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On Fuzzy Cosets of Gamma Nearrings

Satyanarayana Bhavanari, Syam Prasad Kuncham

Abstract

In this paper, we consider fuzzy notion of a Γ -near ring, introduce the notion of a fuzzy coset and obtained some related important fundamental isomorphism theorems.

Key Words: Gamma nearring, ideal, fuzzy ideal, fuzzy coset, Gamma nearring homomorphism.

Introduction

A non-empty set N with two binary operations $+$ and \cdot is called a *nearring* if it satisfies the following axioms.

- (i) $(N, +)$ is a group (not necessarily Abelian);
- (ii) (N, \cdot) is a semi-group;
- (iii) $(a + b)c = ac + bc$ for all $a, b, c \in N$.

Precisely speaking, it is a right near ring. Moreover, a near ring N is said to be a *zero-symmetric* nearring if $n0 = 0$ for all $n \in N$, where 0 is the additive identity in N . The concept of Γ -nearring, a generalization of both the concepts nearring and Γ -ring was introduced by Satyanarayana [10]. Later, several authors such as Satyanarayana [11, 12], Booth [1-3] and Booth and Groenewald [4] studied the ideal theory of Γ -nearrings.

Let $(M, +)$ be a group (not necessarily Abelian) and Γ a non-empty set. Then M is said to be a Γ -nearring if there exists a mapping $M \times \Gamma \times M \rightarrow M$ (the image of (a, α, b) is denoted by $a\alpha b$), satisfying the following conditions:

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(i) $(a + b)\alpha c = a\alpha c + b\alpha c$;

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Moreover, M is said to be *zero-symmetric* if $a\alpha 0 = 0$ for all $a \in M$ and $\alpha \in \Gamma$, where 0 is the additive identity in M .

A normal subgroup $(I, +)$ of $(M, +)$ is called

(i) a *left ideal*, if $a\alpha(b + i) - a\alpha b \in I$ for all $a, b \in M, \alpha \in \Gamma, i \in I$;

(ii) a *right ideal*, if $i\alpha a \in I$ for all $a \in M, \alpha \in \Gamma, i \in I$;

(iii) an *ideal*, if it is both a left and a right ideal.

It is clear that if M is a Γ -nearring, then the elements of Γ act as binary operations on M such that the system $(M, +, \gamma)$ is a nearring for all $\gamma \in \Gamma$. The relations between the concepts Γ -nearring and nearring were studied in Section 1 of Satyanarayana [12]. Throughout this paper, M stands for a zero-symmetric Γ -nearring. The ideal generated by an element $a \in M$ is denoted by $\langle a \rangle$. For other definitions and preliminary results on Γ -nearrings we refer to [7, 11, 12].

The concept of fuzzy subset was introduced by Zadeh [14]. A fuzzy set in a set A is a function $\mu: A \rightarrow [0, 1]$. For any $t \in [0, 1]$, the set μ_t defined by $\mu_t = \{x \in A | \mu(x) \geq t\}$ is called as a *level subset* of μ . For any two fuzzy sets μ, σ in A , we write $\mu \subseteq \sigma$ if

$\mu(x) \leq \sigma(x)$ for all $x \in A$. (In this case, we also say that μ is a subset of σ). Let X and Y be two non empty sets, $f: X \rightarrow Y$, μ and σ be fuzzy subsets of X and Y respectively. Then $f(\mu)$, the *image* of μ under f is a fuzzy subset of Y defined by

$$(f(\mu))(y) = \begin{cases} \sup_{f(x)=y} \mu(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi. \end{cases}$$

$f^{-1}(\sigma)$, the *preimage* of σ under f is a fuzzy subset of X defined by $(f^{-1}(\sigma))(x) = \sigma(f(x))$ for all $x \in X$.

Jun, Sapanici and Ozturk [7] introduced the concept “fuzzy ideal” in Γ -nearrings and studied some fundamental properties. It is clear that a fuzzy set μ in a Γ -nearring M is a mapping $f: M \rightarrow [0, 1]$.

A fuzzy set μ in a Γ -nearring M is called a *fuzzy left (resp., right) ideal* of (or in) M if

(i) μ is a fuzzy normal subgroup with respect to addition (that is, $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, and $\mu(y + x - y) \geq \mu(x)$);

(ii) $\mu(x\alpha(y+z) - x\alpha y) \geq \mu(z)$ (resp., $\mu(x\alpha y) \geq \mu(x)$) for all $x, y, z \in M$ and $\alpha \in \Gamma$.

