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MASHHOOR REFAI

FIDA A. M. MOH'D

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Characterizations of Augmented Graded Rings

Mashhoor Refai, Fida A. M. Moh'd

Abstract

In this paper, we introduce some characterizations for augmented graded rings in special cases.

Key Words: Graded rings, Augmented graded rings, Strongly graded rings.

Introduction

Let G be a group with identity e . A ring R is said to be a G -graded ring if there exist additive subgroups R_g of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The

G -graded ring R is denoted by (R, G) . We denote by $\text{supp}(R, G)$ the support of G which is defined to be $\{g \in G : R_g \neq 0\}$. The elements of R_g are called homogeneous of degree g . For $x \in R$, x can be written uniquely as $\sum_{g \in G} x_g$ where x_g is the component of x in R_g .

Also we write $h(R) = \bigcup_{g \in G} R_g$.

In this paper, we give some characterizations for the augmented graded rings for the case where $\text{supp}(R, G)$ is a subgroup of G . The general case is left open. One of the characterizations has a connection with the second strong property.

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1. Preliminaries

In this section, we give some basic facts of graded rings. For more details, one can look in [3, 4, 5].

Lemma 1.1 *Let R be a G -graded ring and $x, y \in R, g \in G$. Then*

1. $(x + y)_g = x_g + y_g$.
2. $(xy)_g = \sum_{h \in G} x_h y_{h^{-1}g}$.

Proposition 1.2 *Let R be a G -graded ring. Then*

1. R_e is a subring of R and $1 \in R_e$.
2. R_g and R are left (resp. right) R_e -modules, for all $g \in G$.

Definition 1.3 *A G -graded ring R is said to be strongly graded if $R_g R_h = R_{gh}$ for all $g, h \in G$.*

Proposition 1.4 *Let R be a G -graded ring. Then (R, G) is strong iff $R_g R_{g^{-1}} = R_e$ for all $g \in G$.*

Corollary 1.5 *(R, G) is strong iff $1 \in R_g R_{g^{-1}}$ for all $g \in G$.*

Definition 1.6 *Let R be a G -graded ring. Then (R, G) is first strong if $R_g R_{g^{-1}} = R_e$ for all $g \in \text{supp}(R, G)$, or equivalently if $1 \in R_g R_{g^{-1}}$ for all $g \in \text{supp}(R, G)$.*

Proposition 1.7 *If (R, G) is first strong, then $\text{supp}(R, G)$ is a subgroup of G .*

Definition 1.8 *Let R be a G -graded ring. Then (R, G) is said to be second strong if $\text{supp}(R, G)$ is a monoid in G and $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$.*

Remark 1.9 *Every first strongly graded ring is second strong but the converse is not true in general (see [5]).*

Definition 1.10 *A ring R is said to be an augmented G -graded ring if it satisfies the following conditions:*

1. $R = \bigoplus_{g \in G} R_g$ where R_g is an additive subgroup of R and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$ (R is a G -graded ring).
2. If R_e is the identity component of the graduation then $R_e = \bigoplus_{g \in G} R_{e-g}$, where R_{e-g} is an additive subgroup of R_e and $R_{e-g} R_{e-h} \subseteq R_{e-gh}$ for all $g, h \in G$ (R_e is a G -graded ring).

3. For each $g \in G$, there exists $r_g \in R_g$ such that $R_g = \bigoplus_{h \in G} R_{e-h}r_g$. We assume $r_e = 1$.
4. If $g, h \in G$ and r_g, r_h are both non-zero, then $r_g r_h = r_{gh}$ and for all $x, y \in R_e$ we have $(xr_g)(yr_h) = xy r_{gh}$.

