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## On Cauchy's Bound for Zeros of a Polynomial

*V. K. Jain*

### Abstract

In this note, we improve upon Cauchy's classical bound, and upon some recent bounds for the moduli of the zeros of a polynomial.

**Key Words:** Zeros, polynomials, upper bound, moduli, refinement.

### 1. Introduction and Statement of Results

Let

$$\begin{aligned} f(z) &= z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \dots + a_1z + a_0, \\ a_i &\neq 0, \text{ for at least one } i \in I, \\ I &= \{0, 1, 2, \dots, n-1\}, \end{aligned}$$

be a polynomial of degree  $n$ , with complex coefficients. Then, according to Cauchy's classical result [1], we have the following theorem.

#### Theorem A

$$Z[f(z)] \subset \overline{B}(\eta) \subset B(1+a),$$

where  $\eta$  is the unique positive root of the equation

$$Q(x) = 0,$$

$$Q(x) = x^n - |a_{n-1}|x^{n-1} - |a_{n-2}|x^{n-2} - \dots - |a_1|x - |a_0|, \quad (1)$$

$Z[f(z)]$  = the set of all zeros of the polynomial  $f(z)$ ,

$$B(r) = \{z : |z| < r\}, \bar{B}(r) = \{z : |z| \leq r\}.$$

and

$$a = \max_{i \in I} |a_i|. \quad (2)$$

Sun and Hsieh [2] obtained certain refinements of Cauchy's classical bound. They proved the next theorem.

**Theorem B**

$$Z[f(z)] \subset \bar{B}(\eta) \subset B(1 + \delta_1) \subset B(1 + \delta_2) \subset B(1 + a),$$

with

$$\eta < 1 + \delta_1 \leq 1 + \delta_2 \leq 1 + a,$$

where  $\delta_1$  is the unique positive root of the equation

$$Q_1(x) = 0,$$

$$Q_1(x) = x^3 + (2 - |a_{n-1}|)x^2 + (1 - |a_{n-1}| - |a_{n-2}|)x - a, \quad (3)$$

and

$$\delta_2 = \frac{1}{2}[(|a_{n-1}| - 1) + \sqrt{(|a_{n-1}| - 1)^2 + 4a}].$$

**Theorem C** Let

$$g_1(z) = (-1)^n f(z)f(-z),$$

$$h(z) = g_1(\sqrt{z}) = \sum_{i=0}^n b_i z^i, \text{ say,}$$

$$b = \max_{i \in I} |b_i|,$$

$$m = \max\{i : i \in I \ \& \ |b_i| = b\},$$

$$\tilde{b} = \max_{i \in I \sim \{m\}} |b_i|,$$

$$\alpha = \begin{cases} \min[(b/\tilde{b})^{1/(2(n-m-1))}, \{b(n-m-1)\}^{1/2(n-m)}]; \tilde{b} \neq 0, m \neq n-1 \text{ \& } b \geq 1 \\ 1; \text{ otherwise} \end{cases},$$

$$g_2(z) = \alpha^{-2n}(-1)^n f(\alpha z) f(-\alpha z),$$

$$C(z) = g_2(\sqrt{z}) = \sum_{i=0}^n c_i z^i, \text{ say, (with } c_n = 1, \text{ obviously),}$$

$$c = \max_{i \in I} |c_i|.$$

Then

$$Z[f(z)] \subset \bar{B}(\alpha\sqrt{\tilde{\eta}}) \subset B(\alpha\sqrt{1+\tilde{\delta}_1}) \subset B(\alpha\sqrt{1+\tilde{\delta}_2}) \subset B(\alpha\sqrt{1+c}), \quad (4)$$

with

$$\tilde{\eta} < 1 + \tilde{\delta}_1 \leq 1 + \tilde{\delta}_2 \leq 1 + c,$$

where  $\tilde{\eta}$  is the unique positive root of the equation

$$\tilde{Q}(x) = 0,$$

$\tilde{\delta}_1$  is the unique positive root of the equation

$$\tilde{Q}_1(x) = 0,$$

$$\tilde{Q}(x) = x^n - |c_{n-1}|x^{n-1} - |c_{n-2}|x^{n-2} - \dots - |c_1|x - |c_0|,$$

$$\tilde{Q}_1(x) = x^3 + (2 - |c_{n-1}|)x^2 + (1 - |c_{n-1}| - |c_{n-2}|)x - c,$$

and

$$\tilde{\delta}_2 = \frac{1}{2}\{(|c_{n-1}| - 1) + \sqrt{(|c_{n-1}| - 1)^2 + 4c}\}.$$

In this note, we have also obtained a refinement of Cauchy's classical bound and then obtained certain other similar bounds also. More precisely, we have proved the following theorem.

