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Variational field equations of a Majorana neutrino coupled to Einstein's theory of general relativity

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Abstract: A consistent variational derivation of the Majorana 4-spinor field equations coupled to Einstein's theory of gravitation is given. The equivalence of the first and the second order variational field equations is explicitly demonstrated. The Lagrange multiplier 2-forms we use turn out to be precisely the Belinfante-Rosenfeld 2-forms that are needed to symmetrize the canonical energy-momentum tensor of the Majorana spinor.

Keywords: Canonical formalism, Lagrangians, variational principles, Majorana neutrino

1. Introduction

E. Majorana considered in a historic paper in 1937, a derivation of the Dirac equation where the electrons and the positrons are treated in a symmetric way [1]. He made use of a self-conjugate representation of the γ -matrices that we now call the Majorana realization. Furthermore he treated the fermion fields in terms of anticommuting variables. Such a theory has no conventional classical interpretation. However, it makes it possible to derive the Dirac equation by a variational principle. In a particular Majorana realization of γ -matrices as generators of the real part of the Clifford algebra $Clif(1, 3)$ over space-time, all components of the 4-spinors turn out to be real. As such they correspond to electrically neutral fermion fields whose quantization may be achieved on a real Hilbert space [2–4].

A candidate constituent particle for a Majorana spinor is a neutrino [5]. It should be noted in the present approach that there is no need to distinguish a separate antineutrino of opposite chirality: a neutrino coming out together with an electron in β -decay would have a negative helicity while that coming out together with a positron in inverse β -decay would have a positive helicity. In fact, as already noted by Majorana himself, the Majorana nature of neutrino can be tested in *neutrino-less double β -decays* [6]. Such processes violate the lepton number conservation by terms proportional to the Majorana mass of the neutrino. Thus, the Majorana nature of a neutrino emerges as the most natural explanation for the surprisingly small observed value of neutrino masses. Furthermore, the nonconservation of the lepton number L leads to speculations that the decay of supermassive Majorana neutrinos in the very early Universe may have given a rise to an asymmetry in L that transforms to the present-day observed baryon number B asymmetry by virtue of the $B - L$ conservation [7].

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Several laboratories around the world support experiments to detect neutrino-less double β -decay, with no success up till now. A new generation of experiments called CUORE is on its way in Italy to resolve this question. A possible discovery would also provide a natural answer to the dominance of matter over antimatter in our universe [8].

Our main goal in this paper is to provide a consistent derivation of the coupled field equations of a Majorana neutrino and Einstein's theory of gravitation. We make essential use in our calculations of the anticommuting nature of the spinor components. Einstein's theory of general relativity is a theory of gravitation determined by the semi-Riemannian geometry of a 4-dimensional spacetime. The coupled field equations of the theory may be obtained by field variational principle from an action. It is well known that the second order principle where the metric variations of the Levi-Civita connections are taken into account; and the first order (Palatini) where independent variations of the action relative to the metric and connection yield, in these cases, the same set of field equations. The fact that the connection is Levi-Civita may be imposed by the method of Lagrange multipliers. In the following, both the first and the second order variational approaches are given and the resulting field equations are derived and then compared.

2. The action

The field equations will be derived from the action

$$I[e, \omega, \psi, \lambda] = \int_M \mathcal{L}, \quad (2.1)$$

where the Lagrangian density 4-form is given by

$$\mathcal{L} = \frac{1}{2} R_{ab} \wedge *(e^a \wedge e^b) + \frac{i}{2} \bar{\psi} * \gamma \wedge \nabla \psi + \frac{i}{2} m (\bar{\psi} \psi) * 1, \quad (2.2)$$

where m is the Majorana mass. The basic field variables are the g -orthonormal basis 1-forms $\{e^a | a = 0, 1, 2, 3\}$ in terms of which the spacetime metric reads

$$g = \eta_{ab} e^a \otimes e^b = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3; \quad (2.3)$$

the metric-compatible connection 1-forms $\{\omega^a_b\}$ so that $\omega_{ab} = -\omega_{ba}$. The Cartan structure equations

$$T^a = de^a + \omega^a_b \wedge e^b, \quad (2.4)$$

and

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b \quad (2.5)$$

determine the torsion 2-forms $\{T^a\}$ and the curvature 2-forms $\{R^a_b\}$, respectively. The Hodge $*$ -map that takes p -forms to $(4-p)$ -forms, $0 \leq p \leq 4$, is fixed for a particular choice of the space-time orientation by the invariant volume 4-form

$$*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \frac{1}{4!} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d. \quad (2.6)$$

