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Abstract

We show that the completion of a uniform hermitian p-normed algebra is a commutative C^*-algebra.

Key Words: Algebra involution, p-norm, uniform algebra, hermitian p-normed algebra, C^*-algebra.

Preliminaries and Introduction

Let E be a complex algebra. A linear p-norm on E, $0 < p \leq 1$, is a non-negative function $x \longmapsto \|x\|_p$ such that $\|x\|_p = 0$ if and only if $x = 0$, $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ and $\|\lambda x\|_p = |\lambda|^p \|x\|_p$, for all x, y in E. By a p-normed algebra $\left(E, \|.\|_p\right)$, we mean an algebra E endowed with a linear p-norm $\|.\|_p$ such that $\|xy\|_p \leq \|x\|_p \|y\|_p$, for all $x, y \in E$. A complete p-normed algebra is called a p-Banach algebra. For a unitary p-normed algebra $\left(E, \|.\|_p\right)$, we denote by $G(E)$ the group of invertible elements in E. A p-normed algebra $\left(E, \|.\|_p\right)$ is called a Q-algebra if $G(E)$ is open. A uniform p-norm, on E, is an algebra p-norm $\|.\|_p$ satisfying $\left\|x^2\right\|_p = \|x\|_p^2$, for every $x \in E$. A vector involution $x \longmapsto x^*$ ([1]) on a complex algebra E is said to be an algebra involution if $(xy)^* = y^*x^*$, for every $x, y \in E$. An element a of E is said to be hermitian (resp., normal) if $a = a^*$ (resp., $a^*a = aa^*$). We designate by $H(E)$ the set of hermitian elements of E. Let $\left(E, \|.\|_p\right)$,

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$0 < p \leq 1$, be a p-normed algebra and $a \in E$. The radius of boundedness β of an element a is defined by $\beta(a) = \inf\left\{\alpha > 0 : \left(\alpha^{-1}a\right)^n,\ n = 1, 2, ..., \text{ is bounded}\right\}$, with $\inf \emptyset = +\infty$. The reader can prove that $\beta(a) = \lim_{n \to +\infty} \|a^n\|_p^{\frac{1}{np}}$, for every $a \in E$. Throughout this paper, all algebras considered will be associative, complex and with a unit element. We denote Pták's function by $|.|$, that is $|a|^2 = \beta(a^*a)$, for every $a \in E$. The spectrum and spectral radius of an element a of E will be denoted by $Sp_E a$ and $\rho_E(a)$ respectively, where $Sp_E a = \{\lambda \in C : \lambda - a$ is not invertible in $E\}$ and $\rho_E(a) = \sup\{|z| : z \in Sp_E a\}$.

Let $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, be a uniform p-normed Q-algebra. In this paper, we give Proposition 2.1 below, in the a p-normed case, as a version of Theorem 1 (ii) of S. J. Bhatt and D. J. Karia in [2, p. 499]. More precisely, we show that if $\|.\|_q$, $0 < q \leq 1$, is a q-norm on E such that $\left(E, \|.\|_q\right)$ is a uniform Q-algebra, then $\|x\|_q^{\frac{1}{q}} = \|x\|_p^{\frac{1}{p}}$, for every $x \in E$. We also prove Proposition 3.1 below, in the non complete p-normed case, a version of the celebrated Theorem of V. Ptàk [7, p. 267], on the characterization of hermiticity on Banach algebras with not necessarily continuous-involution, in terms of the (so called Ptàk) inequality $\beta(a) \leq |a| = \beta(a^*a)^{\frac{1}{2}}$, for every $a \in E$. In the main result (Theorem 3.4) of this paper, we prove that the completion of a uniform, hermitian p-normed algebra is a commutative C^*-algebra. A counter-example (see Remark 3.6) shows that hermiticity is not superfluous.

1. Some General Results

Proposition 1.1 *Let* $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, *be a p-normed algebra. Then*

1. $\beta(a) \leq \rho_E(a)$, *for every* $a \in E$.

2. *If* $\left(E, \|.\|_p\right)$ *is a Q-algebra, then* $\rho_E(a) \leq \|a\|_p^{\frac{1}{p}}$, *for every* $a \in E$.

Proof. **1.** It follows from the equality

$$\beta(a) = \sup\left\{|\lambda| : \lambda \in Sp_{\widehat{Ep}}a\right\},$$

where $Sp_{\widehat{Ep}}a$ is the spectrum of a in the completion $\widehat{E}_p$ of $(E, \|\cdot\|_p)$.

