

1-1-2005

Common Fixed Point Theorems for Fuzzy Mappings in Quasi-Pseudo-Metric Spaces

İLKER ŞAHİN

HAKAN KARAYILAN

MUSTAFA TELCİ

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>



Part of the [Mathematics Commons](#)

Recommended Citation

ŞAHİN, İLKER; KARAYILAN, HAKAN; and TELCİ, MUSTAFA (2005) "Common Fixed Point Theorems for Fuzzy Mappings in Quasi-Pseudo-Metric Spaces," *Turkish Journal of Mathematics*: Vol. 29: No. 2, Article 3. Available at: <https://journals.tubitak.gov.tr/math/vol29/iss2/3>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

Common Fixed Point Theorems for Fuzzy Mappings in Quasi-Pseudo-Metric Spaces

(Dedicated to the Memory of the Late Professor Dr. Y. A. Verdiyev)

*İlker Şahin, Hakan Karayılan and Mustafa Telci**

Abstract

In this paper, we obtain some common fixed point theorems for pairs of fuzzy mappings in left K -sequentially complete quasi-pseudo-metric spaces and right K -sequentially complete quasi-pseudo-metric spaces, respectively. Well-known theorems are special cases of our results.

Key words and phrases: Fuzzy mapping; Fixed point; Quasi-pseudo-metric; Left K -sequentially complete; Right K -sequentially complete.

1. Introduction

Heilpern [5] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of Nadler's [6] fixed point theorem for multivalued mappings. Bose and Shani [2], in their first theorem, extended the result of Heilpern to a pair of generalized fuzzy contraction mappings. Park and Jeong [7] proved some common fixed point theorems for fuzzy mappings satisfying contractive-type conditions and a rational inequality in complete metric spaces, which are the fuzzy extensions of some theorems in [1, 8]. Recently, Gregori and Pastor [3] proved a fixed point theorem for fuzzy contraction mappings in left K -sequentially complete

2000 *AMS Mathematics Subject Classification:* 54A40, 54H25

*Corresponding author

quasi-pseudo-metric spaces. Their result is a generalization of the result of Heilpern. In [11] the authors extended the results of [3] and [5]. On the other hand, Gregori and Romaguera [4] obtained some interesting fixed point theorems for fuzzy mappings in Smyth-complete and left K -sequentially complete quasi-metric spaces, respectively. Some well known theorems are special cases of their results. In [10] the authors considered a generalized contractive type condition involving fuzzy mappings in left K -sequentially complete quasi-metric spaces and established a fixed point theorem which is an extension of Theorem 2 in [4]. Also, the result of [10] is a quasi-metric version of Theorem 1 in [4].

In this paper, we establish some generalized common fixed point theorems involving pair of fuzzy mappings in left K -sequentially complete quasi-pseudo-metric spaces and right K -sequentially complete quasi-pseudo-metric spaces, respectively, which are generalization of some results in [3, 5, 11]. Also some well known theorems as in [3, 5, 7] are special cases of our results.

2. Preliminaries

Throughout this paper the letter \mathbf{N} denotes the set of positive integers. If A is a subset of a topological space (X, τ) , we will denote by $\text{cl}_\tau A$ the closure of A in (X, τ) .

A *quasi-pseudo-metric* on a nonempty set X is a nonnegative real valued function d on $X \times X$ such that, for all $x, y, z \in X$:

- (i) $d(x, x) = 0$, and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A pair (X, d) is called a *quasi-pseudo-metric space*, if d is a quasi-pseudo-metric on X .

Each quasi-pseudo-metric d on X induces a topology $\tau(d)$ which has as a base the family of all d -balls $B_\varepsilon(x)$, where $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$.

If d is a quasi-pseudo-metric on X , then the function d^{-1} , defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ is also a quasi-pseudo-metric on X . By $d \wedge d^{-1}$ and $d \vee d^{-1}$ we denote $\min\{d, d^{-1}\}$ and $\max\{d, d^{-1}\}$, respectively.

Let d be a quasi-pseudo-metric on X . A sequence $(x_n)_{n \in \mathbf{N}}$ in X is said to be

(i) *left K -Cauchy* [9], if for each $\varepsilon > 0$ there is a $k \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \mathbf{N}$ with $m \geq n \geq k$.