We are going to employ a Majorana (real) realization of the γ -matrices $\{\gamma_a\}$ and let $\gamma = \gamma_a e^a$. We set

$$\gamma_0 = \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \gamma_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

so that

$$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}. \quad (2.7)$$

The charge conjugation matrix \mathcal{C} should satisfy

$$\mathcal{C}^{-1} = \mathcal{C}^T, \quad \mathcal{C}^2 = -I, \quad \mathcal{C} \gamma_a \mathcal{C}^{-1} = \gamma_a^T. \quad (2.8)$$

We made the choice $\mathcal{C} = \gamma_0$ in our Majorana realization. Given a 4-spinor ψ , its charge conjugated spinor is defined to be

$$\psi_C \equiv \mathcal{C} \bar{\psi}^T, \quad (2.9)$$

where $\bar{\psi} = \psi^\dagger \mathcal{C}$. A Majorana 4-spinor is by definition a self-charge conjugate 4-spinor:

$$\psi_C = \psi. \quad (2.10)$$

In the Majorana realization of γ -matrices above, a Majorana spinor is a real spinor:

$$\psi_{Majorana} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbf{R}^4. \quad (2.11)$$

We also note that in a coordinate patch $\{x^\mu\}$, the spinor components $\{\psi_\alpha | \alpha = 1, 2, 3, 4\}$ are odd-Grassmann valued, real functions of all coordinates. That is they are each nilpotent and anticommute among themselves:

$$\psi_\alpha \psi_\beta + \psi_\beta \psi_\alpha = 0. \quad (2.12)$$

The covariant exterior derivative of a Majorana 4-spinor is given by

$$\nabla \psi = d\psi + \frac{1}{2} \omega^{ab} \sigma_{ab} \psi, \quad (2.13)$$

where $\sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]$ are the Lie algebra generators of the spin cover $Spin(1, 3)$ of the local Lorentz group $SO(1, 3)$. In particular,

$$*(\gamma \wedge *\nabla) = (\gamma \cdot \nabla) \quad (2.14)$$

is the Dirac operator on the real spin bundle over space-time.

3. Variational field equations

3.1. Zero-torsion constrained, second-order variations

We impose the zero-torsion condition by the method of Lagrange multipliers so that the connection 1-forms are going to be the Levi-Civita connection determined up to local Lorentz transformations by the metric. To that end we introduce a set $\{\lambda_a\}$ of Lagrange multiplier 2-forms upon whose variations the zero-torsion constraint is imposed on the geometry of space-time. Then the connection becomes the Levi-Civita connection $\{\hat{\omega}^a_b\}$. Let the constraint Lagrangian density 4-form be given by

$$\mathcal{L}_C = T^a \wedge \lambda_a. \quad (3.1)$$

Then, the variation of the total action density is given by

$$\mathcal{L}_T = \mathcal{L} + \mathcal{L}_C, \quad (3.2)$$

reads up to a closed form as follows:

$$\begin{aligned} \dot{\mathcal{L}}_T &= \dot{e}^a \wedge \frac{1}{2} R^{bc} \wedge *(e_a \wedge e_b \wedge e_c) + \frac{1}{2} \dot{\omega}^{ab} \wedge *(e_a \wedge e_b \wedge e_c) \wedge T^c \\ &+ \dot{e}^a \wedge \frac{i}{2} m(\bar{\psi}\psi) * e_a + \dot{e}^a \wedge D\lambda_a + \frac{1}{2} \dot{\omega}^{ab} \wedge (e_b \wedge \lambda_a - e_a \wedge \lambda_b) \\ &+ \frac{i}{2} \dot{e}^a \wedge \bar{\psi} * (e^b \wedge e_a) \wedge \gamma_b \nabla \psi - \frac{i}{4} \dot{\omega}^{ab} \wedge \bar{\psi} * \gamma \wedge \sigma_{ab} \psi \\ &+ \frac{i}{2} \bar{\psi} * \gamma \wedge \nabla \psi + \frac{i}{2} \bar{\psi} * \gamma \wedge \nabla \dot{\psi} + T^a \wedge \dot{\lambda}_a + im\bar{\psi}\psi * 1. \end{aligned} \quad (3.3)$$

(1) The variations of $\dot{\lambda}_a$ impose the zero-torsion constraint

$$T^a = 0, \quad (3.4)$$

that implies that the connection is the unique Levi-Civita connection 1-forms $\hat{\omega}^a_b$ and the remaining variational equations should be solved under this constraint.

