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p-Elastica in the 3-Dimensional Lorentzian Space Forms

Nevin Gürbüz

Abstract

R Huang worked the p-elastic in a Riemannian manifold with constant sectional curvature [1]. In this work, we solve the Euler-Lagrange equation by quadrature and study the Frenet equation of the p-elastica by using the Killing field in the three dimensional Lorentzian space forms.

1. Introduction

Definition 1.1 *Let L be a 3-dimensional Lorentzian space. If (x_1, x_2, x_3) and (y_1, y_2, y_3) are the components of X and Y with respect to an allowable coordinate system, then*

$$\langle X, Y \rangle |_L = x_1y_1 + x_2y_2 - x_3y_3$$

which is called a Lorentzian inner product. Furthermore, a Lorentz exterior product $X \times Y$ is given by

$$X \times Y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2).$$

Then, for any $x \in L^3$ it holds ([2])

$$\langle X \times Y, X \times Y \rangle = \langle X, Y \rangle^2 - \langle X, X \rangle \langle Y, Y \rangle.$$

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Definition 1.2 A semi-Riemannian manifold M has constant curvature if its sectional curvature function is constant. If M constant curvature C , then ([3])

$$R_{xy}z = C\{\langle z, x \rangle y - \langle z, y \rangle x\}.$$

Definition 1.3 The norm of $\vec{X} \in R_1^3$ is denoted by $\|\vec{X}\|$ and defined as ([3])

$$\|\vec{X}\| = \sqrt{|\langle \vec{X}, \vec{X} \rangle|}.$$

Theorem 1.1 Let $\gamma(s)$ be a unit speed curve in R_1^3 , s the arclength parameter. Consider the Frenet frame $\{T = \gamma', N, B\}$ attached to the curve $\gamma = \gamma(s)$ such that is T is the unit tangent vector field, N is the principal normal vector field and $B = T \times N$ is the binormal vector field. The Frenet -Serret formulas is given by

$$\begin{bmatrix} \nabla_T T = T' \\ \nabla_T N = N' \\ \nabla_T B = B' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 \kappa & 0 \\ -\varepsilon_1 \kappa & 0 & -\varepsilon_3 \tau \\ 0 & \varepsilon_2 \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (1.1)$$

where $\langle T, T \rangle = \varepsilon_1, \langle N, N \rangle = \varepsilon_2, \langle B, B \rangle = \varepsilon_3$. ∇ is the semi Riemannian connection on M and $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion functions of γ , respectively.

2. Killing Fields

This section is taken from [4], [5]. Let $\gamma(t)$ be a nonnull immersed curve in three dimensional Lorentzian space form M with sectional curvature C . Let us consider a variation of γ , $\gamma = \gamma(w, t) : (-\varepsilon, \varepsilon) \times I \longrightarrow M$ with $\gamma(0, t) = \gamma(t)$. Associated with γ are two vector fields along γ , $W(w, t) = (\partial\gamma/\partial w)(w, t)$ and $V(w, t) = (\partial\gamma/\partial t)(w, t)$. $W = (\partial\gamma/\partial w)(0, t)$ is the variation vector field along γ and $V = (\partial\gamma/\partial t)$ velocity vector field. $T = T(t)$ will be denoted the unit tangent vector field and the speed of γ will be $v = |\langle V, V \rangle|^{1/2}$. The curvature of γ is defined by $\kappa(t) = |\langle \nabla_T T, \nabla_T T \rangle|^{1/2}$. We will use the notation $W = W(w, t)$, $v = v(w, t)$. If s denote the arclength parameter of the t -curves, we write $v(w, s)$, $\kappa(w, s)$, etc., for the corresponding reparametrizations. Then $s \in [0, L]$, where L is the length of γ . With a direct computation is given following lemma.