If μ is both left and right ideal, then μ is said to be a *fuzzy ideal* in M .

It is easy to verify that if μ is a fuzzy ideal of M , then the following three conditions hold: (i) $\mu(0) \geq \mu(x)$ (in other words, $\mu(0) = \max \{\mu(x) / x \in M\}$);

(ii) $\mu(x + y) = \mu(y + x)$; and

(iii) $\mu(x-y) = \mu(0)$ implies $\mu(x) = \mu(y)$, for all $x, y \in M$.

Jun, Sapançi and Oztürk [7] proved that for a fuzzy set μ in M , μ is a fuzzy left (resp., right) ideal of M if and only if each level subset μ_t , $t \in \text{im}(\mu)$, of μ is a left (resp. right) ideal of M .

A fuzzy (left, right) ideal μ of M with $\mu(0) = 1$ is called a *normal* (left, right) ideal of M . Jun, Kim and Oztürk [5] introduced the fuzzy maximal ideals and some related properties were studied. A fuzzy ideal μ of M is said to be a *fuzzy maximal* ideal if it satisfies the two conditions: (i) μ is non-constant; and (ii) μ^* is a maximal element among all the normal fuzzy ideals of M where $\mu^*(x) = \mu(x) + 1 - \mu(0)$, for all $x \in M$.

For other preliminary definitions and results related to fuzzyness, see [7].

This paper is divided into three sections. In Section 1, we prove a result on fuzzy ideals.

In Section 2, we introduce the concept fuzzy coset in Γ -nearrings and prove that the set of all fuzzy cosets forms a Γ -nearring (theorem 2.4). In the last section, that is, in Section 3, we prove the following important fundamental results related to isomorphism theorems on Γ -nearrings;

(i). M/μ is isomorphic to the quotient Γ -nearring M/M_μ , where M/μ is the Γ -nearring of all the cosets of M with respect to the fuzzy ideal μ , and

$$M_\mu = \{x \in M / \mu(x) = \mu(0)\}.$$

(ii) There exists an order preserving bijection between the set P of all fuzzy ideals σ of M such that $\sigma \supseteq \mu$ and $\sigma(0) = \mu(0)$ and the set Q of all fuzzy ideals θ of M/μ such that $\theta \supseteq \theta_\mu$, where θ_μ is a fuzzy ideal of M/μ defined by $\theta_\mu(x+\mu) = \mu(x)$ for all $x \in M$.

(iii) Let $h: M \rightarrow M^1$ be a Γ -nearring epimorphism and let σ be a fuzzy ideal of M^1 and $\mu = h^{-1}(\sigma)$. Then the map $\psi: M/\mu \rightarrow M^1/\sigma$ defined by $\psi(x+\mu) = h(x) + \sigma$ is a Γ -near ring isomorphism.

As a consequence of (iii), we obtain the following result: If μ and σ are two fuzzy ideals of M such that $\mu \subseteq \sigma$ and $\sigma(0) = \mu(0)$, then $M/\sigma \cong (M/\mu)/(\sigma/\mu)$.

1. Fuzzy ideals

Theorem 1.1 *If μ is a fuzzy ideal of M , and $a \in M$ then $\mu(x) \geq \mu(a)$ for all $x \in \langle a \rangle$.*

Proof. By straightforward verification, we conclude that for $a \in M$, $\langle a \rangle = \bigcup_{i=0}^{\infty} A_i$,

where $A_{k+1} = A_k^* \cup A_k^+ \cup A_k^0 \cup A_k^{++}$, $A_0 = \{a\}$ and

$$A_k^* = \{n+x-n \mid n \in \mathbb{N}, x \in A_k\};$$

$$A_k^+ = \{n_1\alpha(n_2+a)-n_1\alpha n_2 \mid n_1, n_2 \in M, a \in A_k, \alpha \in M\};$$

$$A_k^0 = \{x-y \mid x, y \in A_k\};$$

$$A_k^{++} = \{x\alpha m \mid x \in A_k, \alpha \in \Gamma \text{ and } m \in M\}.$$