(ii) *right K -Cauchy* [9], if for each $\varepsilon > 0$ there is a $k \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \in \mathbf{N}$ with $n \geq m \geq k$.

A quasi-pseudo-metric space (X, d) is said to be *left (right) K -sequentially complete* [9], if each left (right) K -Cauchy sequence in (X, d) converges to some point in X (with respect to the topology $\tau(d)$).

Now let (X, d) be a quasi-pseudo-metric space and let A and B be nonempty subsets of X . Then the *Hausdorff distance* between subsets A and B is defined by

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} \quad (\text{see}[3]),$$

where $d(a, B) = \inf\{d(a, x) : x \in B\}$.

Note that $H(A, B) \geq 0$ with $H(A, B) = 0$ iff $clA = clB$, $H(A, B) = H(B, A)$ and $H(A, B) \leq H(A, C) + H(C, B)$ for any nonempty subset A, B and C of X . When d is a metric on X , clearly H is the usual Hausdorff distance.

A *fuzzy set* on X is an element of I^X where $I = [0, 1]$. The α -*level set* of a fuzzy set A , denoted by A_α , is defined by

$$A_\alpha = \{x \in X : A(x) \geq \alpha\} \text{ for each } \alpha \in (0, 1], \text{ and } A_0 = cl(\{x \in X : A(x) > 0\}).$$

For $x \in X$ we denote by $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of X .

Definition 2.1. Let (X, d) be a quasi-pseudo-metric space. The families $W^*(X)$ and $W'(X)$ of fuzzy sets on (X, d) are defined by

$$\begin{aligned} W^*(X) &= \{A \in I^X : A_1 \text{ is nonempty } d\text{-closed and } d^{-1}\text{-compact}\} \text{ (see}[3]), \\ W'(X) &= \{A \in I^X : A_1 \text{ is nonempty } d\text{-closed and } d\text{-compact}\}. \end{aligned}$$

In [5] it is defined the family $W(X)$ of fuzzy sets on metric linear space (X, d) , as follows: $A \in W(X)$ iff A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $\sup_{x \in X} A(x) = 1$.

If (X, d) is a metric linear space, then we have

$$W(X) \subset W^*(X) = W'(X) = \{A \in I^X : A_1 \text{ is nonempty and } d\text{-compact}\} \subset I^X.$$

Definition 2.2. Let (X, d) be a quasi-pseudo-metric space and let $A, B \in W^*(X)$ or $A, B \in W'(X)$ and $\alpha \in [0, 1]$. Then we define,

$$\begin{aligned} p_\alpha(A, B) &= \inf\{d(x, y) : x \in A_\alpha, y \in B_\alpha\} = d(A_\alpha, B_\alpha), \\ D_\alpha(A, B) &= H(A_\alpha, B_\alpha), \end{aligned}$$

where H is the Hausdorff distance deduced from the quasi-pseudo-metric d on X ,

$$p(A, B) = \sup\{p_\alpha(A, B) : \alpha \in [0, 1]\},$$

$$D(A, B) = \sup\{D_\alpha(A, B) : \alpha \in [0, 1]\}.$$

It is easy to see that p_α is non-decreasing function of α , and $p_1(A, B) = d(A_1, B_1) = p(A, B)$ where $d(A_1, B_1) = \inf\{d(x, y) : x \in A_1, y \in B_1\}$.

Definition 2.3. [3] Let X be an arbitrary set and Y be any quasi-pseudo-metric space. F is said to be a *fuzzy mapping* if F is a mapping from the set X into $W^*(Y)$ or $W'(Y)$.

This definition is more general than the one given in [5].

Definition 2.4. We say that x is a *fixed point* of the mapping $F : X \longrightarrow I^X$, if $\{x\} \subset F(x)$.

Note that, If $A, B \in I^X$, then $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$.

3. Lemmas

Before establishing our main results, we need the lemmas presented in the next section.

The following four lemmas were proved by Gregory and Pastor [3].