2. The proof is the same as in the normed case. □

Proposition 1.2 *Let* $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, *be a p-normed algebra. The following are equivalent:*

1. $\left(E, \|.\|_p\right)$ *is a uniform p-normed algebra.*

2. $\beta(a) = \|a\|_p^{\frac{1}{p}}$, *for every* $a \in E$.

Proof. **1.** $\Longrightarrow$ **2.** By iteration, one has $\left\|a^{2^n}\right\|_p = \|a\|_p^{2^n}$, for all n. Hence $\beta(a) = \lim\limits_{n \to +\infty} \left\|a^{2^n}\right\|_p^{\frac{1}{2^n p}} = \|a\|_p^{\frac{1}{p}}$, for every $a \in E$.

2. $\Longrightarrow$ **1.** Since $\beta(a^2) = \beta(a)^2$, for every $a \in E$, the reader can prove that **2)** implies the uniformity of the algebra $\left(E, \|.\|_p\right)$. □

Let $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, be a p-normed algebra. Denote by $\widehat{E}_p$ the completion of the p-normed algebra $\left(E, \|.\|_p\right)$. The p-norm in $\widehat{E}_p$ will also be designated by $\|.\|_p$.

Definition 1.3. A p-normed algebra $\left(E, \|.\|_p\right)$ with involution $x \longmapsto x^*$ is said to be hermitian if $Sp_{\widehat{E_p}}(h) \subset R$, for every $h \in H(E)$, where $Sp_{\widehat{E_p}}(h)$ is the spectrum of h in $\hat{E}_p$.

It is clear that if $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, is hermitian in the classical sense (the spectrum of every hermitian element is real), then it is hermitian in the sense of definition 1.3. The converse is false in general as the following example shows.

Example 1.4. Let B be a radical and involutive Banach algebra in which 0 is the only nilpotent element (e.g. $L^1([0,1])$). Let $x_0 \neq 0$ be a hermitian element of B and let A be the algebra $C[x_0]$ endowed with the induced norm and involution. Then x_0 is not an algebraic element in A. Indeed, in the contrary case, there exists $f \in C[X]$, $f \neq 0$, such that $f(x_0) = 0$. Write $f = X^r g$, where $g \in C[X]$, and $g(0) \neq 0$. Then $x_0^r g(x_0) = 0$. Since $g(0) \neq 0$ and B is a radical Banach algebra, it follows that $g(x_0)$ is invertible. Whence $x_0^r = 0$, which is impossible, since 0 is the only nilpotent element of B. Consider the map $\varphi : A \longrightarrow XC[X]$: $\sum_{i=1}^n a_i x_0^i \longmapsto \sum_{i=1}^n a_i X^i$. It is easy to verify that φ is an isomorphism of A into $XC[X]$. Since the spectrum of every non-zero element of $XC[X]$ equals to C, it follows that

$$Sp_A x = C, \text{ for every } x \in A,\ x \neq 0.$$

On the other hand, since B is a radical Banach algebra, $x^n \underset{n}{\longrightarrow} 0$, for every $x \in A$. Thus any non zero character of A should be no continuous. This implies that the completion $\hat{A}$, of the normed algebra A, is a radical Banach algebra. Whence

$$Sp_{\hat{A}} x = \{0\}, \text{ for every } x \in \hat{A}.$$

To give a non trivial example, let E be any hermitian Banach algebra. Consider the algera $A \times E$ with usual operations and involution. Then the algebra $A \times E$ is hermitian in the sense of definition 1.3 but not hermitian in the classical sense.

2. Uniform p-Normed Algebras

The structure of a uniform p-normed, $0 < p \leq 1$, Q-algebra is unique as the following result shows.

Proposition 2.1 *Let $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, be a p-normed Q-algebra. If $\|.\|_q$, $0 < q \leq 1$, is a uniform q-norm on E, then*

1. $\|x\|_q^{\frac{1}{q}} \leq \|x\|_p^{\frac{1}{p}}$, *for every $x \in E$,*
2. *If moreover, $\|.\|_p$ is a uniform p-norm and $\|.\|_q$ is a Q-algebra q-norm, then $\|x\|_q^{\frac{1}{q}} = \|x\|_p^{\frac{1}{p}}$, for every $x \in E$.*

Proof. **1.** Since $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, is a p-normed Q-algebra, $\rho_E(x) \leq \|x\|_p^{\frac{1}{p}}$, for every $x \in E$. On the other hand, denote by $\widehat{E}_p$ the completion of the q-normed algebra $\left(E, \|.\|_q\right)$. Then, for every $x \in E$, we have

$$\|x\|_q^{\frac{1}{q}} = \lim_{n \to +\infty} \left[\left(\left\| x^{2^n} \right\|_q \right)^{\frac{1}{2^n}} \right]^{\frac{1}{q}} = \sup \left\{ |\lambda| : \lambda \in Sp_{\widehat{E_q}} x \right\} \leq \rho_E(x).$$

Thus $\|x\|_q^{\frac{1}{q}} \leq \|x\|_p^{\frac{1}{p}}$, for every $x \in E$.