Remark 1.11 It follows from the last definition that

1. Condition 3 of the definition implies $R_h = R_e r_h$ for all $h \in G$.
 2. R_g is a G -graded R_e -module with the usual multiplication on R and with the gradation $R_{g-h} = R_{e-h}r_g$ for all $h \in G$
 3. $R_{g-h}R_{g'-h'} \subseteq R_{gg'-hh'}$ for all $g, g', h, h' \in G$. $R_{g-h}R_{g'-h'} = R_{e-h}r_g R_{e-h'}r_{g'}$
- If $r_g, r_{g^{-1}}$ are both non-zero then $r_g R_e = R_e r_g = R_g$.

Proposition 1.12 Let R be an augmented G -graded ring such that $\text{supp}(R, G)$ is a subgroup of G . Then (R, G) is first strong.

Proof Let $g \in \text{supp}(R, G)$. Then $g^{-1} \in \text{supp}(R, G)$, i.e., $R_g \neq 0$ and $R_{g^{-1}} \neq 0$.

Since $R_g = R_e r_g$ and $R_{g^{-1}} = R_e r_{g^{-1}}$ we get $r_g \neq 0, r_{g^{-1}} \neq 0$ and hence

$r_g r_{g^{-1}} = r_{gg^{-1}} = r_e = 1$. Thus $1 \in R_g R_{g^{-1}}$, i. e., R is first strong. □

2. Characterizations of Augmented Graded Rings

In this section, we give characterizations for the augmented graded rings in the case where $\text{supp}(R, G)$ is a subgroup of G . The general case is still open.

Lemma 2.1 Let $f : S \rightarrow G$ be a group isomorphism and R be a G -graded ring. Then R is S -graded ring with: $R_s = R_{f(s)}$ for all $s \in S$.

Proof. Trivial.

Notation 2.2 Suppose (R, G) is an augmented G -graded ring and $r_g \in R_g$ such that $R_g = \bigoplus_{h \in G} R_{e-h}r_g$. We let $F = \{r_g : g \in \text{supp}(R, G), r_g \text{ is fixed for each } g \in G\}$. It is easy to show $|F| = |\text{supp}(R, G)|$. □

Lemma 2.3 Let R be an augmented G -graded ring such that $\text{supp}(R, G)$ is a subgroup of G . Then F is a multiplicative group with the multiplication of R restricted on F . Furthermore, F is isomorphic to $\text{supp}(R, G)$.

Proof. $F \neq \emptyset$ for $1 = r_e \in F$. Let $g, h \in \text{supp}(R, G)$. Then $r_g r_h = r_{gh} \in F$ because $gh \in \text{supp}(R, G)$. Hence, F is closed under multiplication.

Let $g \in \text{supp}(R, G)$ then $g^{-1} \in \text{supp}(R, G)$ and hence $r_g r_{g^{-1}} = r_{g^{-1}} r_g = r_e = 1$, i.e., r_g has an inverse in F . Namely, $r_g^{-1} = r_{g^{-1}}$. Since F inherits the associativity from R , F is a multiplicative group.

One can show that $f : \text{supp}(R, G) \rightarrow F$ given by $f(g) = r_g$ is a group isomorphism. \square

Lemma 2.4 *Suppose R is an augmented G -graded ring and F given in Notation 2.2 is a multiplicative group. Then $\text{supp}(R, G)$ is subgroup of G and hence F is isomorphic to $\text{supp}(R, G)$.*

Proof. Suppose (R, G) is augmented and F is a multiplicative group.

Let $g, h \in \text{supp}(R, G)$. Then $r_g r_h = r_{gh} \in F$ and hence $gh \in \text{supp}(R, G)$. Thus $\text{supp}(R, G)$ is a monoid in G . Let $g \in \text{supp}(R, G)$. Then $r_g \in F$; So $r_g r_h = 1$ for some $r_h \in F$ and $h \in \text{supp}(R, G)$. So, $r_{gh} = r_g r_h = 1 = r_e$ and hence $gh = e$ and $h = g^{-1}$. Therefore, $g^{-1} \in \text{supp}(R, G)$. By Lemma 2.3, $\text{supp}(R, G)$ is isomorphic to F . \square