Lemma 3.1. *Let (X, d) be a quasi-pseudo-metric space and let $x \in X$ and $A \in W^*(X)$. Then $\{x\} \subset A$ if and only if $p_1(x, A) = 0$.*

Lemma 3.2. *Let (X, d) be a quasi-pseudo-metric space and let $A \in W^*(X)$. Then $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for any $x, y \in X$ and $\alpha \in [0, 1]$.*

Lemma 3.3. *Let (X, d) be a quasi-pseudo-metric space and let $\{x_0\} \subset A$. Then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $A, B \in W^*(X)$ and $\alpha \in [0, 1]$.*

Lemma 3.4. *Suppose $K \neq \emptyset$ is compact in the quasi-pseudo-metric space (X, d^{-1}) . If $z \in X$, then there exists $k_0 \in K$ such that $d(z, K) = d(z, k_0)$.*

Above Lemma 3.1, Lemma 3.2 and Lemma 3.3 were proved by Heilpern [5] for the family $W(X)$ in a metric space.

We will use also the following lemmas.

Lemma 3.5. *Let (X, d) be a quasi-pseudo-metric space and let $x \in X$ and $A \in W'(X)$. Then $\{x\} \subset A$ if and only if $p_1(A, x) = 0$.*

Lemma 3.6. *Let (X, d) be a quasi-pseudo-metric space and let $A \in W'(X)$. Then $p_\alpha(A, x) \leq p_\alpha(A, y) + d(y, x)$ for any $x, y \in X$ and $\alpha \in [0, 1]$.*

Lemma 3.7. *Let (X, d) be a quasi-pseudo-metric space and let $\{x_0\} \subset A$. Then $p_\alpha(B, x_0) \leq D_\alpha(B, A)$ for each $A, B \in W'(X)$ and $\alpha \in [0, 1]$.*

The proofs of these lemmas are similar to the proofs of lemmas in [5] and omitted.

Lemma 3.8. *Suppose $K \neq \emptyset$ is compact in the quasi-pseudo-metric space (X, d) . If $z \in X$, then there exists $k_0 \in K$ such that $d(K, z) = d(k_0, z)$.*

Proof. By a method similar to that in the proof of Lemma 2.9 in [3], the result follows.

4. Common fixed point theorems

We now prove the following theorem.

Theorem 4.1. *Let (X, d) be a left K -sequentially complete quasi-pseudo-metric space and let F_1 and F_2 be fuzzy mappings from X to $W^*(X)$ satisfying the inequality*

$$\begin{aligned} [1 + r(d \vee d^{-1})(x, y)]D(F_1(x), F_2(y)) \leq \\ \leq r \max\{p(x, F_1(x))p(y, F_2(y)), p(x, F_2(y))p(y, F_1(x))\} + \\ + h \max\{(d \wedge d^{-1})(x, y), p(x, F_1(x)), p(y, F_2(y)), \\ \frac{1}{2}[p(x, F_2(y)) + p(y, F_1(x))]\} \end{aligned} \quad (1)$$

for each $x, y \in X$, where $r \geq 0$ and $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

Proof. Suppose x_0 is an arbitrary point in X such that $\{x_1\} \subset F_1(x_0)$. Since $(F_2(x_1))_1$ is d^{-1} -compact, it follows from Lemma 3.4, there exists $x_2 \in (F_2(x_1))_1$ such that $d(x_1, x_2) = d(x_1, (F_2(x_1))_1)$. Thus we have

$$d(x_1, x_2) = d(x_1, (F_2(x_1))_1) \leq H(x_1, (F_2(x_1))_1) \leq D(F_1(x_0), F_2(x_1)). \quad (2)$$

Similarly, we can find $x_3 \in X$ such that

$$\{x_3\} \subset F_1(x_2) \text{ and } d(x_2, x_3) \leq D(F_2(x_1), F_1(x_2)).$$

Continuing in this way, we can obtain a sequence $(x_n)_{n \in \mathbf{N}}$ in X such that

$$\begin{aligned} \{x_{2n+1}\} &\subset F_1(x_{2n}), \{x_{2n+2}\} \subset F_2(x_{2n+1}), \\ d(x_{2n+1}, x_{2n+2}) &\leq D(F_1(x_{2n}), F_2(x_{2n+1})) \end{aligned}$$