We prove that $\mu(u) \geq \mu(a)$ for all $u \in A_m$ for $m \geq 1$. For this, we use induction on m . It is obvious if $m = 0$. Suppose the induction hypothesis for k . That is., $\mu(x) \geq \mu(a)$ for all $x \in A_k$. Now let $v \in A_k^* \cup A_k^+ \cup A_k^0 \cup A_k^{++}$. Suppose $v \in A_k^*$. Then $v = n+x-n$ for some $x \in A_k$. Now $\mu(v) = \mu(n+x-n) \geq \mu(x)$ (since μ is a fuzzy ideal of N) $\geq \mu(a)$ (by induction hypothesis). Let $v \in A_k^+$. Then $v = n_1\alpha(n_2+a)-n_1\alpha n_2$ for some $n_1, n_2 \in M, a \in A_k$ and $\alpha \in \Gamma$. Now $\mu(v) = \mu(n_1\alpha(n_2+a)-n_1\alpha n_2) \geq \mu(x)$ (since μ is a fuzzy ideal of M) $\geq \mu(a)$ (by induction hypothesis on k).

Suppose that $v \in A_k^+$. Then $v = n_1\alpha(n_2+x)-n_1\alpha n_2$ for some $n_1, n_2 \in M, x \in A_k$ and $\alpha \in \Gamma$. Now $\mu(v) = \mu(n_1\alpha(n_2+x)-n_1\alpha n_2) \geq \mu(x)$ (since μ is a fuzzy ideal of M) $\geq \mu(a)$ (by induction hypothesis on k).

Suppose $v \in A_k^{++}$. Then $v = x\alpha m$ for some $x \in A_k, \alpha \in \Gamma$ and $m \in M$. Now $\mu(v) = \mu(x\alpha m) \geq \mu(x)$ (since μ is a fuzzy ideal of M) $\geq \mu(a)$ (by induction hypothesis on k).

Thus in all cases we proved that $\mu(v) \geq \mu(a)$ for all $v \in A_{k+1}$. Hence by the principle of mathematical induction, we conclude that $\mu(v) \geq \mu(a)$ for all $v \in A_m$ and for all positive integers m . Hence $\mu(x) \geq \mu(a)$ for all $x \in \langle a \rangle$. \square

Corollary 1.2 *If I is an ideal of N with $I = \langle a \rangle = \langle b \rangle$, then $\mu(a) = \mu(b)$.*

Proof. Since $a \in \langle b \rangle$ and $b \in \langle a \rangle$, we have $\mu(a) \geq \mu(b)$ and $\mu(b) \geq \mu(a)$, so $\mu(a) = \mu(b)$. \square

2. Fuzzy Cosets

Definition 2.1 Let μ be a fuzzy ideal of M and $m \in M$. Then the fuzzy subset $m+\mu$ defined by $(m+\mu)(m^1) = \mu(m^1-m)$ for all $m^1 \in M$ is called a fuzzy coset of the fuzzy ideal μ .

Lemma 2.2 Let μ be a fuzzy ideal of M . Then for $x, y, z \in M$ we have the following:

- (i) $x+\mu = y+\mu$ if and only if $\mu(x-y) = \mu(0)$;
- (ii) If $x + \mu = y + \mu$, then $\mu(x) = \mu(y)$;
- (iii) $\mu(x+y) = \mu(y+x)$;
- (iv) $M_\mu = \{x \in M / \mu(x) = \mu(0)\}$ is an ideal of M ;
- (v) Every fuzzy coset of a fuzzy ideal μ of M is constant on M_μ ;
- (vi) If $z \in M_\mu$, then $(x+\mu)(z) = \mu(x)$.

Proof. (i), (ii), (iii) have straightforward verifications.

(iv) Proved in Jun, Sapanci and Ozturk [7].