Lemma 2.1 *By the above notation, the following formulas holds:*

$$\begin{aligned}
 (1) \quad & [W, V] = [W, vT] = 0 \\
 (2) \quad & (\partial v / \partial w)(0, t) = -\varepsilon_1 g v \quad \text{with } g = \langle \nabla_T W, T \rangle \\
 (3) \quad & (\partial \kappa^2 / \partial w)(0, t) = 2\varepsilon_2 \langle \nabla_T^2 W, \nabla_T T \rangle + 4\varepsilon_1 g \kappa^2 + 2\varepsilon_2 \langle R(W, T)T, \nabla_T T \rangle = 0. \\
 (4) \quad & (\partial \tau^2 / \partial w)(0, t) = -2\varepsilon_2 \left\langle \begin{array}{l} (1/\kappa) \nabla_T^3 W - (\kappa'/\kappa^2) \nabla_T^2 W + \varepsilon_1 (\varepsilon_2 \kappa + (C/\kappa)) \nabla_T W \\ -\varepsilon_1 (C\kappa'/\kappa^2) W, \tau B \end{array} \right\rangle
 \end{aligned} \tag{2.1}$$

where \langle, \rangle denotes the Lorentzian metric of M and $\kappa' = (\partial \kappa / \partial w)(0, t)$.

Let M be a complete, simply connected, Lorentzian space form and γ a nonnull immersed curve in M . Killing vector fields along γ are characterized by the equations

$$\frac{\partial v}{\partial w}(0, t) = \frac{\partial \kappa^2}{\partial w}(0, t) = \frac{\partial \tau^2}{\partial w}(0, t) = 0. \tag{2.2}$$

W is a Killing vector field along γ if and only if it satisfies the following conditions:

$$\begin{aligned}
 (1) \quad & \langle \nabla_T W, T \rangle = 0 \\
 (2) \quad & \langle \nabla_T^2 W, N \rangle + \varepsilon_1 C \langle W, N \rangle = 0 \\
 (3) \quad & \langle (1/\kappa) \nabla_T^3 W - (\kappa'/\kappa^2) \nabla_T^2 W + \varepsilon_1 (\varepsilon_2 \kappa + (C/\kappa)) \nabla_T W - \varepsilon_1 C (\kappa'/\kappa^2) W, \tau B \rangle = 0.
 \end{aligned} \tag{2.3}$$

3. Equilibrium Equations and the p-elastica

We take $P(\kappa)$ a polynomial of κ with degree ≥ 2 and consider the following curvature energy functional in Minkowski space:

$$\int_0^{L(w)} p(\kappa) ds|_{w=0}. \tag{3.1}$$

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Now, we want to compute the first derivative restriction of this curvature energy functional:

$$\begin{aligned} \frac{\partial}{\partial w} \int_0^{L(w)} p(\kappa) ds \Big|_{w=0} &= \frac{\partial}{\partial w} \int_0^{L(w)} p(\kappa) v dt \Big|_{w=0} \\ &= \int_I [p'(\kappa)W(\kappa)v + p(\kappa)\frac{\partial v}{\partial w}] dt \Big|_{w=0} \\ &= \int_0^L [p'(\kappa)(2\varepsilon_2 \langle \nabla_T^2 W, \nabla_T T \rangle + 4\varepsilon_1 g \kappa^2 + 2\varepsilon_2 \langle R(W, T)T, \nabla_T T \rangle) - p(\kappa)\varepsilon_1 g] ds, \end{aligned}$$

where $W(0, 0) = W(0, L) = 0, \nabla_T W(0, 0) = \nabla_T W(0, L) = 0$ and $L(w)$ is the arclength of $\gamma(w, t)$. Then we can give the first variational formula:

$$\begin{aligned} \frac{\partial}{\partial w} \int_0^{L(w)} p(\kappa) ds \Big|_{w=0} &= \int_0^L [\varepsilon_2 \langle R(W, T)T + \nabla_T^2 W, \frac{p'(\kappa)}{\kappa} \nabla_T T \rangle + \varepsilon_1 (2\kappa p'(\kappa) - p(\kappa))g] ds \\ &= \int_0^L \left\langle \varepsilon_2 \nabla_T^2 \left(\frac{p'(\kappa)}{\kappa} \nabla_T T \right) + \varepsilon_2 \frac{p'(\kappa)}{\kappa} C \nabla_T T + \varepsilon_1 \nabla_T [(2\kappa p'(\kappa) - p(\kappa))T], W \right\rangle ds. \end{aligned} \quad (3.2)$$

