcap_{\substack{j=1 \\ j \neq i}}^n Q_j$  and  $a \notin Q_i$ . Since  $a \in Q_j$  for all

$j \neq i$ ,  $(Q_j : a) = R$  for all  $j \neq i$ . Hence  $(I : a) = \left( \bigcap_{j=1}^n Q_j : a \right) = \bigcap_{j=1}^n (Q_j : a) =$

$\bigcap_{\substack{j=1 \\ j \neq i}}^n (Q_j : a)(Q_i : a) = (Q_i : a)$ . Thus  $\text{Gr}((I : a)) = \text{Gr}((Q_i : a)) = P_i$ . Also,  $I = Q'_1 \cap \dots$

$\cap Q'_{n'}$  is a minimal graded primary  $G$ -decomposition of  $I$  and  $P = P_i$  satisfies  $\text{Gr}((I : a)) = P = P_i$  for some  $a \in \mathfrak{h}(R)$ . By Proposition 2.6, there exists  $j \in \{1, \dots, n'\}$  such that  $P'_j = P = P_i$ . Therefore,  $\{P_1, \dots, P_n\} \subseteq \{P'_1, \dots, P'_{n'}\}$ .  $\square$

The other inclusion is similar and hence  $\mathbf{n} = n'$

**Definition 2.8** Let  $I$  be a  $G$ -decomposable graded ideal of  $(R, G)$ . Let  $I = Q_1 \cap \dots \cap Q_n$ , where  $\text{Gr}(Q_i) = P_i$  for  $i = 1, \dots, n$ , be a minimal graded primary  $G$ -decomposition of  $I$ . Then the set  $\{P_1, \dots, P_n\}$  (which is independent of the choice of minimal graded primary  $G$ -decomposition of  $I$  by corollary 2.7) is called the set of associated graded prime ideals of  $I$  and denoted by  $G\text{-ass}_R(I)$ .

**Lemma 2.9** Let  $I = Q_1 \cap \dots \cap Q_n$  such that  $\text{Gr}(Q_i) = P_i$  for  $i = 1, \dots, n$  be a minimal graded primary  $G$ -decomposition of  $I$ . If  $P \in V(I)$ , then there exists  $P' \in G\text{-ass}_R(I)$  such that  $P' \subseteq P$ .

**Proof.** Let  $P \in V(I)$ . Then  $I \subseteq P$  and hence  $\text{Gr}(I) \subseteq \text{Gr}(P) = P$ . Also,  $\text{Gr}(I) = \text{Gr}(Q_1 \cap \dots \cap Q_n) = \text{Gr}(Q_1) \cap \dots \cap \text{Gr}(Q_n) = \bigcap_{i=1}^n p_i \subseteq P$ . By Proposition 1.4,  $P_j \subset P$  for some  $j$  with  $1 \leq j \leq n$ . □

**Proposition 2.10** *Let  $I$  be a  $G$ -decomposable graded ideal of  $(R, G)$ , and let  $P \in GX$ . Then  $P$  is a minimal graded prime ideal of  $I$  iff  $P$  is a minimal member of  $G\text{-ass}(I)$ , with respect to inclusion. In particular, all the minimal graded prime ideals of  $I$  belongs to  $G\text{-ass}(I)$ .*

**Proof.** Assume that  $P$  is a minimal graded prime ideal of  $I$ . By Lemma 2.9,  $P \supseteq P'$  for some  $P' \in G\text{-ass}(I)$ . But  $G\text{-ass}(I) \subseteq V(I)$ , and so  $P = P'$  must be a minimal member of  $G\text{-ass}(I)$  with respect to inclusion .