(v) Let $y, z \in M_\mu$. We show that $(x+\mu)(y) = (x+\mu)(z)$. Since $y, z \in M_\mu$, we have that $\mu(y) = \mu(0)$ and $\mu(z) = \mu(0)$. Since M_μ is an ideal, we have that $y-z \in M_\mu$. So $\mu(y-z) = \mu(0)$. Now $(x+\mu)(y) = \mu(y-x)$ (by the definition of fuzzy coset)

$$\begin{aligned}
 &= \mu(-(x-y)) \text{ (since } \mu \text{ is a fuzzy ideal of } M) \\
 &= \mu(y-x) = \mu(-z+y-x+z) \text{ (since } \mu \text{ is a fuzzy normal subgroup)} \\
 &\geq \min \{ \mu(-z+y), \mu(x-z) \} \text{ (since } \mu \text{ is a fuzzy ideal of } M) \\
 &= \min \{ \mu(y-z), \mu(x-z) \} = \min \{ \mu(0), \mu(x-z) \} \text{ (since } \mu(y-z) = \mu(0)) \\
 &= \mu(x-z) \text{ (since } \mu(0) \geq \mu(x-z)) \\
 &= (x+\mu)(z) \text{ (by definition of fuzzy coset).}
 \end{aligned}$$

Therefore $(x+\mu)(y) \geq (x+\mu)(z)$.

Similarly by interchanging y and z in above part, we can show that $(x+\mu)(z) \geq (x+\mu)(y)$. Hence $(x+\mu)(y) = (x+\mu)(z)$ for all $y, z \in M_\mu$.

(vi) Let $z \in M_\mu$. Then $\mu(z) = \mu(0)$. Since $z, 0 \in M_\mu$, we have $(x+\mu)(z) = (x+\mu)(0)$ (by (v)) $\Rightarrow \mu(z-x) = \mu(0-x) = \mu(-x) = \mu(x)$ (since μ is a fuzzy ideal of M). Therefore $\mu(z-x) = \mu(x)$. Hence $(x+\mu)(z) = \mu(x)$. \square

Notation 2.3 We write $M/\mu = \{m + \mu \mid m \in M\}$, the set of all fuzzy cosets of μ .

Theorem 2.4 *Let μ be a fuzzy ideal of M . Then the set M/μ of all fuzzy cosets of μ is a Γ -nearring with respect to the operations defined by*

$$(x+\mu)+(y+\mu) = (x+y)+\mu; \text{ and } (x+\mu)\alpha(y+\mu) = x\alpha y+\mu \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma.$$

Proof. First we verify that “+” is well defined. Suppose $x+\mu = u+\mu$, $y+\mu = v+\mu$. Then by Lemma 2.2 (i), $\mu(x-u) = \mu(y-v) = \mu(0)$. Now $\mu\{(x+y)-(u+v)\} = \mu\{(x+y-v-u)\} = \mu\{(x+y-v)-u\} = \mu\{-u+(x+y-v)\} = \mu\{(-u+x)+(y-v)\} \geq \min\{\mu(-u+x), \mu(y-v)\}$ (since μ is a fuzzy ideal of M) $= \min\{\mu(x-u), \mu(y-v)\} = \mu(0)$. Also it is clear that $\mu(0) \geq \mu\{(x+y)-(u+v)\}$. Therefore $\mu\{(x+y)-(u+v)\} = \mu(0)$. Hence by Lemma 2.2 (i), $(x+y)+\mu = (u+v)+\mu$. This shows that “+” is well defined.

Next we verify that “.” is well defined. Now $\mu(x\alpha y-u\alpha v) = \mu(u\alpha v-x\alpha y) = \mu(u\alpha v-x\alpha v+x\alpha v-x\alpha y) = \mu((u-x)\alpha v+x\alpha(y+(-y+v))-x\alpha y) \geq \min\{\mu(u-x), \mu(-y+v)\}$ (since μ is a fuzzy ideal of M) $= \min\{\mu(0), \mu(0)\} = \mu(0) \geq \mu(x\alpha y-u\alpha v)$. This shows that $\mu(x\alpha y-u\alpha v) = \mu(0)$. By Lemma 2.2 (i), $x\alpha y+\mu = u\alpha v+\mu$.

Now we verify that $M/\mu = \{x+\mu / x \in M\}$ is a Γ -nearring with respect to the above operations defined. A direct verification shows that $(M/\mu, +)$ is a group.