So, we obtain Euler-Lagrange equations:

$$E = \varepsilon_2 \nabla_T^2 \left(\frac{p'(\kappa)}{\kappa} \nabla_T T \right) + \varepsilon_2 \frac{p'(\kappa)}{\kappa} C \nabla_T T + \varepsilon_1 \nabla_T [(2\kappa p'(\kappa) - p(\kappa))T] = 0. \quad (3.3)$$

Definition 3.1 *A regular unit-speed curve is called a p -elastica if it satisfies the above Euler-Lagrange equation (3.3).*

We give the variational formulas by using Theorem 1.1:

$$\begin{aligned} E &= [\varepsilon_2 p'''(\kappa) \kappa'^2 + \varepsilon_2 p''(\kappa) \kappa'' + p'(\kappa)(-\varepsilon_3 \tau^2 + \varepsilon_1 \kappa^2 + C) \\ &\quad - \varepsilon_1 \varepsilon_2 p(\kappa) \kappa] N - \varepsilon_2 \varepsilon_3 (2p''(\kappa) \kappa' \tau + p'(\kappa) \tau') B \end{aligned} \quad (3.4)$$

Then, we obtain the Euler-Lagrange equation

$$\begin{aligned}
 & \varepsilon_2 p'''(\kappa) \kappa'^2 + \varepsilon_2 p''(\kappa) \kappa'' \\
 & + \varepsilon_2 p'(\kappa) (-\varepsilon_2 \varepsilon_3 \tau^2 + \varepsilon_1 \varepsilon_2 \kappa^2 + C) - \varepsilon_1 \varepsilon_2 p(\kappa) \kappa = 0, \\
 & -\varepsilon_2 \varepsilon_3 (2p''(\kappa) \kappa' \tau + p'(\kappa) \tau') = 0.
 \end{aligned} \tag{3.5}$$

For κ and τ is constant, from Eq. (3.5),

$$\varepsilon_2 p'(\kappa) (\varepsilon_1 \varepsilon_2 \kappa^2 - \varepsilon_2 \varepsilon_3 \tau^2 + \varepsilon_2 C) - \varepsilon_1 \varepsilon_2 p(\kappa) \kappa = 0. \tag{3.6}$$

In this situation, we can give the formula without intermediaries.

If κ is not constant, from the second of Eq. (3.5),

$$p'(\kappa)^2 \tau = k_1. \tag{3.7}$$

Here, k_1 is a constant. From the integral of the first of Eq. (3.5), we have,

$$\varepsilon_1 (\kappa p'(\kappa) - p(\kappa))^2 + \varepsilon_2 (p''(\kappa) \kappa')^2 + C p'(\kappa)^2 - \varepsilon_3 \frac{k_1^2}{p'(\kappa)^2} = k_2. \tag{3.8}$$

Here k_2 constant. Then, we can give the curvature $\kappa(s)$ by quadratures

$$\pm \sqrt{\frac{\varepsilon_2 p'(\kappa)^2 p''(\kappa)^2}{p'(\kappa)^2 (k_2 - C p'(\kappa)^2 - \varepsilon_1 (\kappa p'(\kappa) - p(\kappa))^2 \varepsilon_3 k_1^2)}} d\kappa = \int ds. \tag{3.9}$$

We construct the Killing field along the p-elastica $\gamma(s)$. Let this Killing field be of the form

$$W = h_1(s)T(s) + h_2(s)N(s) + h_3(s)B(s),$$

where the functions h_1, h_2 and h_3 must satisfy the following equations:

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$$\begin{aligned}
& \varepsilon_1 h'_1 - \kappa h_2 = 0 \\
& h_1 \kappa' + \varepsilon_2 h'_1 \kappa + h'_1 \kappa + \varepsilon_2 h''_2 + h_2(-\varepsilon_1 \kappa^2 - \varepsilon_3 \tau^2 + \varepsilon_1 \varepsilon_2 C) + 2h'_3 \tau + h_3 \tau' = 0 \quad (3.10) \\
& -\varepsilon_2 h_1(\kappa' \kappa \tau^2 + \kappa^2 \tau' \tau) + h'_2(-2\varepsilon_2 \varepsilon_3 \kappa \tau^2 - \tau \kappa \tau' - \tau^2 \kappa' - \varepsilon_2 \varepsilon_3 \tau' \kappa') + \\
& h_2(\varepsilon_1 \varepsilon_2 \tau^2 \kappa^3 + \varepsilon_2 \varepsilon_3 \tau^4 \kappa - \varepsilon_2 \varepsilon_3 \tau'' \tau \kappa - \varepsilon_1(\varepsilon_2 \kappa^2 + C)\kappa \tau^2 - \varepsilon_2 \varepsilon_3 \tau' \kappa' \tau) - \\
& 2\tau^2 \kappa h''_2 - 3\varepsilon_2 \kappa^2 \tau h'_1 + \varepsilon_3 \tau \kappa h'_3 + \varepsilon_3 \kappa' \tau h''_3 + \\
& h_3(-\varepsilon_2 \tau' \tau^2 \kappa - 2\tau^2 \tau' \kappa - \varepsilon_2 \kappa' \tau^3 - \varepsilon_1 \varepsilon_3 C \kappa' \tau) + \\
& h'_3(-3\varepsilon_2 \tau^3 \kappa + \varepsilon_1 \varepsilon_3(\varepsilon_2 \kappa^3 + C\kappa)\tau) + \\
& h'_3(-3\varepsilon_2 \tau^3 \kappa + \varepsilon_1 \varepsilon_3(\varepsilon_2 \kappa^3 + C\kappa)\tau) = 0
\end{aligned}$$

With aid these equations and the Euler-lagrange Eq.(3.5), we obtained that the vector fields

$$J_\gamma = \varepsilon_1(p'(k)k - p(k))T + p''(k)k'N - \varepsilon_3 p'(k)\tau B \quad (3.11)$$

and

$$H_\gamma = -\varepsilon_2 p'(k)B \quad (3.12)$$

are Killing along the p-elastica γ in 3-dimensional Lorentzian space form. The solutions of Eq. (3.10) constitute a six dimensional linear space. So we give the following theorem.

Theorem 3.1 *Let M a simply connected manifold with constant sectional curvature C in Lorentzian space form, and let γ be a p-elastica in M . Vector fields $J_\gamma = \varepsilon_1(p'(k)k - p(k))T + p''(k)k'N - \varepsilon_3 p'(k)\tau B$ and $H_\gamma = -\varepsilon_2 p'(k)B$ can expanded to Killing fields J'_γ and H'_γ on M .*

In 3-dimensional Lorentzian space form, we construct a system of cylindrical coordinates using the Killing fields J_γ and H_γ . The Euler Lagrange equation and its first integral intimate are written

$$\nabla_T J_\gamma = -C p'(k)N \quad (3.13)$$

and

$$\begin{aligned}\|J_\gamma\|^2 &= |\langle J_\gamma, J_\gamma \rangle| \\ &= |\varepsilon_1(p'(k)k - p(k))^2 + \varepsilon_2(p''(k)k')^2 - \varepsilon_3(p'(k)\tau)^2| = |k_2 - Cp'(k)^2|.\end{aligned}\quad (3.14)$$

In R_1^3 , $\nabla_T J_\gamma = 0$. Then the Killing field J_γ is a translation field. We can find one coordinate field $\frac{\partial}{\partial z} = \frac{J_\gamma}{|\langle J_\gamma, J_\gamma \rangle|^{1/2}}$. Due to

$$\langle J_\gamma, H_\gamma \rangle = \varepsilon_2 p'(k)^2 \tau = \varepsilon_2 k_1, \quad (3.15)$$