Corollary 2.5 *Let R be an augmented G -graded ring. Then, $\text{supp}(R, G)$ is a subgroup of G iff F is a multiplicative group. Moreover, F is isomorphic to $\text{supp}(R, G)$.*

Proposition 2.6 *Let R be a G -graded ring such that $\text{supp}(R, G)$ is a subgroup of G . Then (R, G) is augmented iff the following conditions hold:*

1. R_e is a G -graded ring by any graduation.
2. For each $g \in \text{supp}(R, G)$ there exists $r_g \in R_g$ such that $R_g = R_e r_g$.
3. For each $g, h \in \text{supp}(R, G)$ we have $r_g r_h = r_{gh}$ and $x r_g = r_g x$ for each $x \in R_e$.

Proof. Suppose (R, G) is augmented then (1), (2) and (3) follow by Remark 1.11.

For the converse, suppose $R_e = \bigoplus_{h \in G} R_{e-h}$.

First, we show that $R_g = \bigoplus_{h \in G} R_{e-h} r_g$ for all $g \in G$. If $g \notin \text{supp}(R, G)$ we have $r_g = 0$.

One can see that $R_g = \bigoplus_{h \in G} R_{e-h} r_g$. Suppose $g \in \text{supp}(R, G)$ and $x \in R_g = R_e r_g \in \sum_{h \in G} R_{e-h} r_g$.

Then $x = s r_g$ and $s \in R_e$. Assume that $s = \sum_{i=1}^n y_{e-h_i}$ where $y_{e-h_i} \in R_{y_{e-h_i}}$ for

$i = 1, 2, \dots, n$. Then, $x = \sum_{i=1}^n y_{e-h_i} r_g$. Hence, $R_g = \sum_{g \in G} R_{e-h} r_g$.

Let $x \in R_{e-\alpha} r_g \cap \sum_{h \in G - \{\alpha\}} R_{e-h} r_g$. Then $x = y_{e-\alpha} r_g = \sum_{h \in G - \{\alpha\}} y_{e-h} r_g$ and hence $\{y_{e-h} - \sum_{h \in G - \{\alpha\}} y_{e-h}\} r_g = 0$. Thus, $\{y_{e-\alpha} - \sum_{h \in G - \{\alpha\}} y_{e-h}\} r_g r_{g^{-1}} = 0$ or $\{y_{e-\alpha} -$

$\sum_{h \in G - \{\alpha\}} y_{e-h} \} = 0$ where $r_g r_{g^{-1}} = r_e = 1$. Hence, $y_{e-\alpha} = 0$ and $y_{e-h} = 0$ for all $h \in G - \{\alpha\}$ because R_e is a G -graded ring. Therefore, $x = 0$ and $R_{e-\alpha} r_g \cap \sum_{h \in G - \{\alpha\}} R_{e-h} r_g = 0$ for all $\alpha \in G$. Thus, we conclude that $R_g = \bigoplus_{h \in G} R_{e-h} r_g$.

Second, we claim $(x r_g)(y r_h) = x y r_{gh}$ for all $x, y \in R_e$ and $g, h \in \text{supp}(R, G)$. Since $r_g r_h = r_{gh}$ for all $g, h \in \text{supp}(R, G)$ and $x r_g = r_g x$ for all $x \in R_e$, and $g \in \text{supp}(R, G)$, we have $(x r_g)(y r_h) = x(r_g y) r_h = x(y r_g) r_h = x y r_g r_h = x y r_{gh}$. \square

Proposition 2.7 *Let R be a G -graded ring such that $H = \text{supp}(R, G)$ is a subgroup of G . Then (R, G) is augmented iff the following conditions hold*

1. R_e is G -graded ring by any graduation.
2. There exists a multiplicative group $F \subset h(R)$ such that F is isomorphic to $\text{supp}(R, G)$, $F \cap R_e = 1$, $R = R_e F$ and $ax = xa$ for each $x \in R_e$ and $a \in F$.