2. By hypothesis, we have $\|x\|_p^{\frac{1}{p}} \leq \rho_E(x)$, for every $x \in E$, for the algebra $\left(E, \|.\|_p\right)$ is uniform. Whence, by **1)**,

$$\|x\|_p^{\frac{1}{p}} = \rho_E(x), \text{ for every } x \in E.$$

Now since $\|.\|_q$ is a Q-algebra q-norm on E, $\rho_E(x) \leq \|x\|_q^{\frac{1}{q}}$, for every $x \in E$. This implies that $\|x\|_p^{\frac{1}{p}} \leq \|x\|_q^{\frac{1}{q}}$, for every $x \in E$. Hence

$$\|x\|_p^{\frac{1}{p}} = \|x\|_q^{\frac{1}{q}}, \text{ for every } x \in E.$$

□

An immediate consequence ($p = 1$) of Proposition 2.1 is the following.

Corollary 2.2 *Let $(E, \|.\|)$ be a Q-normed algebra. If $\|.\|'$ is a uniform norm on E, then*

1. *$\|x\|' \leq \|x\|$, for every $x \in E$,*
2. *If moreover, $\|.\|$ is a uniform norm and $\|.\|'$ is a Q-algebra norm, then $\|x\|' = \|x\|$, for every $x \in E$.*

3. Uniform Hermitian p-Normed Algebras

In [7, p. 267], V. Ptàk has proved that $\rho_E(a) \leq \rho_E(a^*a)^{\frac{1}{2}}$, for every $a \in E$, is the fundamental inequality in the theory of hermitian Banach algebras. In the non complete p-normed case, an analog of the preceding inequality is the following.

Proposition 3.1 *Let $\left(E, \|.\|_p\right)$, $0 < p \leq 1$ be a p-normed algebra with involution $x \longmapsto x^*$. Then the following conditions are equivalent:*

1. *E is hermitian.*
2. *$\beta(a) \leq |a|$, for every $a \in E$.*

Proof. **1.** $\Longrightarrow$ **2.** The inequality of **2.** is equivalent the following implication

$$(\lambda \in C, |\lambda| > |x|) \Longrightarrow \lambda - x \text{ is invertible in } \widehat{E_p}. \tag{3.1}$$

It suffices, however, to prove the following weaker implication:

$$(\lambda \in C, |\lambda| > |x|) \Longrightarrow \lambda - x \text{ has a left inverse in } \widehat{E_p}. \tag{3.2}$$

Indeed, since

$$|x|^2 = \beta(x^*x) = \sup\left\{|z| : z \in Sp_{\widehat{E_p}}(x^*x)\right\} = \sup\left\{|z| : z \in Sp_{\widehat{E_p}}(xx^*)\right\} = \beta(xx^*) = |x^*|^2,$$

the assumption $|\lambda| > |x|$ gives $\left|\overline{\lambda}\right| > |x^*|$ so that $\overline{\lambda} - x^*$ has a left inverse in $\widehat{E_p}$. Hence both $(\lambda - x)$ and $(\lambda - x)^*$ have left-inverse in $\widehat{E_p}$. Thus $\lambda - x$ has an inverse in $\widehat{E_p}$. Let us now show (3.2). We have

$$\begin{aligned} \left(\overline{\lambda} + x^*\right)(\lambda - x) &= |\lambda|^2 - x^*x + \lambda x^* - \overline{\lambda}x \\ &= \left(|\lambda|^2 - |x|^2\right) + \left(|x|^2 - x^*x\right) + i\left(i\overline{\lambda}x - i\lambda x^*\right). \end{aligned}$$

Since $|x|^2 - x^*x \geq 0$ and $i\overline{\lambda}x - i\lambda x^* \in H(E)$, it follows that

$$Sp_{\widehat{E_p}}\left[\left(|x|^2 - x^*x\right) + i\left(i\overline{\lambda}x - i\lambda x^*\right)\right] \subset R^+ + iR$$

and

$$ReSp_{\widehat{E_p}}\left[\left(\overline{\lambda} + x^*\right)(\lambda - x)\right] \geq |\lambda|^2 - |x|^2 > 0,$$

so that $\left(\overline{\lambda} + x^*\right)(\lambda - x)$ is invertible in $\widehat{E_p}$, hence also $(\lambda - x)$ is invertible in $\widehat{E_p}$.

