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A Fractal Example of a Continuous Monotone Function with Vanishing Derivatives on a Dense Set and Infinite Derivatives on Another Dense Set

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Abstract

Inspired by the theory of analysis on fractals, we construct an example of a continuous, monotone function on an interval, which has vanishing derivatives on a dense set and infinite derivatives on another dense set. Although such examples could be constructed by classical means of probability and measure theory, this one is more elementary and emerges naturally as a byproduct of some new fractal constructions.

Key words and phrases: Sierpinki Gasket, harmonic function.

1. Introduction

Fractal analysis has been a rising field in analysis in the last decade. One of the key features of this new theory has been the invention of Laplacians and harmonic functions on fractals [1]–[4]. The typical example is the Sierpinski-gasket (SG) and the by-now classical construction of (real-valued) harmonic functions on this fractal goes as follows: Let α_0, α_1, α_2 be arbitrary real numbers and set $F(p_i) = \alpha_i$ on vertices of SG. Then define $F(q_0) = (\alpha_0 + 2\alpha_1 + 2\alpha_2)/5$, $F(q_1) = (2\alpha_0 + \alpha_1 + 2\alpha_2)/5$,
$F(q_2) = (2\alpha_0 + 2\alpha_1 + \alpha_2)/5$. (see Figure 1).

Applying this procedure iteratively one gets a function on the set of junction points

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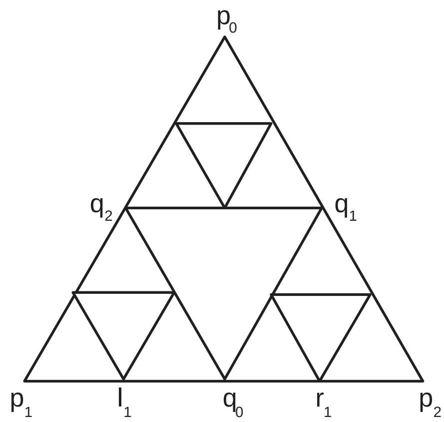

Figure 1. First stages of the SG.

and extending this continuously to the whole of SG one obtains a so-called harmonic function on SG with very nice properties.

The restrictions of this function to line-segments inside SG (for example to $[p_1p_2]$) are worth understanding. This restriction is known to be monotone (or piece-wise monotone on two pieces) [1] and hence differentiable almost everywhere. But nothing is known more specificly about differentiability at given points. We will show below that there is a dense set (of the segment $[p_1p_2]$) where the derivative vanishes and another dense set where the derivative exists improperly.

To simplify the analysis, we give a direct construction of this function (the restriction of the harmonic function on SG to the segment $[p_1p_2]$) independent of the above-mentioned harmonic theory on SG. This is a useful procedure to study the restrictions of harmonic functions to segments and amounts to expressing the values at l_1 and r_1 not in terms of the values at p_0, p_1 and p_2, but in terms of values at p_1, q_0, p_2 as used in [1]. To this end, let α, β and γ be real numbers, satisfying the inequalities

$$\alpha < \beta < \gamma, \qquad \frac{1}{4} < \frac{\gamma - \beta}{\beta - \alpha} < 4. \tag{1.1.1}$$

We define the function $f : [0,1] \longrightarrow \mathbb{R}$ first iteratively on points of the form $k/2^n$ and then extend it to whole of $[0,1]$. For convenience let

$$J = \{\frac{k}{2^n} \,|\, 0 \le k \le 2^n, \quad k, n \in \mathbb{N}\}.$$

First we set $f(0) = \alpha, f(1/2) = \beta$ and $f(1) = \gamma$. Then we define

$$f(\frac{1}{4}) = \frac{8\alpha + 20\beta - 3\gamma}{25}, \qquad f(\frac{3}{4}) = \frac{8\gamma + 20\beta - 3\alpha}{25}. \tag{1.1.2}$$

Now consider the interval $[0,1/2]$. The function f is already defined on the endpoints and midpoint of this interval and we apply the above rule again:

$$f(\frac{1}{8}) = \frac{8f(0) + 20f(\frac{1}{4}) - 3f(\frac{1}{2})}{25}, \qquad f(\frac{3}{8}) = \frac{8f(\frac{1}{2}) + 20f(\frac{1}{4}) - 3f(0)}{25}.$$

Similarly, applying the same rule to $[1/2,1]$ we define

$$f(\frac{5}{8}) = \frac{8f(\frac{1}{2}) + 20f(\frac{3}{4}) - 3f(1)}{25}, \qquad f(\frac{7}{8}) = \frac{8f(1) + 20f(\frac{3}{4}) - 3f(\frac{1}{2})}{25}.$$

In the next step we apply this procedure to the four intervals $[0,1/4]$, $[1/4,1/2], [1/2,3/4], [3/4,1]$ making the function defined at all $k/2^3 \quad (0 \le k \le 8)$.