(2) Einstein field equations follow from the variations \dot{e}^a of co-frame 1-forms:

$$-\frac{1}{2} \hat{R}^{bc} \wedge *(e_a \wedge e_b \wedge e_c) = \frac{i}{2} \bar{\psi} * (e^b \wedge e_a) \wedge \gamma_b \hat{\nabla} \psi + \frac{i}{2} m(\bar{\psi}\psi) * e_a + \hat{D}\lambda_a. \quad (3.5)$$

(3) The Dirac equation satisfied by the Majorana 4-spinor is determined from the variations of the Lagrangian density 4-form with respect to ψ . The relevant terms subject to the zero-torsion constraint are

$$\frac{i}{2} \bar{\psi} * \gamma \wedge \hat{\nabla} \psi + \bar{\psi} * \gamma \wedge \hat{\nabla} \dot{\psi}. \quad (3.6)$$

We open up the covariant derivative in the second term and use Majorana flip identities to write it as

$$\bar{\psi} * \gamma \wedge \hat{\nabla} \dot{\psi} = \frac{i}{2} d\bar{\psi} \wedge * \gamma \dot{\psi} - \frac{i}{2} \hat{\omega}^{ab} \wedge \bar{\psi} \sigma_{ab} * \gamma \dot{\psi}. \quad (3.7)$$

Then we differentiate the first term on the right by parts, use γ -matrix identities and the zero-torsion constraint to pass the covariant exterior derivative action on $\bar{\psi}$ at the left to the covariant derivative action on ψ at the right. Then up to a closed form we get the variation

$$i\bar{\psi} * \gamma \wedge \hat{\nabla} \psi. \quad (3.8)$$

Therefore, the massless Dirac equation satisfied by the Majorana 4-spinor field follows:

$$i * \gamma \wedge \hat{\nabla} \psi + im\psi * 1 = 0. \quad (3.9)$$

(4) The variation of the connections $\{\omega^a_b\}$ gives the field equations

$$e_a \wedge \lambda_b - e_b \wedge \lambda_a = \Sigma_{ab}, \quad (3.10)$$

where

$$\Sigma_{ab} = -\frac{i}{2} \bar{\psi} * \gamma \sigma_{ab} \psi. \quad (3.11)$$

This is a system of linear algebraic equations that can be solved uniquely for the Lagrange multiplier 2-forms λ_a . Let us hit both sides with the interior product operators $\iota^a \equiv \eta^{ab} \iota_{X_b}$. Then

$$\lambda_a + e_a \wedge \iota^b \lambda_b = \iota^b \Sigma_{ba}. \quad (3.12)$$

We hit this expression with ι^a once again so that

$$\iota^a \lambda_a = \frac{1}{4} \iota^a \iota^b \Sigma_{ba}. \quad (3.13)$$

Now we use the γ -matrix identity

$$2\gamma_c \sigma_{ab} = \eta_{ca} \gamma_b - \eta_{cb} \gamma_a + \epsilon_{abcd} \gamma_5 \gamma^d; \quad (3.14)$$

the Majorana flip identity

$$\bar{\psi} \gamma_a \psi = 0, \quad (3.15)$$

satisfied by any odd-Grassmann valued (real) Majorana 4-spinor, and after a series of simplifications we arrive at the final expression

$$\lambda_a = \frac{i}{8} e_a \wedge (\bar{\psi} \gamma_5 \gamma \psi). \quad (3.16)$$

It is not difficult to verify that

$$\hat{D} \lambda_a = \frac{i}{4} e_a \wedge (\bar{\psi} \gamma_5 \gamma \wedge \hat{\nabla} \psi). \quad (3.17)$$

Finally the Einstein field equations become

$$G_a = \frac{i}{2}\bar{\psi} * (e^b \wedge e_a) \wedge \gamma_b \hat{\nabla}\psi + \frac{i}{4}e_a \wedge (\bar{\psi}\gamma_5\gamma \wedge \hat{\nabla}\psi) + \frac{i}{2}m(\bar{\psi}\psi) * e_a, \quad (3.18)$$

where the Einstein 3-forms

$$G_a \equiv G_{ab} * e^b = -\frac{1}{2}\hat{R}^{bc} \wedge *(e_a \wedge e_b \wedge e_c). \quad (3.19)$$