2. $\Longrightarrow$ **1.** Suppose that there exists $a \in H(E)$ and $\gamma + i\delta \in Sp_{\widehat{E_p}}a$, where $\gamma, \delta \in R$, $\delta \neq 0$. Then, for every real t, we have

$$\gamma + i\delta + it \in Sp_{\widehat{E_p}}(a + it) \text{ and } \beta(a + it)^2 \geq \gamma^2 + (\delta + t)^2.$$

On the other hand

$$|a + it|^2 = \beta\left[(a - it)(a + it)\right] = \beta\left(a^2 + t^2\right) \leq \beta(a^2) + t^2.$$

Thus

$$(\delta + t)^2 \leq \beta\,(a + it)^2 \leq |a + it|^2 \leq \beta(a^2) + t^2.$$

So that $2\delta t + \delta^2 \leq \beta(a)^2$ for each real number t. This is impossible unless $\delta = 0$. Hence $Sp_{\widehat{E_p}}\, a \subset R$. □

It is known that every uniform normed algebra is automatically semi-simple and commutative (see [6], p. 275 for any uniform *lmc* algebra). This result remain valid for p-normed algebras, $0 < p \leq 1$, as the following result shows.

Lemma 3.2 *Let* $\left(E, \|.\|_p\right)$, $0 < p \leq 1$ *be a p-normed algebra such that*

$$\left\|a^2\right\|_p = \|a\|_p^2 \text{ for every } a \in E, \tag{3.3}$$

then E is commutative and semi-simple.

Proof. Observe first that, from hypothesis, one has

$$\|a\|_p^{\frac{1}{p}} \leq \rho_E(a), \text{ for every } a \in E.$$

And since the Jacobson's radical of E is contained in the set of quasi-nilpotent elements of E, we deduce that the algebra E is semi-simple. It remains to show that the algebra E is commutative. Let $\widehat{E}_p$ be the completion of E with respect to the p-norm $\|.\|_p$. The p-norm in $\widehat{E}_p$ will also be designated by $\|.\|_p$. Then we have

$$\left\|a^2\right\|_p = \|a\|_p^2 \text{ for every } a \in \widehat{E}_p. \tag{3.4}$$

In the p-Banach algebra $\left(\widehat{E}_p, \|.\|_p\right)$, the spectral radius $\rho_{\widehat{E}_p}$ satisfies, for every $a \in \widehat{E}_p$,

$$\rho_{\widehat{E}_p}(a)^p = \lim_n \left\|a^{2^n}\right\|_p^{2^{-n}} = \|a\|_p\,, \tag{3.5}$$

which yields in particular that $\widehat{E}_p$ is semi-simple. Now we will show that the algebra $\widehat{E}_p$ is commutative. By the previous relation, we have

$$\|ab\|_p = \|ba\|_p \text{ for every } a \in \left(\widehat{E}_p\right)^1 = \widehat{E}_p \oplus C \text{ and } b \in \widehat{E}_p. \tag{3.6}$$

For any $x \in \widehat{E}_p$, put

$$\|x\| = \inf \sum_{i=1}^{n} \|x_i\|_p^{\frac{1}{p}} ,$$

where the infimum is taken over all decompositions of $x = \sum_{i=1}^{n} x_i$, $x_i \in \widehat{E}_p$. By [8, p. 262], $\|.\|$ is a submultiplicative semi-norm on $\widehat{E}_p$ such that

$$\rho_{\widehat{E}_p}(x) \leq \|x\| \leq \|x\|_p^{\frac{1}{p}} , \text{ for every } x \in \widehat{E}_p. \tag{3.7}$$

It follows from (3.6) and (3.7), that

$$\|xy\| \leq \|yx\|_p^{\frac{1}{p}} , \text{ for every } x \in \left(\widehat{E}_p\right)^1 \text{ and } y \in \widehat{E}_p. \tag{3.8}$$

If $\widehat{E}_p$ is not unitary, consider its unitization $\left(\widehat{E}_p\right)^1$. For $a, b \in \widehat{E}_p$, consider the map f defined, on C, by

$$f(\lambda) = (\exp(\lambda a))\, b \exp(-\lambda a).$$

One checks that, for any φ in the topological dual of $\left(\widehat{E}_p, \|.\|\right)$, $\varphi \circ f$ is holomorphic. It is also bounded by (3.8). By Liouville's theorem $\varphi \circ f$ is constant, and so the coefficient of λ in the power series expansion of $\varphi \circ f$ is zero, i.e., $\varphi\,(ab - ba) = 0$. Whence $ab - ba \in Rad\widehat{E}_p$, by Hahn-Banach theorem and (3.7). But $\widehat{E}_p$ is semi-simple, hence $ab = ba$. Whence the commutativity of $\hat{E}_p$. It follows that E is commutative too. This completes the proof. □

Remark 3.3 In [2, p. 499], S. J. Bhatt and D. J. Karia have proved that a seminorm with square property on a commutative algebra is automatically submultiplicative. In [4, p. 52], H. V. Dedania has proved the same result in the noncommutative case. In [5], we extend this result to the p-seminorm case.