and

$$d(x_{2n+2}, x_{2n+3}) \leq D(F_2(x_{2n+1}), F_1(x_{2n+2}))$$

for $n = 0, 1, 2, \dots$

Now using inequalities (1) and (2) we have,

$$\begin{aligned} [1 + rd(x_0, x_1)]d(x_1, x_2) &\leq [1 + r(d \vee d^{-1})(x_0, x_1)]D(F_1(x_0), F_2(x_1)) \leq \\ &\leq r \max\{p(x_0, F_1(x_0))p(x_1, F_2(x_1)), p(x_0, F_2(x_1))p(x_1, F_1(x_0))\} + \\ &\quad + h \max\{(d \wedge d^{-1})(x_0, x_1), p(x_0, F_1(x_0)), p(x_1, F_2(x_1)), \\ &\quad \frac{1}{2}[p(x_0, F_2(x_1)) + p(x_1, F_1(x_0))]\}. \end{aligned}$$

Since $x_1 \in (F_1(x_0))_1$ and $x_2 \in (F_2(x_1))_1$, we have $p(x_0, F_1(x_0)) \leq d(x_0, x_1)$, $p(x_1, F_2(x_1)) \leq d(x_1, x_2)$, $p(x_0, F_2(x_1)) \leq d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$ and $p(x_1, F_1(x_0)) = 0$.

Thus we have,

$$\begin{aligned} [1 + rd(x_0, x_1)]d(x_1, x_2) &\leq rd(x_0, x_1)d(x_1, x_2) + \\ &\quad + h \max\{d(x_0, x_1), d(x_1, x_2), \frac{1}{2}[d(x_0, x_1) + d(x_1, x_2)]\}. \end{aligned}$$

and it follows that

$$d(x_1, x_2) \leq h \max\{d(x_0, x_1), d(x_1, x_2), \frac{1}{2}[d(x_0, x_1) + d(x_1, x_2)]\} = hd(x_0, x_1)$$

since $h < 1$. Thus

$$d(x_1, x_2) \leq hd(x_0, x_1).$$

Similarly,

$$d(x_2, x_3) \leq hd(x_1, x_2) \leq h^2d(x_0, x_1)$$

and, in general,

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \quad \text{for all } n \in \mathbf{N}.$$

For $n < m$, we have

$$d(x_n, x_m) \leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=0}^{m-1} h^i d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1).$$

Since $0 < h < 1$, it follows that $(x_n)_{n \in \mathbf{N}}$ is a left K -Cauchy sequence in the left K -sequentially complete quasi-pseudo-metric space (X, d) and so there exists $x^* \in X$ such that $\lim_{n \rightarrow \infty} x_n = x^*$.

Now, by Lemma 3.2, we have $p_1(x^*, F_2(x^*)) \leq d(x^*, x_{2n+1}) + p_1(x_{2n+1}, F_2(x^*))$ for all $n \in \mathbf{N}$. So, by Lemmas 3.3 and inequality (1),

$$\begin{aligned} p_1(x^*, F_2(x^*)) &\leq d(x^*, x_{2n+1}) + D_1(F_1(x_{2n}), F_2(x^*)) \leq \\ &\leq d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*)) \leq \\ &\leq d(x^*, x_{2n+1}) + \frac{r \max\{p(x_{2n}, F_1(x_{2n}))p(x^*, F_2(x^*)), \\ &\quad \frac{p(x_{2n}, F_2(x^*))p(x^*, F_1(x_{2n}))\} + h \max\{(d \wedge d^{-1})(x_{2n}, x^*), \\ &\quad \frac{p(x_{2n}, F_2(x^*))p(x^*, F_1(x_{2n}))\}}{1 + r(d \vee d^{-1})(x_{2n}, x^*)}\}}{1 + r(d \vee d^{-1})(x_{2n}, x^*)}. \end{aligned}$$

Since

$$(d \vee d^{-1})(x_{2n}, x^*) \geq d^{-1}(x_{2n}, x^*) = d(x^*, x_{2n})$$

and

$$(d \wedge d^{-1})(x_{2n}, x^*) \leq d^{-1}(x_{2n}, x^*) = d(x^*, x_{2n}),$$

we have

$$\begin{aligned} p_1(x^*, F_2(x^*)) &\leq d(x^*, x_{2n+1}) + \\ &+ \frac{r \max\{p(x_{2n}, F_1(x_{2n}))p(x^*, F_2(x^*)), p(x_{2n}, F_2(x^*))p(x^*, F_1(x_{2n}))\}}{1 + rd(x^*, x_{2n})} + \\ &+ \frac{h \max\{d(x^*, x_{2n}), p(x_{2n}, F_1(x_{2n})), p(x^*, F_2(x^*)), \\ &\quad \frac{1}{2}[p(x_{2n}, F_2(x^*)) + p(x^*, F_1(x_{2n}))]\}}{1 + rd(x^*, x_{2n})}, \end{aligned}$$