Proof. Suppose (R, G) is an augmented G -graded ring. Then condition (1) is clear. Let $R_g = \bigoplus_{h \in G} R_{e-h} r_g$ for some $r_g \in R_g$ and $g \in G$. Fixing this r_g for each $g \in G$, taking $F = \{r_g : g \in H\}$ and using Lemma 2.3, we have F is a multiplicative group such that $F \subset h(R)$, F isomorphic to $\text{supp}(R, G)$, $F \cap R_e = \{1\}$ and $R = \bigoplus_{g \in G} R_e r_g =$

$$\bigoplus_{g \in H} R_e r_g = R_e F.$$

By Remark 1.11, $r_g x = x r_g$ for all $x \in R_e$ and $g \in G$ (or $g \in H$).

Conversely, let $f : H \rightarrow F$ be a group isomorphism. We show (R, G) is augmented step by step.

Step1: If $g_1, g_2 \in H$ and $g_1 \neq g_2$ then $\sigma_1 \neq \sigma_2$ where $f(g_i) = R_{\sigma_i}$, for $i = 1, 2$. Otherwise, if $\sigma_1 = \sigma_2$ then $f(g_i) \in R_{\sigma_i}$, for $i = 1, 2$, and one can show that $f(g_i^{-1}) = R_{\sigma_i^{-1}}$. So, $f(g_1^{-1})f(g_2) \in R_{\sigma_1^{-1}}R_{\sigma_1} \subset R_e$ or $f(g_1^{-1}g_2) \in R_e \cap F = \{1\}$. Hence, $f(g_1^{-1}g_2) = 1$ and then $g_1^{-1}g_2 = e$, i.e., $g_1 = g_2$.

Step2: We show $R_\sigma \cap F \neq \emptyset$ for each $\sigma \in H$.

Let $K = \{\sigma \in H : R_\sigma \cap F = \emptyset\}$. Then $R = R_e F = \sum_{g \in H} R_e f(g)$ and $R_e f(g) \subset R_{\sigma_g}$ where $f(g) \in R_{\sigma_g}$. Also, if $g_1 \neq g_2$ then $\sigma_{g_1} \neq \sigma_{g_2}$ and hence $R = \bigoplus_{g \in H} R_e f(g)$. Let $g \in H$

and $m \in R_{\sigma_g}$. Then, $m = \sum_{i=1}^n x_i f(g_i)$ where $x_i \in R_e$ and $g_i \in H$ for all $i = 1, \dots, n$. Thus,

$n = 1$ and $g_1 = g$ with $f(g) \in R_{\sigma_g}$, i.e., $m = x_1 f(g) \in R_e f(g)$. Therefore, $R_{\sigma_g} = R_e f(g)$ and hence $R_{\sigma_g} \cap F \neq \emptyset$. So, $\sigma_g \in H - K$ for all $g \in H$. Since $R = \bigoplus_{g \in H} R_e f(g) =$

$\bigoplus_{g \in H} R_{\sigma_g} \subset \bigoplus_{\sigma \in H-K} R_{\sigma} \subset \bigoplus_{\sigma \in H} R_{\sigma} = R$ we have $R = \bigoplus_{\sigma \in H} R_{\sigma} = \bigoplus_{\sigma \in H-K} R_{\sigma}$ and hence $\bigoplus_{\sigma \in K} R_{\sigma} = 0$. Since $K \subset H$, $K = \emptyset$. Therefore, $R_{\sigma} \cap F \neq \emptyset$ for all $\sigma \in H$.

Step3: Define $\zeta : H \rightarrow H$ by $\zeta(g) = \sigma_g$ where $f(g) \in R_{\sigma_g}$. Our aim now is to show that ζ is a group isomorphism.

Clearly ζ is well-defined and monomorphism.