Here the Einstein tensor $G = G_{ab}e^a \otimes e^b$ is symmetric by construction. We look at the two terms on the right hand side now. The first term gives the canonical energy-momentum tensor $T[can] = T_{ab}[can]e^a \otimes e^b$ of the Majorana 4-spinor ψ :

$$\tau_a[can] \equiv T_{ab}[can] * e^b = \frac{i}{2}\bar{\psi} * (e^b \wedge e_a) \wedge \gamma_b \hat{\nabla}\psi + \frac{i}{2}m(\bar{\psi}\psi) * e_a. \quad (3.20)$$

The canonical energy-momentum tensor $T[can]$ is asymmetric in general. In fact, we calculate its skew symmetric part as

$$e_a \wedge \tau_b[can] - e_b \wedge \tau_a[can] = \frac{i}{2}\bar{\psi}(\eta_{ac}\gamma_b - \eta_{bc}\gamma_a)\psi * e^c. \quad (3.21)$$

On the other hand, the skew symmetric part of the second term on the right hand side of the Einstein field equations reads

$$e_a \wedge \hat{D}\lambda_b - e_b \wedge \hat{D}\lambda_a = \frac{i}{2}e_a \wedge e_b \wedge \bar{\psi}\gamma_5\gamma \wedge \hat{\nabla}\psi. \quad (3.22)$$

Thus, adding the last two equalities side by side and using the γ -matrix identity

$$2\sigma_{ab}\gamma_c = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a - \epsilon_{abcd}\gamma_5\gamma^d, \quad (3.23)$$

we prove that the skew symmetric part of the total energy-momentum tensor of a Majorana 4-spinor becomes

$$e_a \wedge \tau_b - e_b \wedge \tau_a = \frac{i}{2}\bar{\psi}\sigma_{ab} * \gamma \wedge \hat{\nabla}\psi. \quad (3.24)$$

Therefore, the total energy-momentum tensor of ψ is symmetric on-shell. Since $e_a \wedge \hat{D}\lambda_b + e_b \wedge \hat{D}\lambda_a = 0$ by inspection, the symmetric part of the total energy-momentum tensor of ψ is read from the symmetric part of the canonical energy-momentum tensor:

$$\begin{aligned} e_a \wedge \tau_b[can] + e_b \wedge \tau_a[can] &= i\eta_{ab}\bar{\psi} * \gamma \wedge \hat{\nabla}\psi + im\eta_{ab}(\bar{\psi}\psi) * 1 \\ &\quad - \frac{i}{2}\bar{\psi}(*e_b\gamma_a + *e_a\gamma_b) \wedge \hat{\nabla}\psi. \end{aligned} \quad (3.25)$$

Then noting that $*e_a \wedge \hat{\nabla}\psi = -\hat{\nabla}_{X_a}\psi * 1$, and working on-shell, we determine

$$T_{ab}[sym] = \frac{i}{2}\bar{\psi}(\gamma_a\hat{\nabla}_{X_b} + \gamma_b\hat{\nabla}_{X_a})\psi + im\eta_{ab}(\bar{\psi}\psi). \quad (3.26)$$

as expected.

3.2. Nonconstrained first-order variations

In this case the variations of the action density read

$$\begin{aligned}
\dot{\mathcal{L}} &= \dot{e}^a \wedge \frac{1}{2} R^{bc} \wedge *(e_a \wedge e_b \wedge e_c) + \frac{1}{2} \dot{\omega}^{ab} \wedge *(e_a \wedge e_b \wedge e_c) \wedge T^c \\
&+ \frac{i}{2} \dot{e}^a \wedge \bar{\psi} * (e^b \wedge e_a) \wedge \gamma_b \nabla \psi + \frac{i}{2} \dot{e}^a \wedge m(\bar{\psi} \psi) * e_a \\
&- \frac{i}{4} \dot{\omega}^{ab} \wedge \bar{\psi} * \gamma \wedge \sigma_{ab} \psi \\
&+ \frac{i}{2} \bar{\psi} * \gamma \wedge \nabla \psi + \frac{i}{2} \bar{\psi} * \gamma \wedge \nabla \dot{\psi} + im \bar{\psi} \dot{\psi} * 1.
\end{aligned} \tag{3.27}$$