and by lemmas 3.2 and 3.3,

$$\begin{aligned}
 p_1(x^*, F_2(x^*)) &\leq d(x^*, x_{2n+1}) + \\
 &+ \frac{r \max\{d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*)), \\
 &\quad [d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})\}}{1 + rd(x^*, x_{2n})} + \\
 &+ \frac{h \max\{d(x^*, x_{2n}), d(x_{2n}, x_{2n+1}), d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*)), \\
 &\quad \frac{1}{2}[d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*)) + d(x^*, x_{2n+1})]\}}{1 + rd(x^*, x_{2n})}.
 \end{aligned}$$

it follows that

$$\begin{aligned}
 p_1(x^*, F_2(x^*)) &\leq d(x^*, x_{2n+1}) + \\
 &+ \frac{r \max\{d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*)), \\
 &\quad [d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})\}}{1 + rd(x^*, x_{2n})} + \\
 &+ \frac{h \max\{d(x^*, x_{2n}), d(x_{2n}, x_{2n+1}), d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))\}}{1 + rd(x^*, x_{2n})}, \tag{3}
 \end{aligned}$$

since $\frac{1}{2}[d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*)) + d(x^*, x_{2n+1})]$ is less than or equal to $d(x_{2n}, x_{2n+1})$ or $d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))$.

Now let

$$\begin{aligned}
 m_n &= \max\{d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*)), \\
 &\quad [d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})\}
 \end{aligned}$$

and

$$M_n = \max\{d(x^*, x_{2n}), d(x_{2n}, x_{2n+1}), d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))\}.$$

Then from inequality (3) we have

$$p_1(x^*, F_2(x^*)) \leq d(x^*, x_{2n+1}) + \frac{rm_n + hM_n}{1 + rd(x^*, x_{2n})}. \tag{4}$$

Now we have to consider, for each $n \in \mathbf{N}$, the following four cases:

Case 1. If $m_n = d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*))$ and M_n is equal to either $d(x^*, x_{2n})$ or $d(x_{2n}, x_{2n+1})$, then since $d(x^*, x_{2n})$ and $d(x_{2n}, x_{2n+1})$ converge to 0 as $n \rightarrow \infty$, we obtain that $m_n \rightarrow 0$ and $M_n \rightarrow 0$. Also, $d(x^*, x_{2n+1})$ converge to 0. Hence, from (4), we obtain $p_1(x^*, F_2(x^*)) = 0$.

Case 2. If $m_n = d(x_{2n}, x_{2n+1})p(x^*, F_2(x^*))$ and $M_n = d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))$, then by inequality (1), we have

$$M_n \leq d(x^*, x_{2n+1}) + \frac{rm_n + hM_n}{1 + rd(x^*, x_{2n})}$$

and it follows that

$$M_n \left[\frac{1 + rd(x^*, x_{2n}) - h}{1 + rd(x^*, x_{2n})} \right] \leq d(x^*, x_{2n+1}) + \frac{rm_n}{1 + rd(x^*, x_{2n})}.$$

Since $d(x^*, x_{2n})$, $d(x^*, x_{2n+1})$ and m_n converge to 0 as $n \rightarrow \infty$, we obtain that $M_n \rightarrow 0$. Thus from (4), we have $p_1(x^*, F_2(x^*)) = 0$.