Let $\sigma \in H$. By Step2, $R_{\sigma} \cap F \neq \emptyset$. Thus there exists $a \in F \cap R_{\sigma}$. Moreover, $a \in F$ and f is onto imply $a = f(g) \in R_{\sigma}$ and hence $\sigma = \zeta(g)$ where $g \in H$, i.e., ζ is onto.

By Lemma 2.1, R is an H -graded ring with $R_{(h)} = R_{\zeta(h)}$ for all $h \in H$. Hence, R is G -graded with $R_{(g)} = R_{\zeta(g)}$ if $g \in H$ and $R_{(g)} = 0$ if $g \notin H$.

Let $R = \bigoplus_{g \in G} R_{(g)}$. Then, by Proposition 2.6, $R = \bigoplus_{g \in G} R_{(g)}$ is an augmented G -graded ring. □

Remark 2.8 Let G be an abelian multiplicative group and $\{H_{\alpha} : \alpha \in \Delta\}$ be a family of subgroups of G . We write $G = \bigotimes_{\alpha \in \Delta} H_{\alpha}$ if for each $g \in G$, $g = \prod_{\alpha \in \Delta} g_{\alpha}$ where $g_{\alpha} \in H_{\alpha}$

and $g_{\alpha} = e$ for all $\alpha \in \Delta$ except finitely many and if $H_{\beta} \cap \left(\bigcap_{\alpha \in \Delta - \{\beta\}} H_{\alpha} \right) = e$ where e is the identity of G , for all $\beta \in \Delta$. Indeed, this is the internal direct product of the multiplicative subgroups of G .

If $g \in G$. Then g has a unique decomposition of the form $g = \prod_{\alpha \in \Delta} g_{\alpha}$.

Proposition 2.9 Let R be a G -graded ring such that $\text{supp}(R, G)$ is an abelian subgroup of G . Suppose $\text{supp}(R, G) = \bigotimes_{\alpha \in \Delta} \langle g_{\alpha} \rangle$, where $g_{\alpha} \in \text{supp}(R, G)$ and $\langle g_{\alpha} \rangle$ is the cyclic group generated by g_{α} for all $\alpha \in \Delta$, and $xy = yx$ for each $x, y \in h(R) - R_e$. Then (R, G) is augmented iff the following conditions hold

1. R_e is G -graded ring with any graduation.
2. (R, G) is second strong.
3. $R_{g_{\alpha}}$ is isomorphic to R_e as a left and right R_e -module for all $\alpha \in \Delta$. In the case $g_{\alpha} = e$ for some $\alpha \in \Delta$ we suppose R_e isomorphic to itself by the identity isomorphism.

Proof. Suppose (R, G) is augmented graded ring. Since $\text{supp}(R, G)$ is subgroup of G it follows by Proposition 1.12, (R, G) is second strong. By Remark 1.11, $xr_g = r_g x$ for all $x \in R_e$ and $g \in G$ where $r_g \in R_g$ with $R_g = \bigoplus_{h \in G} R_{e-h}r_g$. Also, $R_g = R_e r_g$ for all $g \in G$ and $r_g \neq 0$ iff $g \in \text{supp}(R, G)$.

Define $f : R_e \rightarrow R_g = R_e R_g$ by $f(x) = xr_g$ for each $x \in R_e$. Then clearly f is well-defined and f is R_e -module isomorphism for all $g \in \text{supp}(R, G)$ and hence for g_α where $\alpha \in \Delta$.

For the converse, assume that conditions (1), (2) and (3) hold. Since $\text{supp}(R, G)$ is a subgroup of G and (R, G) is second strong then (R, G) is first strong. Let $f_\alpha : R_e \rightarrow R_{g_\alpha}$ be an R_e -module isomorphism. For each $x \in R_e$, $f_\alpha(x) = x f_\alpha(1) = f_\alpha(1)x$. Let $r_{g_\alpha} = f_\alpha(1)$ for each $\alpha \in \Delta$. Then for each $x_{g_\alpha} \in R_{g_\alpha}$ there exists $x \in R_e$ such that $x_{g_\alpha} = f_\alpha(x) = xr_\alpha = r_\alpha x$, and x is unique because f_α is 1-1. Hence $R_{g_\alpha} = R_e r_{g_\alpha} = r_{g_\alpha} R_e$ for all $\alpha \in \Delta$.