Continuing this, in the n^{th} step we apply the algorithm to the subintervals of the form

$$[\frac{2k}{2^n}, \frac{2k+2}{2^n}], \quad (0 \le k \le 2^{n-1} - 1) \tag{1.1.3}$$

getting the values

$$f(\frac{4k+1}{2^{n+1}}) = \frac{8\alpha' + 20\beta' - 3\gamma'}{25} \quad \text{and} \quad f(\frac{4k+3}{2^{n+1}}) = \frac{8\gamma' + 20\beta' - 3\alpha'}{25}$$

for

$$f(\frac{2k}{2^n}) = \alpha', \quad f(\frac{2k+1}{2^n}) = \beta', \quad f(\frac{2k+2}{2^n}) = \gamma'.$$

In this way the function f becomes defined on all of J.

The following property can easily be verified.

Lemma 1.1 *Let* $\lambda = f(1/4)$ *and* $\mu = f(3/4)$. *Then the triples* (α, λ, β) *and* (β, μ, γ) *satisfy the same inequalities as in* (1.1.1) *also, i.e.*

$$\alpha < \lambda < \beta \quad , \qquad \frac{1}{4} < \frac{\beta - \lambda}{\lambda - \alpha} < 4$$

$$\beta < \mu < \gamma \quad , \qquad \frac{1}{4} < \frac{\gamma - \mu}{\mu - \beta} < 4$$

($f(1/4)$ *and* $f(3/4)$ *are defined in* (1.1.2)*).*

Corollary 1.2 f *is strictly monotone on* J.

2. Continuity of the Function

Now we shall show that f is continuous on J. First we prove the following lemma.

Lemma 2.1 *Let* $l_m = 1/2 - 1/2^{m+1}$, $\quad r_m = 1/2 + 1/2^{m+1}$, $\qquad (m = 1, 2, ...)$. *Then*

$$f(\frac{1}{2^m}) = \frac{1 + 5^m - 2 \cdot 3^m}{5^m}\alpha + \frac{3^m - 1}{2 \cdot 5^{m-1}}\beta + \frac{3 - 3^m}{2 \cdot 5^m}\gamma, \tag{2.2.1}$$

$$f(1 - \frac{1}{2^m}) = \frac{3 - 3^m}{2 \cdot 5^m}\alpha + \frac{3^m - 1}{2 \cdot 5^{m-1}}\beta + \frac{1 + 5^m - 2 \cdot 3^m}{5^m}\gamma, \tag{2.2.2}$$

$$f(l_m) = \frac{3^{m+1} + 7}{10 \cdot 5^m}\alpha + \frac{5^m - 1}{5^m}\beta + \frac{3 - 3^{m+1}}{10 \cdot 5^m}\gamma, \tag{2.2.3}$$

$$f(r_m) = \frac{3 - 3^{m+1}}{10 \cdot 5^m}\alpha + \frac{5^m - 1}{5^m}\beta + \frac{3^{m+1} + 7}{10 \cdot 5^m}\gamma. \tag{2.2.4}$$

Proof. Formula (2.2.1) can be proven by induction: It is true for $m = 1$ and $m = 2$. Assume that it is true for $m - 1$ and m and prove it for $m + 1$. By extension of formula (1.1.2), we obtain

$$f(\frac{1}{2^{m+1}}) = \frac{8}{25}f(0) + \frac{20}{25}f(\frac{1}{2^m}) - \frac{3}{25}f(\frac{1}{2^{m-1}}).$$

Inserting $f(0) = \alpha$ and by induction hypothesis the values for $f(\frac{1}{2^m})$ and $f(\frac{1}{2^{m-1}})$ we get after simplifying

$$f(\frac{1}{2^{m+1}}) = \frac{1 + 5^{m+1} - 2 \cdot 3^{m+1}}{5^{m+1}}\alpha + \frac{3^{m+1} - 1}{2 \cdot 5^m}\beta + \frac{3 - 3^{m+1}}{2 \cdot 5^{m+1}}\gamma$$

and (2.2.1) is proved.

(2.2.2) is obtained by symmetry: Substitute the values γ, β and α for α, β and γ in (2.2.1).

(2.2.4) can be obtained from (2.2.1) by substituting the values $\beta, f(\frac{3}{4})$ and γ for α, β and γ.