$$\begin{aligned} & \text{Let } x, y, z \in M \text{ and } \alpha, \beta \in \Gamma. \text{ Now } ((x+\mu)+(y+\mu))\alpha(z+\mu) \\ &= ((x+y)+\mu)\alpha(z+\mu) \\ &= ((x+y)\alpha z)+\mu \text{ (by definition of multiplication)} \\ &= ((x\alpha z)+(y\alpha z))+\mu \text{ (by right distributive law in } M) \\ &= ((x\alpha z)+\mu)+((y\alpha z)+\mu) \\ &= (x+\mu)\alpha(z+\mu)+(y+\mu)\alpha(z+\mu). \end{aligned}$$

$$\begin{aligned} & \text{Also } ((x+\mu)\alpha(y+\mu))\beta(z+\mu) = ((x\alpha y)+\mu)\beta(z+\mu) \text{ (by definition of addition)} \\ &= (x\alpha y)\beta(z+\mu) = x\alpha(y\beta z)+\mu = (x+\mu)\alpha(y\beta z+\mu) = (x+\mu)\alpha((y+\mu)\beta(z+\mu)). \end{aligned}$$

Hence M/μ is a Γ -nearring. □

Notation 2.5 *Let M be a fuzzy ideal. We define $\theta_\mu: M/\mu \rightarrow [0, 1]$ by $\theta_\mu(x+\mu) = \mu(x)$ for all $x \in M$.*

Lemma 2.6 *If μ is a fuzzy ideal, then θ_μ is a fuzzy ideal of M/μ .*

Proof. Given that $\theta_\mu(x+\mu) = \mu(x)$. Suppose $x+\mu = y+\mu$. Then $\mu(x-y) = \mu(0)$. This implies $\mu(x) = \mu(y)$. That is, $\theta_\mu(x+\mu) = \theta_\mu(y+\mu)$. Therefore θ_μ is well defined.

We verify that θ_μ is a fuzzy ideal of M/μ .

$$\begin{aligned} \text{(i) } \theta_\mu((x+\mu)+(y+\mu)) &= \theta_\mu(x+y+\mu) = \mu(x+y) \text{ (by definition of } \theta_\mu) \\ &\geq \min \{ \mu(x), \mu(y) \} \text{ (since } \mu \text{ is a fuzzy ideal of } M) \\ &= \min \{ \theta_\mu(x+\mu), \theta_\mu(y+\mu) \}. \end{aligned}$$

Therefore $\theta_\mu((x+\mu)+(y+\mu)) \geq \min \{ \theta_\mu(x+\mu), \theta_\mu(y+\mu) \}$.

(ii) $\theta_\mu(x+\mu) = \mu(x) = \mu(-x)$ (since μ is a fuzzy ideal of M) $= \theta_\mu(-x+\mu)$, by definition of θ_μ .

$$\text{(iii) } \theta_\mu((y+\mu)+(x+\mu)-(y+\mu)) = \theta_\mu((y+x-y)+\mu) = \mu(y+x-y) = \mu(x) = \theta_\mu(x+\mu)$$

(iv) $\theta_\mu((x+\mu)\alpha(y+\mu)) = \theta_\mu(x\alpha y+\mu) = \mu(x\alpha y) \geq \mu(x) = \theta_\mu(x+\mu)$, by definition of θ_μ .

$$\begin{aligned} \text{(v) } \theta_\mu\{(x+\mu)\alpha((y+\mu)+(z+\mu))-(x+\mu)\alpha(y+\mu)\} &= \theta_\mu\{(x+\mu)\alpha((y+z)+\mu)- \\ (x+\mu)\alpha(y+\mu)\} &= \theta_\mu\{(x\alpha(y+z)+\mu)-(x\alpha y+\mu)\} = \theta_\mu\{(x\alpha(y+z)-(x\alpha y)+\mu)\} \\ &= \mu\{x\alpha(y+z)-(x\alpha y)\} \geq \mu(z) = \theta_\mu(z+\mu). \end{aligned}$$

Hence θ_μ is a fuzzy ideal of M/μ . □

3. Some Isomorphism Theorems

Theorem 3.1 (*Jun, Sapanci and Ozturk [7]*): *If μ is a fuzzy (left, right) ideal of M then the set $M_\mu = \{x \in M / \mu(x) = \mu(0)\}$ is a fuzzy (left, right) ideal of M .*

Definition 3.2 *Let M and N be Γ -nearrings. A map $\theta: M \rightarrow N$ is called a Γ -nearring homomorphism if $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover if θ is one-one (onto, bijection, respectively) then θ is called as monomorphism (epimorphism, isomorphism, respectively).*

Now we prove the following theorem.

