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Turk J Math
29 (2005) , 129 – 140.

Common Fixed Point Theorems for Fuzzy Mappings in Quasi-Pseudo-Metric Spaces

(Dedicated to the Memory of the Late Professor Dr. Y. A. Verdiyev)

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Abstract

In this paper, we obtain some common fixed point theorems for pairs of fuzzy mappings in left K-sequentially complete quasi-pseudo-metric spaces and right K-sequentially complete quasi-pseudo-metric spaces, respectively. Well-known theorems are special cases of our results.

Key words and phrases: Fuzzy mapping; Fixed point; Quasi-pseudo-metric; Left K-sequentially complete; Right K-sequentially complete.

1. Introduction

Heilpern [5] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of Nadler's [6] fixed point theorem for multivalued mappings. Bose and Shani [2], in their first theorem, extended the result of Heilpern to a pair of generalized fuzzy contraction mappings. Park and Jeong [7] proved some common fixed point theorems for fuzzy mappings satisfying contractive-type conditions and a rational inequality in complete metric spaces, which are the fuzzy extensions of some theorems in [1, 8]. Recently, Gregori and Pastor [3] proved a fixed point theorem for fuzzy contraction mappings in left K-sequentially complete

2000 *AMS Mathematics Subject Classification:* 54A40, 54H25
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quasi-pseudo-metric spaces. Their result is a generalization of the result of Heilpern. In [11] the authors extended the results of [3] and [5]. On the other hand, Gregori and Romaguera [4] obtained some interesting fixed point theorems for fuzzy mappings in Smyth-complete and left K-sequentially complete quasi-metric spaces, respectively. Some well known theorems are special cases of their results. In [10] the authors considered a generalized contractive type condition involving fuzzy mappings in left K-sequentially complete quasi-metric spaces and established a fixed point theorem which is an extension of Theorem 2 in [4]. Also, the result of [10] is a quasi-metric version of Theorem 1 in [4].

In this paper, we establish some generalized common fixed point theorems involving pair of fuzzy mappings in left K-sequentially complete quasi-pseudo-metric spaces and right K-sequentially complete quasi-pseudo-metric spaces, respectively, which are generalization of some results in [3, 5, 11]. Also some well known theorems as in [3, 5, 7] are special cases of our results.

2. Preliminaries

Throughout this paper the letter $\mathbf{N}$ denotes the set of positive integers. If A is a subset of a topological space (X, τ), we will denote by $\mathrm{cl}_\tau A$ the closure of A in (X, τ).

A *quasi-pseudo-metric* on a nonempty set X is a nonnegative real valued function d on $X \times X$ such that, for all $x, y, z \in X$:

(i) $d(x,x) = 0$, and (ii) $d(x,y) \leq d(x,z) + d(z,y)$.

A pair (X, d) is called a *quasi-pseudo-metric space*, if d is a quasi-pseudo-metric on X.

Each quasi-pseudo-metric d on X induces a topology $\tau(d)$ which has as a base the family of all d-balls $B_\varepsilon(x)$, where $B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$.

If d is a quasi-pseudo-metric on X, then the function d^{-1}, defined on $X \times X$ by $d^{-1}(x,y) = d(y,x)$ is also a quasi-pseudo-metric on X. By $d \wedge d^{-1}$ and $d \vee d^{-1}$ we denote $\min\{d, d^{-1}\}$ and $\max\{d, d^{-1}\}$, respectively.

Let d be a quasi-pseudo-metric on X. A sequence $(x_n)_{n \in \mathbf{N}}$ in X is said to be

(i) *left K-Cauchy* [9], if for each $\varepsilon > 0$ there is a $k \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \mathbf{N}$ with $m \geq n \geq k$.

(ii) *right K-Cauchy* [9], if for each $\varepsilon > 0$ there is a $k \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \mathbf{N}$ with $n \geq m \geq k$.

A quasi-pseudo-metric space (X, d) is said to be *left* (*right*) *K-sequentially complete* [9], if each left (right) K-Cauchy sequence in (X, d) converges to some point in X (with respect to the topology $\tau(d)$).

Now let (X, d) be a quasi-pseudo-metric space and let A and B be nonempty subsets of X. Then the *Hausdorff distance* between subsets A and B is defined by

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \ \sup_{b \in B} d(b, A)\} \qquad \text{(see[3])},$$

where $d(a, B) = \inf\{d(a, x) : x \in B\}$.