Since R is first strong, $R_{g_\alpha} R_{g_\alpha^{-1}} = R_{g_\alpha^{-1}} R_{g_\alpha} = R_e$ for all $\alpha \in \Delta$. So, $(r_{g_\alpha} R_e) R_{g_\alpha^{-1}} = R_{g_\alpha^{-1}} (R_e r_{g_\alpha})$ or $r_{g_\alpha} (R_e R_{g_\alpha^{-1}}) = (R_{g_\alpha^{-1}} R_e) r_{g_\alpha}$ and hence we obtain $r_{g_\alpha} R_{g_\alpha^{-1}} = R_{g_\alpha^{-1}} r_{g_\alpha} = R_e$. Therefore there exist $x, y \in R_{g_\alpha^{-1}}$ such that $1 = xr_{g_\alpha} = r_{g_\alpha} y$. Clearly, $xr_{g_\alpha}, r_{g_\alpha} y \in R_e$. Thus $f_\alpha(xr_{g_\alpha}) = f_\alpha(r_{g_\alpha} y)$ and hence $r_{g_\alpha} (xr_{g_\alpha}) = (r_{g_\alpha} y) r_{g_\alpha}$. Multiplying both sides by x from the left to get $(xr_{g_\alpha})(xr_{g_\alpha}) = (xr_{g_\alpha})(yr_{g_\alpha})$ which gives $xr_{g_\alpha} = yr_{g_\alpha}$. Multiplying both sides from the right by y to get $x(r_{g_\alpha} y) = y(r_{g_\alpha} y)$ and so $x = y$. Thus, $xr_g = r_{g_\alpha} x = 1$, i.e., r_{g_α} is a unit in R , for each $\alpha \in \Delta$. Since $R_{g_\alpha} R_{g_\alpha^{-1}} = R_e$, $r_{g_\alpha} R_{g_\alpha^{-1}} = R_e$ and hence $R_{g_\alpha^{-1}} = r_{g_\alpha}^{-1} R_e$.

Similarly, $R_{g_\alpha^{-1}} = R_e r_{g_\alpha}^{-1}$ for each $\alpha \in \Delta$. We define $r_{g_\alpha^{-1}} = r_{g_\alpha}^{-1}$; $\alpha \in \Delta$. Thus $R_e = R_{g_\alpha} R_{g_\alpha^{-1}} = R_e r_{g_\alpha} r_{g_\alpha^{-1}}$. We let $r_e = r_{g_\alpha} r_{g_\alpha^{-1}} = 1$.

If $\alpha \in \Delta$ and $n \in N$ then $R_{g_\alpha^n} = R_{g_\alpha} \cdots R_{g_\alpha}$ (n -times) and hence $R_{g_\alpha^n} = (R_e r_{g_\alpha}) \cdots (R_e r_{g_\alpha})$ (n -times) which gives $R_{g_\alpha^n} = R_e r_{g_\alpha} \cdots r_{g_\alpha}$ where r_{g_α} is product with itself n -times. We define $r_{g_\alpha^n} = r_{g_\alpha}^n$.

If $n \in Z - (N \cup \{0\})$, i.e., $n < 0$ we have $R_{g_\alpha^n} = R_{(g_\alpha^{-1})^n} = R_{g_\alpha^{-1}} \cdots R_{g_\alpha^{-1}}$ ($|n|$ -times) and hence $R_{g_\alpha^n} = (R_e r_{g_\alpha^{-1}}) \cdots (R_e r_{g_\alpha^{-1}})$ ($|n|$ -times) and so $R_{g_\alpha^n} = R_e r_{g_\alpha^{-1}} \cdots r_{g_\alpha^{-1}} = R_e r_{g_\alpha}^{|n|}$. We define $r_{g_\alpha^n} = r_{g_\alpha}^{|n|} = (r_{g_\alpha}^{-1})^{|n|} = (r_{g_\alpha})^{-|n|} = r_{g_\alpha}^n$ for all $\alpha \in \Delta$.