Theorem 3.3 *If μ is a fuzzy ideal of M then the map $\theta: M \rightarrow M/\mu$, defined by $\theta(x) = x+\mu$, $x \in M$, is a Γ -near-ring epimorphism with kernel M_μ where*

$M_\mu = \{x \in M / \mu(x) = \mu(0)\}$. Moreover M/M_μ is isomorphic to M/μ (under the mapping $x+M_\mu \rightarrow x+\mu$).

Proof. $\theta(x+y) = \theta(x) + \theta(y)$ is clear.

Now $\theta(x\alpha y) = (x\alpha y)+\mu$ (by definition of θ) $= (x+\mu)\alpha(y+\mu)$
 $= \theta(x)\alpha\theta(y)$. Therefore θ is a homomorphism.

Now $x \in \text{kernel } \theta \Leftrightarrow \theta(x) = 0 = 0 + \mu \Leftrightarrow x + \mu = 0 + \mu \Leftrightarrow \mu(x - 0) = \mu(0)$, (by Lemma 2.2(i)) $\Leftrightarrow \mu(x) = \mu(0) \Leftrightarrow x \in M_\mu$. This shows that $\text{kernel } \theta = M_\mu$. \square

Notation 3.4 Let μ and σ be two fuzzy ideals of M such that $\mu \subseteq \sigma$ and $\sigma(0) = \mu(0)$. Then we define a fuzzy set $\theta_\sigma: M/\mu \rightarrow [0, 1]$ by $\theta_\sigma(x + \mu) = \sigma(x)$ for all $x + \mu \in M/\mu$.

Lemma 3.5 θ_σ is a fuzzy ideal of M/μ such that $\theta_\mu \subseteq \theta_\sigma$ where θ_σ and θ_μ are given by the above notation. Also $\theta_\mu(0) = \theta_\sigma(0)$.

Proof. A direct verification shows that θ_σ is well-defined and is a fuzzy normal subgroup of M/μ . Now we verify that θ_σ is a fuzzy ideal of M/μ .

$$\begin{aligned} \theta_\sigma((x + \mu)\alpha(y + \mu)) &= \theta_\sigma(x\alpha y + \mu) = \sigma(x\alpha y) \text{ (by definition of } \theta_\sigma) \geq \sigma(x) = \theta_\sigma(x + \mu). \\ \theta_\sigma\{(x + \mu)\alpha((y + \mu) + (z + \mu)) - (x + \mu)\alpha(y + \mu)\} \\ &= \theta_\sigma\{(x + \mu)\alpha((y + z) + \mu) - (x\alpha y + \mu)\} \\ &= \theta_\sigma\{(x\alpha(y + z) + \mu) - (x\alpha y + \mu)\} \\ &= \theta_\sigma\{(x\alpha(y + z) - x\alpha y) + \mu\} \\ &= \sigma(x\alpha(y + z) - x\alpha y) \quad \text{(by definition of } \theta_\sigma) \\ &\geq \sigma(z) \quad \text{(since } \sigma \text{ is a fuzzy ideal of } M) \\ &= \theta_\sigma(z + \mu) \text{ (by definition of } \theta_\sigma). \end{aligned}$$

Also $\theta_\sigma(x + \mu) = \sigma(x) \geq \mu(x) = \theta_\mu(x + \mu)$. Hence $\theta_\mu \subseteq \theta_\sigma$. \square

Notation 3.6 (i) The fuzzy ideal θ_σ of M/μ is denoted by σ/μ . Note that $\mu \subseteq \sigma$ with $\sigma(0) = \mu(0)$.

(ii) Let μ be a fuzzy ideal of M and θ a fuzzy ideal of M/μ such that $\theta_\mu \subseteq \theta$ and $\theta_\mu(0) = \theta(0)$. Then we define $\sigma_\theta: M \rightarrow [0, 1]$ by $\sigma_\theta(x) = \theta(x + \mu)$ for all $x \in M$.

Lemma 3.7 σ_θ (defined above in notation 3.6), is a fuzzy ideal of M such that $\mu \subseteq \sigma_\theta$ and $\mu(0) = \sigma_\theta(0)$.