Note that $H(A, B) \geq 0$ with $H(A, B) = 0$ iff $clA = clB$, $H(A, B) = H(B, A)$ and $H(A, B) \leq H(A, C) + H(C, B)$ for any nonempty subset A, B and C of X. When d is a metric on X, clearly H is the usual Hausdorff distance.

A *fuzzy set* on X is an element of I^X where $I = [0, 1]$. The *α-level set* of a fuzzy set A, denoted by A_α, is defined by
$A_\alpha = \{x \in X : A(x) \geq \alpha\}$ for each $\alpha \in (0, 1]$, and $A_0 = cl(\{x \in X : A(x) > 0\})$.

For $x \in X$ we denote by $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of X.

Definition 2.1. Let (X, d) be a quasi-pseudo-metric space. The families $W^*(X)$ and $W'(X)$ of fuzzy sets on (X, d) are defined by

$W^*(X) = \{A \in I^X : A_1$ is nonempty $d-$ closed and d^{-1}-compact$\}$ (see[3]),
$W'(X) = \{A \in I^X : A_1$ is nonempty $d-$ closed and d-compact$\}$.

In [5] it is defined the family $W(X)$ of fuzzy sets on metric linear space (X, d), as follows: $A \in W(X)$ iff A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $\sup_{x \in X} A(X) = 1$.

If (X, d) is a metric linear space, then we have

$$W(X) \subset W^*(X) = W'(X) = \{A \in I^X : A_1 \text{ is nonempty and } d\text{-compact }\} \subset I^X.$$

Definition 2.2. Let (X, d) be a quasi-pseudo-metric space and let $A, B \in W^*(X)$ or $A, B \in W'(X)$ and $\alpha \in [0, 1]$. Then we define,

$$p_\alpha(A, B) = \inf\{d(x, y) : x \in A_\alpha, y \in B_\alpha\} = d(A_\alpha, B_\alpha),$$

$$D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$

where H is the Hausdorff distance deduced from the quasi-pseudo-metric d on X,

$$p(A,B) = \sup\{p_\alpha(A,B) : \alpha \in [0,1]\},$$

$$D(A,B) = \sup\{D_\alpha(A,B) : \alpha \in [0,1]\}.$$

It is easy to see that p_α is non-decreasing function of α, and $p_1(A,B) = d(A_1,B_1) = p(A,B)$ where $d(A_1,B_1) = \inf\{d(x,y) : x \in A_1, y \in B_1\}$.

Definition 2.3. [3] Let X be an arbitrary set and Y be any quasi-pseudo-metric space. F is said to be a *fuzzy mapping* if F is a mapping from the set X into $W^*(Y)$ or $W'(Y)$.

This definition is more general than the one given in [5].

Definition 2.4. We say that x is a *fixed point* of the mapping $F : X \longrightarrow I^X$, if $\{x\} \subset F(x)$.

Note that, If $A, B \in I^X$, then $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.

3. Lemmas

Before establishing our main results, we need the lemmas presented in the next section.

The following four lemmas were proved by Gregory and Pastor [3].

Lemma 3.1. *Let (X,d) be a quasi-pseudo-metric space and let $x \in X$ and $A \in W^*(X)$. Then $\{x\} \subset A$ if and only if $p_1(x,A) = 0$.*

Lemma 3.2. *Let (X,d) be a quasi-pseudo-metric space and let $A \in W^*(X)$. Then $p_\alpha(x,A) \leq d(x,y) + p_\alpha(y,A)$ for any $x, y \in X$ and $\alpha \in [0,1]$.*

Lemma 3.3. *Let (X,d) be a quasi-pseudo-metric space and let $\{x_0\} \subset A$. Then $p_\alpha(x_0,B) \leq D_\alpha(A,B)$ for each $A, B \in W^*(X)$ and $\alpha \in [0,1]$.*

Lemma 3.4. *Suppose $K \neq \emptyset$ is compact in the quasi-pseudo- metric space (X,d^{-1}). If $z \in X$, then there exists $k_0 \in K$ such that $d(z,K) = d(z,k_0)$.*

Above Lemma 3.1, Lemma 3.2 and Lemma 3.3 were proved by Heilpern [5] for the family $W(X)$ in a metric space.

We will use also the following lemmas.