The coframe variations give the Einstein field equations

$$-\frac{1}{2} R^{bc} \wedge *(e_a \wedge e_b \wedge e_c) = \frac{i}{2} \bar{\psi} * (e^b \wedge e_a) \wedge \gamma_b \nabla \psi + im(\bar{\psi} \psi) * e_a \tag{3.28}$$

while the variations of the connection 1-forms provide the field equations satisfied by the torsion 2-forms:

$$*(e_a \wedge e_b \wedge e_c) \wedge T^c = \frac{i}{2} \bar{\psi} * \gamma \wedge \sigma_{ab} \psi. \tag{3.29}$$

These are linear algebraic equations that determine the torsion 2-forms uniquely as

$$T^a = \frac{i}{4} * e^a_b (\bar{\psi} \gamma_5 \gamma^b \psi). \tag{3.30}$$

Then the contortion 1-forms will be

$$K^a_b = \frac{i}{8} * e^a_{bc} (\bar{\psi} \gamma_5 \gamma^c \psi). \tag{3.31}$$

The simplification of the Dirac equation needs some further manipulations. Variation of the action density gives

$$\frac{i}{2} \bar{\psi} * \gamma \wedge \nabla \psi + \frac{i}{2} \bar{\psi} * \gamma \wedge (d\dot{\psi} + \frac{1}{2} \omega^{ab} \sigma_{ab} \dot{\psi}). \tag{3.32}$$

We use Majorana flip identities on the second term and get

$$\frac{i}{2} \bar{\psi} * \gamma \wedge \nabla \psi + \frac{i}{2} d\bar{\psi} \wedge * \gamma \psi - \frac{i}{4} \bar{\psi} \sigma_{ab} * \gamma \psi \wedge \omega^{ab}. \tag{3.33}$$

We now differentiate the second term by parts and use γ -matrix identities on the third term and get our expression simplified to (up to a closed form)

$$i\bar{\psi} * \gamma \wedge \nabla \psi + \frac{i}{2} \bar{\psi} * e^a_b \gamma_a \dot{\psi} \wedge T^b + (\text{closed form}). \tag{3.34}$$

Therefore, our Majorana 4-spinor field ψ satisfies the variational (massless) Dirac equation

$$i * \gamma \wedge \nabla \psi + \frac{i}{2} T^a \wedge * e_a^b \gamma_b \psi + im\psi * 1 = 0. \quad (3.35)$$

We now note that connection 1-forms with torsion can be uniquely decomposed as the sum of the Levi-Civita connection 1-forms, determined only by the metric and the contortion 1-forms determined by our Majorana 4-spinor:

$$\omega^a_b = \hat{\omega}^a_b + K^a_b. \quad (3.36)$$

Then the curvature 2-forms can be similarly decomposed

$$R^a_b = \hat{R}^a_b + \hat{D}K^a_b + K^a_c \wedge K^c_b. \quad (3.37)$$

Therefore, the Einstein field equations can be written as

$$\begin{aligned} -\frac{1}{2} \hat{R}^{bc} \wedge * e_{abc} &= \frac{i}{2} \bar{\psi} * (e^b \wedge e_a) \wedge \gamma_b \hat{\nabla} \psi + \frac{i}{2} m(\bar{\psi} \psi) * e_a + \frac{1}{2} \hat{D}K^{bc} \wedge * e_{abc} \\ &+ \frac{i}{4} \bar{\psi} * (e^b \wedge e_a) \wedge \gamma_b K^{cd} \sigma_{cd} \psi + \frac{1}{2} K^b_d \wedge K^{dc} \wedge * e_{abc}. \end{aligned} \quad (3.38)$$

It is not difficult to check that

$$\frac{1}{2} \hat{D}K^{bc} \wedge * e_{abc} = \frac{i}{4} e_a \wedge (\bar{\psi} \gamma_5 \gamma \wedge \hat{\nabla} \psi). \quad (3.39)$$

Therefore, the first, second and third terms on the right hand side above add up to the symmetric stress-energy-momentum 3-forms of the Majorana spinor. The sum of the fourth and fifth terms gives a particular quartic self-coupling of the Majorana spinor. We explicitly work it out to be

$$\frac{3}{64} (\bar{\psi} \gamma_5 \gamma_b \psi) (\bar{\psi} \gamma_5 \gamma^b \psi) * e_a. \quad (3.40)$$