Proof. It is easy to verify that σ_θ is a fuzzy normal subgroup of M . Now

$$\begin{aligned} \sigma_\theta(x\alpha y) &= \theta(x\alpha y + \mu) = \theta((x + \mu)\alpha(y + \mu)) \geq \theta(x + \mu) \text{ (since } \theta \text{ is a fuzzy right ideal of } M/\mu) \\ &= \sigma_\theta(x) \quad \text{(by definition of } \sigma_\theta). \end{aligned}$$

Therefore σ_θ is a fuzzy right ideal of M .

$$\begin{aligned} \text{Also } \sigma_\theta\{x\alpha(y+z)-x\alpha y\} &= \theta\{(x\alpha(y+z)-x\alpha y)+\mu\} \\ &= \theta\{(x+\mu)\alpha((y+\mu)+(z+\mu))-(x+\mu)\alpha(y+\mu)\} \\ &\geq \theta(z+\mu) \quad (\text{since } \theta \text{ is a fuzzy left ideal of } M/\mu) \\ &= \sigma_\theta(z) \quad (\text{by definition of } \sigma_\theta). \end{aligned}$$

Therefore σ_θ is a fuzzy left ideal of M .

This shows that σ_θ is a fuzzy ideal of M .

Now we have $\sigma_\theta(x) = \theta(x+\mu) \geq \theta_\mu(x+\mu)$ (since $\theta_\mu \subseteq \theta$) = $\mu(x)$ and so $\mu \subseteq \sigma_\theta$.

Also $\sigma_\theta(0) = \theta(0+\mu) = \theta(0) = \theta_\mu(0) = \theta_\mu(0+\mu) = \mu(0)$. □

Notation 3.8 Let μ be a fuzzy ideal of M . We write $P = \{\sigma/\sigma \text{ is a fuzzy ideal of } M, \mu \subseteq \sigma, \sigma(0) = \mu(0)\}$ and $Q = \{\theta / \theta \text{ is a fuzzy ideal of } M/\mu, \theta_\mu \subseteq \theta \text{ and } \theta(0) = \theta_\mu(0)\}$.

Theorem 3.9 Let μ be a fuzzy ideal of M . There exists an order preserving bijective mapping between the sets P and Q .

Proof. Define $\eta: P \rightarrow Q$ by $\eta(\sigma) = \theta_\sigma$. By the lemma 3.5, $\eta(\sigma) = \theta_\sigma$ is a fuzzy ideal of M/μ such that $\theta_\mu \subseteq \theta_\sigma$ and $\theta_\mu(0) = \theta_\sigma(0)$. By the definition of θ_σ , the mapping η is well defined. Suppose $\eta(\sigma) = \eta(\beta) \Rightarrow \theta_\sigma = \theta_\beta$.

Now $\sigma(x) = \theta_\sigma(x+\mu) = \theta_\beta(x+\mu) = \beta(x)$ for all $x \in M$. We have proved that $\eta(\sigma) = \eta(\beta) \Rightarrow \sigma = \beta$. Therefore η is one-one.

Let $\theta \in Q$. Consider $\sigma_\theta: M \rightarrow [0, 1]$ defined in notation 3.6 (ii). By Lemma 3.7, $\sigma_\theta \in P$. Now we have to show that $\eta(\sigma_\theta) = \theta$.

$(\eta(\sigma_\theta))(x+\mu) = \sigma_\theta(x)$ (by definition of η and the definition of θ_σ in notation 3.4) = $\theta(x+\mu)$ (by the definition of σ_θ in notation 3.6 (ii)).

This is true for all $x+\mu \in M/\mu$. Hence $\eta(\sigma_\theta) = \theta$ and so η is onto.

Thus $\eta: P \rightarrow Q$ is a bijection.

Let $\sigma, \beta \in P$ such that $\sigma \subseteq \beta$. Now $(\eta(\sigma))(x+\mu) = \theta_\sigma(x+\mu)$ (by the definition of η) $\sigma(x) \leq \beta(x)$ (since $\sigma \subseteq \beta$) = $\theta_\beta(x+\mu) = (\eta(\beta))(x+\mu)$.

Since this is true for all $x+\mu \in M/\mu$, we have that $\eta(\sigma) \subseteq \eta(\beta)$.