Lemma 3.5. *Let (X,d) be a quasi-pseudo-metric space and let $x \in X$ and $A \in W'(X)$. Then $\{x\} \subset A$ if and only if $p_1(A,x) = 0$.*

Lemma 3.6. *Let (X, d) be a quasi-pseudo-metric space and let $A \in W'(X)$. Then $p_\alpha(A, x) \leq p_\alpha(A, y) + d(y, x)$ for any $x, y \in X$ and $\alpha \in [0, 1]$.*

Lemma 3.7. *Let (X, d) be a quasi-pseudo-metric space and let $\{x_0\} \subset A$. Then $p_\alpha(B, x_0) \leq D_\alpha(B, A)$ for each $A, B \in W'(X)$ and $\alpha \in [0, 1]$.*

The proofs of these lemmas are similar to the proofs of lemmas in [5] and omitted.

Lemma 3.8. *Suppose $K \neq \emptyset$ is compact in the quasi-pseudo- metric space (X, d). If $z \in X$, then there exists $k_0 \in K$ such that $d(K, z) = d(k_0, z)$.*

Proof. By a method similar to that in the proof of Lemma 2.9 in [3], the result follows.

4. Common fixed point theorems

We now prove the following theorem.

Theorem 4.1. *Let (X, d) be a left K-sequentially complete quasi-pseudo-metric space and let F_1 and F_2 be fuzzy mappings from X to $W^*(X)$ satisfying the inequality*

$$\begin{aligned}[1 + r(d \vee d^{-1})(x, y)]D(F_1(x), F_2(y)) &\leq \\ \leq r \max\{p(x, F_1(x))p(y, F_2(y)), p(x, F_2(y))p(y, F_1(x))\} &+ \\ +h \max\{(d \wedge d^{-1})(x, y), p(x, F_1(x)), p(y, F_2(y)), &\\ \frac{1}{2}[p(x, F_2(y)) + p(y, F_1(x))]\}& \qquad (1)\end{aligned}$$

for each $x, y \in X$, where $r \geq 0$ and $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

Proof. Suppose x_0 is an arbitrary point in X such that $\{x_1\} \subset F_1(x_0)$. Since $(F_2(x_1))_1$ is d^{-1}-compact, it follows from Lemma 3.4, there exists $x_2 \in (F_2(x_1))_1$ such that $d(x_1, x_2) = d(x_1, (F_2(x_1))_1)$. Thus we have

$$d(x_1, x_2) = d(x_1, (F_2(x_1))_1) \leq H(x_1, (F_2(x_1))_1) \leq D(F_1(x_0), F_2(x_1)). \qquad (2)$$

Similarly, we can find $x_3 \in X$ such that

$$\{x_3\} \subset F_1(x_2) \text{ and } d(x_2, x_3) \leq D(F_2(x_1), F_1(x_2)).$$

Continuing in this way, we can obtain a sequence $(x_n)_{n\in\mathbf{N}}$ in X such that

$$\{x_{2n+1}\} \subset F_1(x_{2n}), \{x_{2n+2}\} \subset F_2(x_{2n+1}),$$

$$d(x_{2n+1}, x_{2n+2}) \leq D(F_1(x_{2n}), F_2(x_{2n+1}))$$

and

$$d(x_{2n+2}, x_{2n+3}) \leq D(F_2(x_{2n+1}), F_1(x_{2n+2}))$$

for $n = 0, 1, 2, ...$

Now using inequalities (1) and (2) we have,

$$\begin{aligned}[1 + rd(x_0, x_1)]d(x_1, x_2) &\leq [1 + r(d \vee d^{-1})(x_0, x_1)]D(F_1(x_0), F_2(x_1)) \leq \\ &\leq r \max\{p(x_0, F_1(x_0))p(x_1, F_2(x_1)), p(x_0, F_2(x_1))p(x_1, F_1(x_0))\} + \\ &+h \max\{(d \wedge d^{-1})(x_0, x_1), p(x_0, F_1(x_0)), p(x_1, F_2(x_1)), \\ &\frac{1}{2}[p(x_0, F_2(x_1)) + p(x_1, F_1(x_0))]\}.\end{aligned}$$

Since $x_1 \in (F_1(x_0))_1$ and $x_2 \in (F_2(x_1))_1$, we have $p(x_0, F_1(x_0)) \leq d(x_0, x_1)$, $p(x_1, F_2(x_1)) \leq d(x_1, x_2), p(x_0, F_2(x_1)) \leq d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$ and $p(x_1, F_1(x_0)) = 0$.