In a similar way, the Dirac equation decomposes as

$$i * \gamma \wedge \hat{\nabla} \psi + im\psi * 1 + \frac{3}{16} (\bar{\psi} \gamma_5 \gamma \psi) \wedge \gamma_5 * \gamma \psi = 0. \quad (3.41)$$

4. Concluding comments

(A) As far as the coupled field equations are concerned the zero-torsion constrained variations of the action density 4-form

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} R_{ab} \wedge * (e^a \wedge e^b) + \frac{i}{2} \bar{\psi} * \gamma \wedge \nabla \psi + \frac{i}{2} m(\bar{\psi} \psi) * 1 \\ &+ \frac{\alpha}{4} (\bar{\psi} \gamma_5 \gamma \psi) \wedge (\bar{\psi} \gamma_5 * \gamma \psi) + T^a \wedge \lambda_a, \end{aligned} \quad (4.1)$$

and the unconstrained (i.e. with nonvanishing torsion) variations of the action density 4-form

$$\begin{aligned} \mathcal{L}' = & \frac{1}{2} R_{ab} \wedge *(e^a \wedge e^b) + \frac{i}{2} \bar{\psi} * \gamma \wedge \nabla \psi + \frac{i}{2} m (\bar{\psi} \psi) * 1 \\ & + \frac{\beta}{4} (\bar{\psi} \gamma_5 \gamma \psi) \wedge (\bar{\psi} \gamma_5 * \gamma \psi) \end{aligned} \quad (4.2)$$

yield the same field equations provided their parameters are related by

$$\alpha = \beta + \frac{3}{16}. \quad (4.3)$$

Of course, the coupling of spinning matter to space-time geometry in these two cases may differ and needs further considerations.

(B) A short remark concerning the physical interpretation of our Lagrange multiplier 2-forms is in order. Consider a generic matter Lagrangian density 4-form

$$\mathcal{L}_{matter}(\psi, e, \hat{\omega}), \quad (4.4)$$

whose gravitational couplings are described through its dependence on both the coframe and the connection. The coupling of Dirac, Weyl or Majorana spinors to gravity provide familiar examples of such couplings. On the other hand, nonminimal couplings of electromagnetic fields to gravity described by generic terms of the type RF^2 in the Lagrangian density 4-form provide rather unfamiliar examples. Then the variational derivatives

$$\frac{\delta \mathcal{L}_{matter}}{\delta e^a} \equiv \tau_a(matter) \quad (4.5)$$

give the energy-momentum 3-forms of matter for which the corresponding canonical energy-momentum tensor is non-symmetric in general. Similarly

$$\frac{\delta \mathcal{L}_{matter}}{\delta \omega^{ab}} \equiv \Sigma_{ab}(matter) \quad (4.6)$$

gives the angular momentum 3-forms of matter. We construct the 2-forms

$$\lambda_a = \iota^b \Sigma_{ba} - \frac{1}{4} e_a \wedge \iota^c \iota^b \Sigma_{bc} \quad (4.7)$$

so that

$$\hat{D} \lambda_a = \hat{D}(\iota^b \Sigma_{ba}) + \frac{1}{4} e_a \wedge d(\iota^c \iota^b \Sigma_{bc}). \quad (4.8)$$

Then the energy-momentum tensor corresponding to the sum

$$\tau_a(matter) + \hat{D} \lambda_a, \quad (4.9)$$

is symmetric. This is the Belinfante-Rosenfeld procedure [9, 10] for the construction of a symmetric energy-momentum tensor when the matter Lagrangian density 4-form carries an explicit dependence

on the connection 1-forms. We will not repeat the proof that the conservation laws associated with both the canonical tensors and the symmetric tensors remain the same [11]. Our main observation here is that our Lagrange multiplier 2-forms λ_a are nothing but the well-known Belinfante-Rosenfeld 2-forms.

(C) Under local Pauli-Gürsey (chiral) rotations

$$\psi \rightarrow e^{\theta\gamma_5}\psi, \quad (4.10)$$

there exists an axial vector current density

$$J_5 = \frac{i}{4}(\bar{\psi}\gamma_5\psi), \quad (4.11)$$

such that

$$*d * J_5 = im(\bar{\psi}\gamma_5\psi), \quad (4.12)$$

on-shell. J_5 is conserved for a massless-neutrino. The generic quartic self-coupling of Majorana neutrinos takes the form $J_5 \wedge *J_5$.

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