Case 3. If $m_n = [d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})$ and M_n is equal to either $d(x^*, x_{2n})$ or $d(x_{2n}, x_{2n+1})$, then by inequality (1), we have

$$m_n \leq \left[d(x_{2n}, x_{2n+1}) + \frac{rm_n + hM_n}{1 + rd(x^*, x_{2n})} \right] d(x^*, x_{2n+1})$$

and it follows that

$$m_n \left[\frac{1 + rd(x^*, x_{2n}) - rd(x^*, x_{2n+1})}{1 + rd(x^*, x_{2n})} \right] \leq \left[d(x_{2n}, x_{2n+1}) + \frac{hM_n}{1 + rd(x^*, x_{2n})} \right] d(x^*, x_{2n+1}).$$

Since $d(x^*, x_{2n})$, $d(x^*, x_{2n+1})$, $d(x_{2n}, x_{2n+1})$ and M_n converge to 0 as $n \rightarrow \infty$, we obtain that $m_n \rightarrow 0$. Thus from (4), we have $p_1(x^*, F_2(x^*)) = 0$.

Case 4. If $m_n = [d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})$ and $M_n = d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))$, then by inequality (1), we have

$$\begin{aligned} D(F_1(x_{2n}), F_2(x^*)) &\leq \frac{rm_n + hM_n}{1 + rd(x^*, x_{2n})} = \\ &= \frac{r[d(x_{2n}, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]d(x^*, x_{2n+1})}{1 + rd(x^*, x_{2n})} + \\ &\quad + \frac{h[d(x^*, x_{2n+1}) + D(F_1(x_{2n}), F_2(x^*))]}{1 + rd(x^*, x_{2n})} \end{aligned}$$

and it follows that

$$\begin{aligned} D(F_1(x_{2n}), F_2(x^*)) \left[\frac{1 + rd(x^*, x_{2n}) - rd(x^*, x_{2n+1}) - h}{1 + rd(x^*, x_{2n})} \right] &\leq \\ &\leq \frac{[rd(x_{2n}, x_{2n+1}) + h]d(x^*, x_{2n+1})}{1 + rd(x^*, x_{2n})}. \end{aligned}$$

Since $d(x^*, x_{2n})$, $d(x^*, x_{2n+1})$ and $d(x_{2n}, x_{2n+1})$ converge to 0 as $n \rightarrow \infty$ and $0 < 1 - h < 1$, we obtain that $D(F_1(x_{2n}), F_2(x^*)) \rightarrow 0$. Hence m_n and M_n converge to 0 as $n \rightarrow \infty$. Thus from (4), we have $p_1(x^*, F_2(x^*)) = 0$.

It now follows from cases 1 – 4 and Lemma 3.1 that $\{x^*\} \subset F_2(x^*)$.

Similarly, it can be shown that $\{x^*\} \subset F_1(x^*)$.

When (X, d) is a right K -sequentially complete quasi-pseudo-metric space, using Lemmas 3.5, 3.6, 3.7 and 3.8 we get the following result.

Theorem 4.2. *Let (X, d) be a right K -sequentially complete quasi-pseudo-metric space and let F_1 and F_2 be fuzzy mappings from X to $W'(X)$ satisfying the inequality*

$$\begin{aligned} [1 + r(d \vee d^{-1})(x, y)]D(F_1(x), F_2(y)) &\leq \\ &\leq r \max\{p(F_1(x), x)p(F_2(y), y), p(F_2(y), x)p(F_1(x), y)\} + \\ &\quad + h \max\{(d \wedge d^{-1})(x, y), p(F_1(x), x), p(F_2(y), y), \\ &\quad \frac{1}{2}[p(F_2(y), x) + p(F_1(x), y)]\} \end{aligned} \tag{5}$$

for each $x, y \in X$, where $r \geq 0$ and $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

The proof of this theorem is similar to the proof of Theorem 4.1 and is omitted.

On noting that

$$\begin{aligned} [p(x, F_1(x))p(y, F_2(y))]^{1/2} &\leq \frac{1}{2}[p(x, F_1(x)) + p(y, F_2(y))] \leq \\ &\leq \max\{(d \wedge d^{-1})(x, y), p(x, F_1(x)), p(y, F_2(y)), \frac{1}{2}[p(x, F_2(y)) + p(y, F_1(x))]\}, \end{aligned}$$

we have the following corollary from Theorem 4.1 with $r = 0$.

Corollary 4.1. *Let (X, d) be a left K -sequentially complete quasi-pseudo-metric space and let F_1 and F_2 be fuzzy mappings from X to $W^*(X)$ satisfying the inequality*

$$D(F_1(x), F_2(y)) \leq h[p(x, F_1(x))p(y, F_2(y))]^{1/2},$$

for each $x, y \in X$, where $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

Similarly, we have the following corollary from Theorem 4.2.