Therefore, for any $\alpha \in \Delta$ and $n \in Z$ we define $r_{g_\alpha^n} = r_{g_\alpha}^n$ and hence $R_{g_\alpha^n} = R_e r_{g_\alpha}^n$. Similarly, we can show that $R_{g_\alpha^n} = r_{g_\alpha}^n R_e$, for all $n \in Z$ and $\alpha \in \Delta$.

Now, let $h \in \text{supp}(R, G)$. By Remark 2.8, h can be written uniquely as $h = \prod_{\alpha \in \Delta} g_\alpha^{n_\alpha}$

where $n_\alpha \in \mathbb{Z}$ and $g_\alpha^{n_\alpha} = e$ for all α except finitely many.

Without loss of generality, suppose $h = g_{\alpha_1}^{n_1} \cdots g_{\alpha_m}^{n_m}$. Then $R_h = R_{g_{\alpha_1}^{n_1}} \cdots R_{g_{\alpha_m}^{n_m}} = (R_e r_{g_{\alpha_1}^{n_1}}) \cdots (R_e r_{g_{\alpha_m}^{n_m}}) = R_e r_{g_{\alpha_1}^{n_1}} \cdots r_{g_{\alpha_m}^{n_m}} = r_{g_{\alpha_1}^{n_1}} \cdots r_{g_{\alpha_m}^{n_m}} R_e$. We define $r_h = r_{g_{\alpha_1}^{n_1}} \cdots r_{g_{\alpha_m}^{n_m}} = r_{g_{\alpha_1}^{n_1}} \cdots r_{g_{\alpha_m}^{n_m}}$. Since $g_\alpha^{n_\alpha} = e$ for each $\alpha \notin \{\alpha_1, \dots, \alpha_m\}$ we have $r_{g_\alpha^{n_\alpha}} = r_e = 1$. So, it is possible to write $r_h = \prod_{\alpha \in \Delta} r_{g_\alpha^{n_\alpha}}$. Clearly, $R_h = R_e r_h$ and similarly $R_h = r_h R_e$.

Let $g, h \in \text{supp}(R, G)$. Then $g = \prod_{\alpha \in \Delta} g_\alpha^{n_\alpha}$ and $h = \prod_{\alpha \in \Delta} g_\alpha^{m_\alpha}$ and hence $gh = \prod_{\alpha \in \Delta} g_\alpha^{n_\alpha + m_\alpha} = \prod_{\alpha \in \Delta} g_\alpha^{m_\alpha + n_\alpha} = hg$.

But $r_{gh} = \prod_{\alpha \in \Delta} r_{g_\alpha^{n_\alpha + m_\alpha}} = \prod_{\alpha \in \Delta} r_{g_\alpha^{n_\alpha + m_\alpha}} = \prod_{\alpha \in \Delta} r_{g_\alpha^{m_\alpha + n_\alpha}} = r_{hg}$ implies $r_{gh} = r_{hg}$ for all $g, h \in \text{supp}(R, G)$. Moreover, if $g, h \in \text{supp}(R, G)$ such that $g = \prod_{\alpha \in \Delta} g_\alpha^{n_\alpha}$ and

$$h = \prod_{\alpha \in \Delta} g_\alpha^{m_\alpha} \text{ then } r_g r_h = \left(\prod_{\alpha \in \Delta} r_{g_\alpha^{n_\alpha}} \right) \left(\prod_{\alpha \in \Delta} r_{g_\alpha^{m_\alpha}} \right) = \prod_{\alpha \in \Delta} r_{g_\alpha^{n_\alpha}} r_{g_\alpha^{m_\alpha}} = \prod_{\alpha \in \Delta} r_{g_\alpha^{n_\alpha + m_\alpha}} = r_{gh}$$

because the homogeneous elements commute, i.e., $r_g r_h = r_{gh}$ for all $g, h \in \text{supp}(R, G)$.