Thus we have,

$$\begin{aligned}[1 + rd(x_0, x_1)]d(x_1, x_2) &\leq rd(x_0, x_1)d(x_1, x_2)+ \\ &+h \max\{d(x_0, x_1), d(x_1, x_2), \frac{1}{2}[d(x_0, x_1) + d(x_1, x_2)]\}.\end{aligned}$$

and it follows that

$$d(x_1, x_2) \leq h \max\{d(x_0, x_1), d(x_1, x_2), \frac{1}{2}[d(x_0, x_1) + d(x_1, x_2)]\} = hd(x_0, x_1)$$

since $h < 1$. Thus

$$d(x_1, x_2) \leq hd(x_0, x_1).$$

Similarly,

$$d(x_2, x_3) \leq hd(x_1, x_2) \leq h^2 d(x_0, x_1)$$

and, in general,

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \qquad \text{for all } n \in \mathbf{N}.$$

For $n < m$, we have

$$d(x_n, x_m) \leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{m-1} h^i d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1).$$

Since $0 < h < 1$, it follows that $(x_n)_{n\in\mathbf{N}}$ is a left K-Cauchy sequence in the left K-sequentially complete quasi-pseudo-metric space (X, d) and so there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$.

Now, by Lemma 3.2, we have $p_1(x^*, F_2(x^*)) \leq d(x^*, x_{2n+1}) + p_1(x_{2n+1}, F_2(x^*))$ for all $n \in \mathbf{N}$. So, by Lemmas 3.3 and inequality (1),

$$\begin{aligned}
&p_1(x^*, F_2(x^*)) \leq d(x^*, x_{2n+1}) + D_1(F_1(x_{2n}), F_2(x^*)) \leq \\
&\leq d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*)) \leq \\
&\leq d(x^*, x_{2n+1}) + \frac{r \max\{p(x_{2n}, F_1(x_{2n}))p(x^*, F_2(x^*)),}{1 + r(d \vee d^{-1})(x_{2n}, x^*))} \\
&\qquad \frac{p(x_{2n}, F_2(x^*))p(x^*, F_1(x_{2n}))\} + h \max\{(d \wedge d^{-1})(x_{2n}, x^*),}{1 + r(d \vee d^{-1})(x_{2n}, x^*))} \\
&\qquad \frac{p(x_{2n}, F_1(x_{2n})), p(x^*, F_2(x^*)), \frac{1}{2}[p(x_{2n}, F_2(x^*)) + p(x^*, F_1(x_{2n}))]\}}{1 + r(d \vee d^{-1})(x_{2n}, x^*))}.
\end{aligned}$$

Since

$$(d \vee d^{-1})(x_{2n}, x^*) \geq d^{-1}(x_{2n}, x^*) = d(x^*, x_{2n})$$

and

$$(d \wedge d^{-1})(x_{2n}, x^*) \leq d^{-1}(x_{2n}, x^*) = d(x^*, x_{2n}),$$

we have

$$\begin{aligned}
&p_1(x^*, F_2(x^*)) \leq d(x^*, x_{2n+1}) + \\
&\quad + \frac{r \max\{p(x_{2n}, F_1(x_{2n}))p(x^*, F_2(x^*)), p(x_{2n}, F_2(x^*))p(x^*, F_1(x_{2n}))\}}{1 + rd(x^*, x_{2n})} + \\
&\qquad + \frac{h \max\{d(x^*, x_{2n}), p(x_{2n}, F_1(x_{2n})), p(x^*, F_2(x^*)),}{1 + rd(x^*, x_{2n})} \\
&\qquad\qquad \frac{\frac{1}{2}[p(x_{2n}, F_2(x^*)) + p(x^*, F_1(x_{2n}))]\}}{1 + rd(x^*, x_{2n})},
\end{aligned}$$

and by lemmas 3.2 and 3.3,

$$\begin{aligned}p_1(x^*, F_2(x^*)) &\le d(x^*, x_{2n+1}) + \\ &+ \frac{r\max\{d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*)),}{1 + rd(x^*, x_{2n})} \\ &\frac{[d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})\}}{1 + rd(x^*, x_{2n})} + \\ &+ \frac{h\max\{d(x^*, x_{2n}), d(x_{2n}, x_{2n+1}), d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*)),}{1 + rd(x^*, x_{2n})} \\ &\qquad\qquad \frac{\frac{1}{2}[d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*)) + d(x^*, x_{2n+1})]\}}{1 + rd(x^*, x_{2n})}.\end{aligned}$$

it follows that

$$\begin{aligned}p_1(x^*, F_2(x^*)) &\le d(x^*, x_{2n+1}) + \\ &+ \frac{r\max\{d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*)),}{1 + rd(x^*, x_{2n})} \\ &\frac{[d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})\}}{1 + rd(x^*, x_{2n})} + \\ &+ \frac{h\max\{d(x^*, x_{2n}), d(x_{2n}, x_{2n+1}), d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))\}}{1 + rd(x^*, x_{2n})},\end{aligned} \tag{3}$$

since $\frac{1}{2}[d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*)) + d(x^*, x_{2n+1})]$ is less then or equal to $d(x_{2n}, x_{2n+1})$ or $d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))$.