We will prove the following theorem.

Theorem 3.4 *(The main result) Let* $\left(E, \|.\|_p\right)$, $0 < p \leq 1$ *be a hermitian p-normed algebra such that*

$$\left\|a^2\right\|_p = \|a\|_p^2 \text{ for every } a \in E,$$

then the completion of $\left(E, \|.\|_p\right)$ *is a commutative* C^**-algebra.*

Proof. By Lemma 3.2, the algebra $\left(E, \|.\|_p\right)$ is commutative and semi-simple. It follows that $\left(\widehat{E}_p, \|.\|_p\right)$ is commutative and semi-simple too and hence $\|.\|_p^{\frac{1}{p}}$ is an algebra norm. It remains to show that $\left(\widehat{E}_p, \|.\|_p\right)$ is a C^*-algebra. Since $\widehat{E}_p$ is semi-simple, the involution $x \longmapsto x^*$ is continuous. Let $\alpha > 0$ such that

$$\|x^*\|_p \leq \alpha \|x\|_p, \text{ for every } x \in \widehat{E}_p.$$

Now since E is hermitian, by proposition 2.1, we have

$$\beta(a) \leq |a|, \text{ for every } a \in E.$$

But

$$|a|^2 = \beta(a^*a) = \rho_E(a^*a) \leq \|a^*a\|_p^{\frac{1}{p}} \leq \alpha \|a\|^{\frac{2}{p}}, \text{ for every } a \in E.$$

Hence

$$|a| \leq \alpha^{\frac{1}{2p}} \|a\|_p^{\frac{1}{p}}, \text{ for every } a \in E.$$

and

$$\beta(a) \leq \alpha^{\frac{1}{2p}} \|a\|_p^{\frac{1}{p}}, \text{ for evrey } a \in E.$$

Whence the continuity of β and $|.|$. This implies that the inequality $\beta(a) \leq |a|$, for every $a \in E$, can be extended to the completion $\widehat{E}_p$ and we have

$$\beta(a) \leq |a|, \text{ for every } a \in \hat{E}_p.$$

So, it follows from Proposition 2.1 that, $\left(\widehat{E_p}, \ \|.\|_p^{\frac{1}{p}}\right)$ is a hermitian Banach algebra. Now, since the involution is continuous, the elements $\exp(ih)$ are normal. So, by (4), we have

$$\|\exp(ih)\|_p^{\frac{1}{p}} = \rho_{\widehat{E}_p}(\exp(ih)) = 1$$

for every hermitian element h of $\widehat{E_p}$. Finally, by [7, p. 284], $\left(\widehat{E_p},\ \|.\|_p^{\frac{1}{p}}\right)$ is a C^*-algebra. This completes the proof. □

An immediate consequence $(p = 1)$ of Theorem 3.4. is the following corollary

Corollary 3.5 *Let* $(E, \|.\|)$ *be a hermitian normed algebra such that*

$$\left\|a^2\right\| = \|a\|^2 \text{ for every } a \in E.$$

Then the completion of $(E, \|.\|)$ *is a commutative* C^**-algebra.*

Remark 3.6 One may ask if the hermiticity of the algebra E, in theorem 3.3, can be omitted. The answer is negative, as the following example shows.

Let $E = C \times C$ with pointwise addition and product. Define scalar multiplication, involution and p-norm, $0 < p \leq 1$, in E by

$$\begin{aligned} \lambda (z_1, z_2) &= \left(\lambda z_1, \overline{\lambda} z_2\right) \\ (z_1, z_2)^* &= (z_2, z_1) \\ \|(z_1, z_2)\|_p &= \max\left(|z_1|^p, |z_2|^p\right). \end{aligned}$$

It is easy to verify that $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, is a commutative p-Banach algebra with involution but $\left(E, \|.\|_p^{\frac{1}{p}}\right)$ is not a C^*-algebra. The algebra E is not hermitian. Indeed let $h = (i, i)$ be an element of E. It is clear that $h \in H(E)$ and $Sp_E h = \{i, -i\}$ is not contained in R.

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