(*) If $g \notin \text{supp}(R, G)$ we let $r_g = 0$. Then we have the following

1. R_e is G -graded with any graduation.
2. For each $g \in G$ there exists $r_g \in R_g$ such that $R_g = R_e r_g$.
3. For all $g, h \in \text{supp}(R, G)$ and noticing (*) we have $r_g r_h = r_{gh}$.

If $x \in R_e$ we have $x r_{g_\alpha} = r_{g_\alpha} x$ for all $\alpha \in \Delta$ and hence $r_{g_\alpha}^{-1} x = x r_{g_\alpha}^{-1}$, i. e., $r_{g_\alpha}^{-1} x = x r_{g_\alpha}^{-1}$; $\alpha \in \Delta$. If $n \in \mathbb{Z}$, $x r_{g_\alpha}^n = x r_{g_\alpha}^n = r_{g_\alpha}^n x = r_{g_\alpha}^n x$ follows by associativity of R , for $\alpha \in \Delta$.

Let $h \in \text{supp}(R, G)$. Then $h = \prod_{\alpha \in \Delta} g_\alpha^{n_\alpha}$ and $r_h = \prod_{\alpha \in \Delta} r_{g_\alpha^{n_\alpha}}$. Without loss of generality, suppose $h = g_{\alpha_1}^{n_1} \cdots g_{\alpha_m}^{n_m}$ and $g_\alpha^{n_\alpha} = e$ for all $\alpha \notin \{\alpha_1, \dots, \alpha_m\}$. Then $x r_h = x r_{g_{\alpha_1}^{n_1}} \cdots r_{g_{\alpha_m}^{n_m}} = r_{g_{\alpha_1}^{n_1}} x r_{g_{\alpha_2}^{n_2}} \cdots r_{g_{\alpha_m}^{n_m}} = r_{g_{\alpha_1}^{n_1}} r_{g_{\alpha_2}^{n_2}} \cdots x r_{g_{\alpha_m}^{n_m}} = r_{g_{\alpha_1}^{n_1}} r_{g_{\alpha_2}^{n_2}} \cdots r_{g_{\alpha_m}^{n_m}} x = r_h x$. Therefore, $x r_h = r_h x$ for all $x \in R_e$ and $h \in \text{supp}(R, G)$. If $h \notin \text{supp}(R, G)$ then $r_h = 0$ and clearly $x r_h = r_h x$.

Therefore, by Proposition 2.6, (R, G) is augmented graded ring. \square

Corollary 2.10 *Suppose (R, G) is commutative ring such that $\text{supp}(R, G)$ is an abelian subgroup of G . Suppose $\text{supp}(R, G) = \bigotimes_{\alpha \in \Delta} \langle g_\alpha \rangle$, where $g_\alpha \in \text{supp}(R, G)$ and $\langle g_\alpha \rangle$*

is the cyclic group generated by g_α for all $\alpha \in \Delta$. Then (R, G) is augmented ring iff the following conditions hold

1. R_e is G -graded ring with any graduation.
2. (R, G) is second strong.
3. R_{g_α} is isomorphic to R_e as an R_e -module for all $\alpha \in \Delta$.

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Mashhoor REFAI
Dean, Faculty of Information
Technology and Computer Sciences,
Yarmouk University Irbid-JORDAN
e-mail: mrefai@yu.edu.jo
Fida A. M. MOH'D
Department of Mathematics
Yarmouk University
Irbid-JORDAN
e-mail: fida@math.mun.ca

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