Now let

$$\begin{aligned}m_n = \max\{&d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*)), \\ &[d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})\}\end{aligned}$$

and

$$M_n = \max\{d(x^*, x_{2n}), d(x_{2n}, x_{2n+1}), d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))\}.$$

Then from inequality (3) we have

$$p_1(x^*, F_2(x^*)) \le d(x^*, x_{2n+1}) + \frac{rm_n + hM_n}{1 + rd(x^*, x_{2n})}. \tag{4}$$

Now we have to consider, for each $n \in \mathbf{N}$, the following four cases:

Case 1. If $m_n = d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*))$ and M_n is equal to either $d(x^*, x_{2n})$ or $d(x_{2n}, x_{2n+1})$, then since $d(x^*, x_{2n})$ and $d(x_{2n}, x_{2n+1})$ converge to 0 as $n \to \infty$, we obtain that $m_n \to 0$ and $M_n \to 0$. Also, $d(x^*, x_{2n+1})$ converge to 0. Hence, from (4), we obtain $p_1(x^*, F_2(x^*)) = 0$.

Case 2. If $m_n = d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*))$ and
$M_n = d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))$, then by inequality (1), we have

$$M_n \leq d(x^*, x_{2n+1}) + \frac{rm_n + hM_n}{1 + rd(x^*, x_{2n})}$$

and it follows that

$$M_n[\frac{1 + rd(x^*, x_{2n}) - h}{1 + rd(x^*, x_{2n})}] \leq d(x^*, x_{2n+1}) + \frac{rm_n}{1 + rd(x^*, x_{2n})}.$$

Since $d(x^*, x_{2n})$, $d(x^*, x_{2n+1})$ and m_n converge to 0 as $n \to \infty$, we obtain that $M_n \to 0$. Thus from (4), we have $p_1(x^*, F_2(x^*)) = 0$.

Case 3. If $m_n = [d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})$ and M_n is equal to either $d(x^*, x_{2n})$ or $d(x_{2n}, x_{2n+1})$, then by inequality (1), we have

$$m_n \leq [d(x_{2n}, x_{2n+1}) + \frac{rm_n + hM_n}{1 + rd(x^*, x_{2n})}]d(x^*, x_{2n+1})$$

and it follows that

$$m_n[\frac{1 + rd(x^*, x_{2n}) - rd(x^*, x_{2n+1})}{1 + rd(x^*, x_{2n})}] \leq [d(x_{2n}, x_{2n+1}) + \frac{hM_n}{1 + rd(x^*, x_{2n})}]d(x^*, x_{2n+1}).$$

Since $d(x^*, x_{2n})$, $d(x^*, x_{2n+1})$, $d(x_{2n}, x_{2n+1})$ and M_n converge to 0 as $n \to \infty$, we obtain that $m_n \to 0$. Thus from (4), we have $p_1(x^*, F_2(x^*)) = 0$.

Case 4. If $m_n = [d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})$ and
$M_n = d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))$, then by inequality (1), we have

$$D(F_1(x_{2n}), F_2(x^*)) \leq \frac{rm_n + hM_n}{1 + rd(x^*, x_{2n})} =$$
$$= \frac{r[d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})}{1 + rd(x^*, x_{2n})} +$$
$$+ \frac{h[d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]}{1 + rd(x^*, x_{2n})}$$

and it follows that

$$D(F_1(x_{2n}), F_2(x^*))[\frac{1 + rd(x^*, x_{2n}) - rd(x^*, x_{2n+1}) - h}{1 + rd(x^*, x_{2n})}] \leq$$
$$\leq \frac{[rd(x_{2n}, x_{2n+1}) + h]d(x^*, x_{2n+1})}{1 + rd(x^*, x_{2n})}.$$

Since $d(x^*, x_{2n})$, $d(x^*, x_{2n+1})$ and $d(x_{2n}, x_{2n+1})$ converge to 0 as $n \to \infty$ and $0 < 1 - h < 1$, we obtain that $D(F_1(x_{2n}), F_2(x^*)) \to 0$. Hence m_n and M_n converge to 0 as $n \to \infty$. Thus from (4), we have $p_1(x^*, F_2(x^*)) = 0$.

