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Corrigendum Uniqueness of Primary Decompositions [Turkish J. Math. 27 (2003), 425--434]

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In [1], the statement of Theorem 6 needs to be amended and also the comment following the proof of Theorem 15. We also take this opportunity to give a clearer proof of Theorem 6. I am grateful to Mr. A. R. Woodward for bringing these matters to my attention.

Theorem 6 *Let R be any ring and let N be a submodule of an R-module M such that N has a primary decomposition. Then the following statements are equivalent for a prime ideal P of R.*

(i) P is an associated prime ideal of N.

(ii) $P = (N : L)$ for some submodule L of M with $L \nsubseteq N$.

(iii) $P = \{r \in R : rRm \subseteq N\}$ for some element $m \in M \backslash N$.

Proof. (i) $\Rightarrow$ (iii) Let $N = K_1 \cap \ldots \cap K_n$ be a normal decomposition of N where K_i is a P_i-primary submodule of M for some prime ideal P_i of R for each $1 \leq i \leq n$. Let $1 \leq i \leq n$ and let $H_i = K_1 \cap \ldots \cap K_{i-1} \cap K_{i+1} \cap \ldots \cap K_n$. There exists a positive integer $k(i)$ such that $P_i^{k(i)} M \subseteq K_i$ and hence $P_i^{k(i)} H_i \subseteq N$. Since $H_i \nsubseteq N$ there exists an integer $1 \leq t(i) \leq k(i)$ such that $P_i^{t(i)} H_i \subseteq N$ but $P_i^{t(i)-1} H_i \nsubseteq N$. Let $L_i = P_i^{t(i)-1} H_i$. Then L_i is a submodule of M such that $L_i \nsubseteq N$ and $P_i L_i \subseteq N$.

Let $m \in L_i \backslash N$ and let $A = \{r \in R : rRm \subseteq N\}$. Then A is an ideal of R and $P_i \subseteq A$. On the other hand, $Am \subseteq N \subseteq K_i$. If $m \in K_i$ then $m \in N$, a contradiction. Thus $A \subseteq P_i$ and it follows that $P_i = A$.

(iii) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (i) As before. $\square$

In the remark after Theorem 15 it is claimed that if P is a prime ideal of a PI-ring then for any ideal A of R such that $A \nsubseteq P$ there exist a finitely generated left ideal C and an ideal $B \nsubseteq P$ such that $B \subseteq C \subseteq A$. Although this is clearly true for commutative rings and for left Noetherian PI-rings, it is not true in general, as the following example shows.

Example *There exists a PI-ring R which contains a prime ideal P such that for some ideal $A \nsubseteq P$ there do not exist a finitely generated left ideal C and an ideal $B \nsubseteq P$ such that $B \subseteq C \subseteq A$.*

Proof. Let $\mathbb{Z}$ denote the ring of rational integers and X the Prüfer p-group for any prime p. Let

$$R = \begin{bmatrix} \mathbb{Z} & X \\ 0 & \mathbb{Z} \end{bmatrix}$$

denote the ring of "matrices

$$\begin{bmatrix} a & x \\ 0 & b \end{bmatrix},$$

where $a, b \in \mathbb{Z},\ x \in X$, and addition and multiplication in R are the usual matrix addition and multiplication, respectively. Let

$$P = \begin{bmatrix} 0 & X \\ 0 & \mathbb{Z} \end{bmatrix} \quad \textit{and} \quad A = \begin{bmatrix} \mathbb{Z} & X \\ 0 & 0 \end{bmatrix}.$$

Then A is an ideal of R which is not contained in the prime ideal P of R. Let C be any finitely generated left ideal of R such that $C \subseteq A$. It is easy to check that

$$C \subseteq \begin{bmatrix} \mathbb{Z} & Y \\ 0 & 0 \end{bmatrix}$$

for some finitely generated (and so finite) submodule Y of X. Let B be any ideal of R such that $B \subseteq A$ and $B \nsubseteq P$. There exists an element

$$b = \begin{bmatrix} a & x \\ 0 & 0 \end{bmatrix} \in B$$

where $0 \neq a \in \mathbb{Z},\ x \in X$. Now $X = aX$ gives that

$$\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \subseteq bR \subseteq B.$$

Hence $B \nsubseteq C$. Thus R has the required properties.

Finally note that in Corollary 16, $H_n = M$ or H_n is a P-primary submodule containing N for each positive integer n. □

References

[1] P.F. Smith, Uniqueness of primary decompositions, Turkish J. Math. 27 (2003), 425-434.

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