Corollary 4.2. *Let (X, d) be a right K -sequentially complete quasi-pseudo-metric space and let F_1 and F_2 be fuzzy mappings from X to $W'(X)$ satisfying the inequality*

$$D(F_1(x), F_2(y)) \leq [p(F_1(x), x)p(F_2(y), y)]^{1/2},$$

for each $x, y \in X$, where $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

Both Corollary 4.1 and Corollary 4.2 are extensions of Theorem 3.2 of [7] in quasi-pseudo-metric space.

When (X, d) is a complete metric space, we get the following corollary.

Corollary 4.3. *Let (X, d) be a complete metric space and let F_1 and F_2 be fuzzy mappings from X to $W'(X)$ satisfying the inequality*

$$\begin{aligned} [1 + rd(x, y)]D(F_1(x), F_2(y)) \leq \\ \leq r \max\{p(x, F_1(x))p(y, F_2(y)), p(x, F_2(y))p(y, F_1(x))\} + \\ + h \max\{d(x, y), p(x, F_1(x)), p(y, F_2(y)), \\ \frac{1}{2}[p(x, F_2(y)) + p(y, F_1(x))]\} \end{aligned} \quad (6)$$

for each $x, y \in X$, where $r \geq 0$ and $0 < h < 1$. Then there exists $x^* \in X$ such that $\{x^*\} \subset F_1(x^*)$ and $\{x^*\} \subset F_2(x^*)$.

Remark 1. Letting $F_1 = F_2$ with $r = 0$ in inequality (1), then Theorem 3.2 of [11] is a consequence of Theorem 4.1. Similarly, notice that Theorem 3.1 of [3] can be obtained from Theorem 4.1.

Remark 2. If we put $r = 0$ in inequality (6), we can see that Theorem 3.1 in [7] is a special case of our Corollary 4.3. Also Theorem 3.2 of [7] can be obtained from Corollary 4.3.

Remark 3. Similarly, if we put $r = 0$ in inequality (6), we can obtain Theorem 3.1 of [5] from Corollary 4.3.

References

- [1] Beg, I., Azam, A.: Fixed point of asymptotically regular multivalued mappings, J. Austral. Math. Soc. 53, 313-326 (1992).
- [2] Bose, R. K., Sahani, D.: Fuzzy mappings and fixed point theorems, Fuzzy Sets and Systems 21, 53-58 (1987).
- [3] Gregori, V., Pastor, J.: A fixed point theorem for fuzzy contraction mappings, Rend. Istit. Math. Univ. Trieste 30, 103-109 (1999).
- [4] Gregori, V., Romaguera, S.: Fixed point theorems for fuzzy mappings in quasi-metric spaces, Fuzzy Sets and Systems 115, 477-483 (2000).
- [5] Heilpern, S.: Fuzzy mappings and fixed point theorem, J. Math. Anal. Appl. 83, 566-569 (1981).
- [6] Nadler, S. B.: Multivalued contraction mappings, Pacific J. Math. 30, 475-488 (1969).
- [7] Park, J. Y., Jeong, J. U.: Fixed point theorems for fuzzy mappings, Fuzzy Sets and Systems 87, 111-116 (1997).
- [8] Popa, V.: Common fixed points for multifunctions satisfying a rational inequality, Kobe J. Math. 2, 23-28 (1985).
- [9] Reilly, I. L., Subrahmanyam, P. V. and Vamanamurthy, M. K.: Cauchy sequences in quasi-pseudo-metric spaces, Monatsh. Math. 93, 127-140 (1982).
- [10] Telci, M., Fisher, B.: On a fixed point theorem for fuzzy mappings in quasi-metric spaces, Thai J. Math. 2, 1-8 (2003).
- [11] Telci, M., Şahin, İ.: A fixed point theorem for fuzzy mappings in quasi-pseudo-metric spaces, J. Fuzzy Math. 10, 105-110 (2002).

İ. ŞAHİN, H. KARAYILAN and M. TELCİ
 Department of Mathematics,
 Faculty of Arts and Sciences,
 Trakya University, 22030 Edirne-TURKEY
 e-mail: mtelci@trakya.edu.tr

Received 04.11.2003