Thus $\eta: P \rightarrow Q$ is an order preserving bijection. □

Theorem 3.10 (Jun, Sapanci and Ozturk [7]) *A Γ -nearing homomorphic pre-image of a fuzzy (left, right) ideal is a fuzzy (left, right) ideal.*

Theorem 3.11 *Let $h: M \rightarrow M^1$ be an epimorphism, σ is a fuzzy ideal of M^1 and $\mu = h^{-1}(\sigma)$. Then the map $\psi: M/\mu \rightarrow M^1/\sigma$ defined by $\psi(x+\mu) = h(x) + \sigma$ is a Γ -near- ring isomorphism.*

Proof. First we show that the mapping ψ is well defined.

Let $z^1 \in M^1$. Since h is an epimorphism, $h(z) = z^1$ for some $z \in M$.

$$\begin{aligned} \text{Now } x+\mu = y+\mu &\Rightarrow (x+\mu)(z) = (y+\mu)(z) \Rightarrow \mu(x-z) = \mu(y-z) \Rightarrow (h^{-1}(\sigma))(x-z) = \\ &(h^{-1}(\sigma))(y-z) \Rightarrow \sigma(h(x-z)) = \sigma(h(y-z)) \Rightarrow \sigma(h(x)-z^1) = \sigma(h(y)-z^1) \\ &\Rightarrow (h(x)+\sigma)(z^1) = (h(y)+\sigma)(z^1) \end{aligned}$$

This is true for all $z^1 \in M^1$. Hence $h(x) + \sigma = h(y) + \sigma$.

Now we proved that $x+\mu = y+\mu \Rightarrow \psi(x+\mu) = \psi(y+\mu)$. Thus ψ is well defined.

It is easy to verify that $\psi((x+\mu)+(y+\mu)) = \psi(x+\mu) + \psi(y+\mu)$.

Now $\psi((x+\mu)\alpha(y+\mu)) = \psi(x\alpha y+\mu) = h(x\alpha y)+\sigma$, by definition of ψ . Since h is a homomorphism, we have $h(x\alpha y)+\sigma = (h(x)\alpha h(y))+\sigma = (h(x)+\sigma)\alpha(h(y)+\sigma) = \psi(x+\mu)\alpha\psi(y+\mu)$, by definition of ψ . Therefore ψ is a homomorphism.

Now we verify that ψ is one-one. Suppose $h(x)+\sigma = h(y)+\sigma$. Then $\sigma[h(x)-h(y)] = \sigma[h(0)]$, by definition. Since h is a homomorphism, $\sigma[h(x-y)] = \sigma(h(0))$. This implies $(h^{-1}(\sigma))(x-y) = (h^{-1}(\sigma))(0)$, which implies $\mu(x-y) = \mu(0)$. By Lemma 2.2 (i), $x+\mu = y+\mu$. This shows that ψ is one-one. Let $y \in M^1/\sigma$. Then $y = h(x)+\sigma$ for some $x \in M$. Now $\psi(x + \mu) = h(x) + \sigma = y$. Therefore ψ is onto. Hence ψ is an isomorphism. \square

As a consequence of Theorem 3.11, we obtain the following corollary.

Corollary 3.12 *Let μ and σ be two fuzzy ideals of M such that $\mu \subseteq \sigma$ and $\sigma(0) = \mu(0)$. Then $M/\sigma \cong (M/\mu)/(\sigma/\mu)$.*

Proof. Define $\psi: M \rightarrow M/\mu$ by $\psi(x) = x + \mu$ for all $x \in M$.

By theorem 3.3, ψ is Γ -nearing epimorphism. From the notation 3.4 and 3.6 we have $\theta_\sigma = \sigma/\mu$ and by Lemma 3.5, σ/μ is a fuzzy ideal of M/μ such that $\theta_\mu \subseteq \theta_\sigma = \sigma/\mu$ and $\theta_\mu(0) = \theta_\sigma(0)$. Now $\psi^{-1}(\sigma/\mu)$ is a fuzzy set in M and for any $x \in M$ we have

$(\psi^{-1}(\sigma/\mu))(x) = \psi^{-1}(\theta_\sigma)(x) = \theta_\sigma(\psi(x)) = \theta_\sigma(x+\mu) = \sigma(x)$. Therefore $\psi^{-1}(\sigma/\mu) = \sigma$ is a fuzzy ideal of M . Define $\psi^*: M/\sigma \rightarrow (M/\mu)/(\sigma/\mu)$ by $\psi^*(x+\sigma) = \psi(x) + (\sigma/\mu)$.

By theorem 3.11, ψ^* is a Γ -nearring isomorphism. This completes the proof. \square

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