Conversely, suppose  $P$  is a minimal member of  $G\text{-ass}(I)$ . Then  $P \supseteq I$ , and so by Corollary 2.3, there exists a minimal graded prime ideal  $P'$  of  $I$  such that  $P \supseteq P' \supseteq I$ . Hence by Lemma 2.9, there exists  $P'' \in G\text{-ass}(I)$  such that  $P' \supseteq P''$ . But then  $P \supseteq P' \supseteq P''$ , and since  $P$  is a minimal member of  $G\text{-ass}(I)$ ,  $P = P' = P''$ . Therefore,  $P = P'$  is a minimal graded prime ideal of  $I$ . □

**Definition 2.11** *Let  $I$  be a  $G$ -decomposable graded ideal of  $(R, G)$ . By Proposition 2.10, the minimal members of  $G\text{-ass}(I)$  are precisely the minimal graded prime ideals of  $I$ . These graded prime ideals are called the minimal or  $G$ -isolated graded primes of  $I$ . The remaining associated graded primes of  $I$ , i.e, the associated graded primes of  $I$  which are not minimal, are called the  $G$ -embedded graded primes of  $I$ .*

*The next Proposition shows that in a minimal graded primary  $G$ -decomposition of  $I$ , the graded primary term corresponding to a  $G$ -isolated graded prime ideal of  $I$  is*

uniquely determined by  $I$  and is independent of the choice of minimal graded primary  $G$ -decomposition.

**Proposition 2.12** *Let  $I$  be a  $G$ -decomposable graded ideal of  $(R, G)$ ,  $G\text{-ass}(I) = \{P_1, \dots, P_n\}$ . Let  $I = Q_1 \cap \dots \cap Q_n$  with  $\text{Gr}(Q_i) = P_i$  for  $i = 1, \dots, n$  and  $I = Q'_1 \cap \dots \cap Q'_n$  with  $\text{Gr}(Q'_i) = P_i$  for  $i = 1, \dots, n$  be two minimal graded primary  $G$ -decompositions of  $I$ . If  $P_i$  is a minimal graded prime ideal belonging to  $I$ , then for each  $i$  with  $1 \leq i \leq n$ ,  $Q_i = Q'_i$ .*

**Proof.** If  $n = 1$ , there is nothing to prove. We therefore, suppose that  $n > 1$ . Let  $P_i$  be a minimal graded prime ideal belonging to  $I$ . Then there exists  $a \in \mathfrak{h}(R)$  such that  $a \in$

$$\bigcap_{\substack{j=1 \\ j \neq i}}^n P_j \cap \mathfrak{h}(R) \setminus P_i \text{ (for otherwise if } \bigcap_{\substack{j=1 \\ j \neq i}}^n P_j \subseteq P_i, \text{ then } P_j \subseteq P_i \text{ for some } j \in \mathbb{N} \text{ with } 1 \leq j \leq n \text{ and } j \neq i \text{ contrary to the fact that } P_i \text{ is a minimal graded prime ideal belonging to } I).$$

For each  $j = 1, \dots, n$  with  $j \neq i$ , there exists  $t_j \in \mathbb{N}$  such that  $a^{t_j} \in Q_j$  because  $a \in P_j = \text{Gr}(Q_j)$  for  $j = 1, \dots, n$  with  $j \neq i$ . Let  $t \geq \max\{t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n\}$ . Then  $a^t \in Q_j$  for  $j = 1, \dots, n$  and  $j \neq i$ . Also,  $a^t \notin P_i$ , implies by Proposition 1.15,  $(Q_j : a^t) = R$  for  $j = 1, \dots, n$  and  $j \neq i$  and  $(Q_i : a^t) = Q_i$ . Hence  $(I : a^t) = \left( \bigcap_{j=1}^n Q_j : a^t \right)$

$$= \bigcap_{j=1}^n (Q_j : a^t) = (Q_i : a^t) = Q_i. \text{ Also, } \text{Gr}(Q'_i) = P_i \text{ for } i = 1, \dots, n. \text{ So, in a similar}$$

way there exists  $m \in \mathbb{N}$  such that  $(I : a^m) = Q'_i$ . Let  $s \geq \max\{t, m\}$ . Then  $(I : a^s) = Q_i = Q'_i$ . □

**Definition 2.13** *Let  $I$  be a graded ideal of  $(R, G)$ . We say that  $I$  is  $G$ -irreducible iff  $I$  is proper and whenever  $I = I_1 \cap I_2$  with  $I_1, I_2$  graded ideals of  $(R, G)$ , then  $I = I_1$  or  $I = I_2$ .*