It now follows from cases 1 – 4 and Lemma 3.1 that $\{x^*\} \subset F_2(x^*)$.

Similarly, it can be shown that $\{x^*\} \subset F_1(x^*)$.

When (X, d) is a right K-sequentially complete quasi-pseudo-metric space, using Lemmas 3.5, 3.6, 3.7 and 3.8 we get the following result.

Theorem 4.2. *Let (X, d) be a right K-sequentially complete quasi-pseudo-metric space and let F_1 and F_2 be fuzzy mappings from X to $W'(X)$ satisfying the inequality*

$$\begin{aligned}&[1 + r(d \vee d^{-1})(x, y)]D(F_1(x), F_2(y)) \leq \\ &\leq r \max\{p(F_1(x), x)p(F_2(y), y), p(F_2(y), x)p(F_1(x), y)\} + \\ &+h \max\{(d \wedge d^{-1})(x, y), p(F_1(x), x), p(F_2(y), y), \\ &\frac{1}{2}[p(F_2(y), x) + p(F_1(x), y)]\}\end{aligned} \qquad (5)$$

for each $x, y \in X$, where $r \geq 0$ and $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

The proof of this theorem is similar to the proof of Theorem 4.1 and is omitted.

On noting that

$$[p(x, F_1(x))p(y, F_2(y))]^{1/2} \leq \frac{1}{2}[p(x, F_1(x)) + p(y, F_2(y))] \leq$$
$$\leq \max\{(d \wedge d^{-1})(x, y), p(x, F_1(x)), p(y, F_2(y)), \frac{1}{2}[p(x, F_2(y)) + p(y, F_1(x))]\},$$

we have the following corollary from Theorem 4.1 with $r = 0$.

Corollary 4.1. *Let (X, d) be a left K-sequentially complete quasi-pseudo-metric space and let F_1 and F_2 be fuzzy mappings from X to $W^*(X)$ satisfying the inequality*

$$D(F_1(x), F_2(y)) \leq h[p(x, F_1(x))p(y, F_2(y))]^{1/2},$$

for each $x, y \in X$, where $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

Similarly, we have the following corollary from Theorem 4.2.

Corollary 4.2. *Let (X, d) be a right K-sequentially complete quasi-pseudo-metric space and let F_1 and F_2 be fuzzy mappings from X to $W'(X)$ satisfying the inequality*

$$D(F_1(x), F_2(y)) \leq [p(F_1(x), x)p(F_2(y), y)]^{1/2},$$

for each $x, y \in X$, where $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

Both Corollary 4.1 and Corollary 4.2 are extensions of Theorem 3.2 of [7] in quasi-pseudo-metric space.

When (X, d) is a complete metric space, we get the following corollary.

Corollary 4.3. *Let (X, d) be a complete metric space and let F_1 and F_2 be fuzzy mappings from X to $W'(X)$ satisfying the inequality*

$$\begin{aligned}
&[1 + rd(x, y)]D(F_1(x), F_2(y)) \leq \\
&\quad \leq r \max\{p(x, F_1(x))p(y, F_2(y)), p(x, F_2(y))p(y, F_1(x))\} + \\
&\qquad\qquad + h \max\{d(x, y), p(x, F_1(x)), p(y, F_2(y)), \\
&\qquad\qquad\qquad \frac{1}{2}[p(x, F_2(y)) + p(y, F_1(x))]\}
\end{aligned} \tag{6}$$

for each $x, y \in X$, where $r \geq 0$ and $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

Remark 1. Letting $F_1 = F_2$ with $r = 0$ in inequality (1), then Theorem 3.2 of [11] is a consequence of Theorem 4.1. Similarly, notice that Theorem 3.1 of [3] can be obtained from Theorem 4.1.

Remark 2. If we put $r = 0$ in inequality (6), we can see that Theorem 3.1 in [7] is a special case of our Corollary 4.3. Also Theorem 3.2 of [7] can be obtained from Corollary 4.3.

Remark 3. Similarly, if we put $r = 0$ in inequality (6), we can obtain Theorem 3.1 of [5] from Corollary 4.3.

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Received 04.11.2003