**Theorem 1**

$$Z[f(z)] \subset \bar{B}(\eta) \subset B(1 + \delta_0) \subset B(1 + \delta_1),$$

with

$$\eta < 1 + \delta_0 \leq 1 + \delta_1,$$

where  $\delta_0$  is the unique positive root of the equation

$$Q_0(x) = 0, \tag{5}$$

$$Q_0(x) = x^4 + (3 - |a_{n-1}|)x^3 + (3 - 2|a_{n-1}| - |a_{n-2}|)x^2 + (1 - |a_{n-1}| - |a_{n-2}| - |a_{n-3}|)x - a. \tag{6}$$

**Remark 1** It is obvious that Theorem 1 is a refinement of Theorem B and therefore, also a refinement of Cauchy's classical bound.

**Theorem 2**

$$Z[f(z)] \subset \bar{B}(\alpha\sqrt{\tilde{\eta}}) \subset B(\alpha\sqrt{1 + \tilde{\delta}_0}) \subset B(\alpha\sqrt{1 + \tilde{\delta}_1}),$$

with

$$\tilde{\eta} < 1 + \tilde{\delta}_0 \leq 1 + \tilde{\delta}_1,$$

where  $\tilde{\delta}_0$  is the unique positive root of the equation

$$\tilde{Q}_0(x) = 0,$$

$$\tilde{Q}_0(x) = x^4 + (3 - |c_{n-1}|)x^3 + (3 - 2|c_{n-1}| - |c_{n-2}|)x^2 + (1 - |c_{n-1}| - |c_{n-2}| - |c_{n-3}|)x - c.$$

**Remark 2** It is obvious that Theorem 2 is a refinement of Theorem C. Therefore, thinking of Theorem 1 and Theorem 2 together, we can say that we have got upper bounds for the moduli of the zeros of the polynomial  $f(z)$ , better than those obtained by Sun and Hsieh [2], and hence, also better than those obtained by Zilovic et al. [3], as suggested by Sun and Hsieh [2].

## 2. Proofs of the Theorems

**Proof of Theorem 1** That equation (5) has a unique positive root  $\delta_0$ , follows by the use of Descartes' rule of signs. Further,

$$\begin{aligned}
 Q(1 + \delta_0) &= (1 + \delta_0)^n - |a_{n-1}|(1 + \delta_0)^{n-1} - |a_{n-2}|(1 + \delta_0)^{n-2} - \\
 &\quad |a_{n-3}|(1 + \delta_0)^{n-3} - |a_{n-4}|(1 + \delta_0)^{n-4} - \dots \\
 &\quad \dots - |a_0|, \text{ (by (1))}, \\
 &\geq (1 + \delta_0)^n - |a_{n-1}|(1 + \delta_0)^{n-1} - |a_{n-2}|(1 + \delta_0)^{n-2} \\
 &\quad - |a_{n-3}|(1 + \delta_0)^{n-3} - a(1 + \delta_0)^{n-4} - \dots \\
 &\quad \dots - a(1 + \delta_0) - a, \text{ (by (2))}, \\
 &= (1 + \delta_0)^n - |a_{n-1}|(1 + \delta_0)^{n-1} - |a_{n-2}|(1 + \delta_0)^{n-2} \\
 &\quad - |a_{n-3}|(1 + \delta_0)^{n-3} - a \left\{ \frac{(1 + \delta_0)^{n-3} - 1}{\delta_0} \right\}, \\
 &> (1 + \delta_0)^{n-3} \left\{ (1 + \delta_0)^3 - |a_{n-1}|(1 + \delta_0)^2 - \right. \\
 &\quad \left. |a_{n-2}|(1 + \delta_0) - |a_{n-3}| - \frac{a}{\delta_0} \right\}, \\
 &= \frac{(1 + \delta_0)^{n-3}}{\delta_0} Q_0(\delta_0), \\
 &= 0,
 \end{aligned}$$

which implies

$$\eta < 1 + \delta_0.$$

Again,

$$\begin{aligned}
 Q_0(\delta_1) &= Q_0(\delta_1) - \delta_1 Q_1(\delta_1) - Q_1(\delta_1) \\
 &= \delta_1(a - |a_{n-3}|), \text{ (by (3) and (6))}, \\
 &\geq 0,
 \end{aligned}$$

thereby implying that

$$\delta_0 \leq \delta_1.$$

And now Theorem 1 follows, by using the fact that  $\eta$  is unique positive root of the equation

$$Q(x) = 0.$$

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*Proof of Theorem 2.* We can prove, as in the proof of Theorem 1, that

$$\begin{aligned}\tilde{\eta} &< 1 + \tilde{\delta}_0, \\ \tilde{\delta}_0 &\leq \tilde{\delta}_1,\end{aligned}$$

and then Theorem 2 follows by using

$$Z[f(z)] \subset \bar{B}(\alpha\sqrt{\tilde{\eta}}) \text{ (by (4)).}$$

### References

- [1] Cauchy, A.L.: Exercices de mathématique, in Oeuvres (2) Vol. 9, (1829), p. 122.
- [2] Sun, Y.J. and Hsieh, J.G.: A note on circular bound of polynomial zeros, *IEEE Trans. Circuits Syst. I* 43 (1996), 476-478.
- [3] Zilovic, M.S., Roytman, L.M., Combettes, P.L. and Swamy, M.N.S.: A bound for the zeros of polynomials, *ibid* 39 (1992), 476-478.

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