**Proposition 2.14** *Let  $R$  be a  $G$ -graded ring being  $gr$ -Noetherian ring. Then every proper graded ideal of  $(R, G)$  can be expressed as an intersection of finitely many  $G$ -irreducible graded ideals of  $(R, G)$ .*

**Proof.** Let  $\Omega$  be the set of all proper graded ideals of  $(R, G)$  that can not be expressed as an intersection of finitely many  $G$ -irreducible graded ideals of  $(R, G)$ . Our aim is to show that  $\Omega = \phi$ . Suppose that this is not the case. Then, since  $(R, G)$  is  $gr$ -Noetherian,  $\Omega$  has a maximal member  $I$  with respect to inclusion. Then  $I$  itself is not  $G$ -irreducible, for we can write  $I = I \cap I$  and  $I$  would not be in  $\Omega$ . Since  $I$  is proper, it therefore, follows that  $I = I_1 \cap I_2$  for some graded ideals  $I_1, I_2$  of  $(R, G)$  for which  $I \subsetneq I_1$  and  $I \subsetneq I_2$ . Note that this implies both of  $I_1$  and  $I_2$  are proper graded ideals. By choice of  $I$ , we must have  $I_i \notin \Omega$  for  $i = 1, 2$ . Since both  $I_1$  and  $I_2$  are proper, it follows that both can be expressed as intersection of finitely many  $G$ -irreducible graded ideals of  $(R, G)$ . Hence  $I = I_1 \cap I_2$  has the same property, and this is a contradiction. Hence  $\Omega = \phi$ .  $\square$

**Proposition 2.15** *Let  $(R, G)$  be a  $gr$ -Noetherian ring and let  $I$  be a  $G$ -irreducible graded ideal of  $(R, G)$ . Then  $I$  is graded primary.*

**Proof.** By definition of  $G$ -irreducible graded ideal,  $I$  is proper. Suppose  $a, b \in h(R)$  such that  $ab \in I$  and  $b \notin I$ . Consider  $(I:a) \subseteq (I:a^2) \subseteq \dots \subseteq (I:a^n) \subseteq \dots$ , to be an ascending chain of graded ideals of  $(R, G)$ . Since  $(R, G)$  is

$gr$ -Noetherian, there exists  $n \in \mathbf{N}$  such that  $(I:a^n) = (I:a^{n+i})$  for all  $i \in \mathbf{N}$ . We show  $I = (I+Ra^n) \cap (I+Rb)$ . It is clear that  $I \subseteq (I+Ra^n) \cap (I+Rb)$ . Let  $r \in (I+Ra^n) \cap (I+Rb)$ , then we can write  $r = g + ca^n = h + db$ , for some  $g, h \in I$  and  $c, d \in R$ . Thus  $ra = ga + ca^{n+1} = ha + dab$ , so that since  $ab, g, h \in I$ , we have  $ca^{n+1} = ha + dab - ga \in I$ . Hence  $c \in (I:a^{n+1}) = (I:a^n)$  (by choice of  $n$ ). So,  $r = g + ca^n \in I$  implies  $I = (I+Ra^n) \cap (I+Rb)$ . Now since  $I$  is  $G$ -irreducible and  $I \neq I+Rb$ ,  $I = I+Ra^n$  which implies  $a^n \in I$  and hence  $a \in Gr(I)$ . Therefore,  $I$  is graded primary.  $\square$

**Corollary 2.16** *Let  $(R, G)$  be a  $gr$ -Noetherian ring. If  $I$  is proper graded ideal of  $(R, G)$ , then  $I$  has a graded primary  $G$ -decomposition.*

**Proof.** Let  $I$  be a proper graded ideal of  $(R, G)$ . Then by Proposition 2.14,  $I$  can be expressed as an intersection of finitely many  $G$ -irreducible graded ideals of  $(R, G)$  and a  $G$ -irreducible graded ideal of  $(R, G)$  is graded primary by Proposition 2.15. Therefore,  $I$  has a graded primary  $G$ -decomposition.  $\square$

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