H_γ shows a rotation along z direction.

$$J_1 = \varepsilon_2 J_\gamma - \frac{1}{k_1} \langle J_\gamma, J_\gamma \rangle H_\gamma \quad (3.16)$$

is a rotation field perpendicular to J_γ . So, for some normalization factor, we have $\frac{\partial}{\partial \varphi} = QJ_1$. Hence

$$\frac{\partial}{\partial r} = \frac{J_\gamma \times B}{|\langle J_\gamma \times B, J_\gamma \times B \rangle|^{1/2}} \quad (3.17)$$

is given. We can write the unit tangent vector as

$$T = r_s(\partial/\partial r) + \varphi_s(\partial/\partial \varphi) + z_s(\partial/\partial z),$$

with

$$r_s = \left\langle T, \frac{\partial}{\partial r} \right\rangle = \frac{\langle T, J_\gamma \times B \rangle}{|\langle J_\gamma \times B, J_\gamma \times B \rangle|^{1/2}} = -\frac{\varepsilon_1 p'(k)p''(k)k'}{|\varepsilon_3 k_2 p'(k)^2 + k_1^2|^{1/2}}, \quad (3.18)$$

$$z_s = \left\langle T, \frac{\partial}{\partial z} \right\rangle = \frac{p'(k)k - p(k)}{|\varepsilon_3 k_2 p'(k)^2 + k_1^2|^{1/2}}. \quad (3.19)$$

QJ_1 has the proper length at the maxima of $\kappa(s)$. Then the length of $\partial/\partial \varphi$ at such a point $\gamma(s)$ is $r = r(s_0)$, the reciprocal of the curvature κ_0 of the circle $r = r(s_0)$, $z = z(s_0)$. At this point, T has vertical component

$$\left\langle T, \frac{J_\gamma}{|\langle J_\gamma, J_\gamma \rangle|^{1/2}} \right\rangle = \frac{p'(k)k - p(k)}{|\langle J_\gamma, J_\gamma \rangle|^{1/2}}$$

and the horizontal component is

$$-\frac{p'(k)\tau}{|\langle J_\gamma, J_\gamma \rangle|^{1/2}}.$$

Note that

$$\begin{aligned} \kappa_0 &= -\frac{|\langle J_\gamma, J_\gamma \rangle|^{1/2}}{p'(k)\tau} \left| \nabla_T \left(\frac{\partial/\partial\varphi}{|\langle \partial/\partial\varphi, \partial/\partial\varphi \rangle|^{1/2}} \right) \right| \\ &= -\frac{|\langle J_\gamma, J_\gamma \rangle|^{3/2}}{k_1 |\langle J_1, J_1 \rangle|}. \end{aligned} \quad (3.20)$$

Hence,

$$Q = \frac{k_1}{|k_2|^{3/2}}$$

and

$$\varphi_s = \frac{\langle T, (\partial/\partial\varphi) \rangle}{|\langle \partial/\partial\varphi, \partial/\partial\varphi \rangle|} = \frac{\varepsilon_2 |k_2|^{3/2} (p'(k)k - p(k))}{k_1 |\langle J_1, J_1 \rangle|}. \quad (3.21)$$

So, we give the following theorem.

Theorem 3.2 *Let (r, φ, z) be cylindrical coordinates given above in R_1^3 , and $\gamma(s) = (r(s), \varphi(s), z(s))$. Then we give*

$$r_s = -\frac{\varepsilon_1 p'(k)p''(k)k'}{|\varepsilon_3 k_2 p'(k)^2 + k_1^2|^{1/2}}, \quad z_s = \frac{p'(k)k - p(k)}{|\varepsilon_3 k_2 p'(k)^2 + k_1^2|^{1/2}}, \quad \varphi_s = \frac{\varepsilon_2 |k_2|^{3/2} (p'(k)k - p(k))}{k_1 |\langle J_1, J_1 \rangle|}.$$

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