(2.2.3) is obtained from (2.2.4) by symmetry again. □

Proposition 2.2 *The function f defined on J is continuous on J.*

Proof. As $m \longrightarrow \infty$, $l_m \longrightarrow 1/2$, $r_m \longrightarrow 1/2$ and
$f(l_m) \longrightarrow f(1/2) = \beta$, $f(r_m) \longrightarrow f(1/2) = \beta$.

Since f is monotone on J, this suffices for the continuity of $f : J \longrightarrow \mathbb{R}$ at the point $p = 1/2$. Applying Lemma 2.1 to all subintervals of the form (1.1.3) (instead of $[0, 1]$) we get the continuity of f at all points $p \in J \backslash \{0, 1\}$. Using (2.2.1), (2.2.2) and monotonicity of $f : J \longrightarrow \mathbb{R}$, we obtain continuity at the points $p = 0$ and $p = 1$ also. □

Indeed, more is true about the continuity of f, as we illustrate with the following lemma.

Lemma 2.3 *For $p, q \in J$, $|p - q| < 1/2^n$ it holds that*

$$|f(p) - f(q)| < 2(\frac{4}{5})^n(\gamma - \alpha).$$

Proof. By the inequalities (1.1.1) one gets

$$\max\{\lambda - \alpha, \beta - \lambda, \mu - \beta, \gamma - \mu\} < \frac{4}{5}\max\{\beta - \alpha, \gamma - \beta\}.$$

When we arrive at a subinterval of length $1/2^n$ by successive application of the bisection process, we thus obtain the upperbound

$$(\frac{4}{5})^{n-1}\max\{\beta - \alpha, \gamma - \beta\}$$

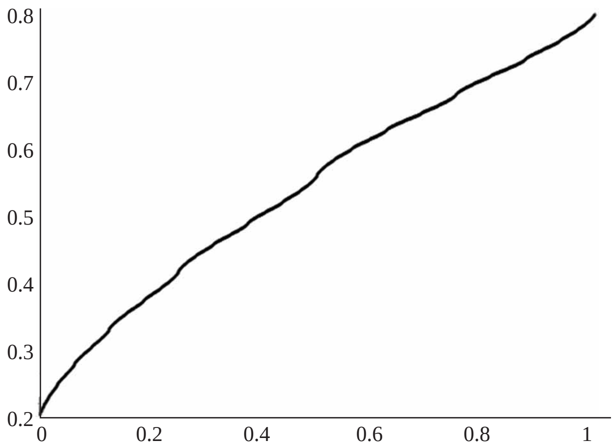

Figure 2. Graph of f for $\alpha = .2$, $\beta = .56, \gamma = .8$.

for the difference of the values of the function at endpoints of such a subinterval. Hence, for $p, q \in J, \qquad |p - q| < 1/2^n$ we get

$$|f(p) - f(q)| < 2(\frac{4}{5})^{n-1} \max\{\beta - \alpha, \gamma - \beta\}$$

by monotonicity of f on J. As

$$\max\{\beta - \alpha, \gamma - \beta\} < \frac{4}{5}(\gamma - \alpha)$$

by (1.1.1), we obtain the desired inequality. □

Corollary 2.4 *f is uniformly continuous on J.*

This enables us to extend f from J to $[0, 1]$, because J is dense in $[0, 1]$, getting still a continuous strictly-monotone function defined on $[0, 1]$ (called again f). This $f : [0, 1] \longrightarrow \mathbb{R}$ will be the promised function. As an example see Figure 2, where we took $\alpha = .2$, $\beta = .56, \gamma = .8$.

3. Improper Derivatives on a Dense Subset

Now we shall prove that f is improperly differentiable at all $p \in J$. Furthermore, in the next section we will give another dense subset of $[0,1]$, where the function is differentiable with vanishing derivative. We need first the following lemma.

Lemma 3.1 *Let $g : [0,1] \longrightarrow \mathbb{R}$ be a monotone function, $x_0 \in [0,1]$, $d \in (0,1)$, $a \neq 0$ and $x_m = x_0 + ad^m$. Assume*

$$\frac{g(x_m) - g(x_0)}{x_m - x_0}$$

is defined and tends to 0 (or $\pm\infty$) as $m \longrightarrow \infty$. If $a < 0$ then the left derivative of g at x_0 exists and is 0 (or $\pm\infty$); If $a > 0$ then the right derivative of g at x_0 exists and is 0 (or $\pm\infty$).

Proof. We consider only the case where g is monotone increasing and $a > 0$. Let $x_0 \in [0,1)$ and $x > x_0$. Then there exists $m \in \mathbb{N}$ such that

$$x_0 + ad^{m+1} \leq x \leq x_0 + ad^m.$$

As x tends to x_0, m tends to infinity and from the inequalities

$$d \cdot \frac{g(x_{m+1}) - g(x_0)}{x_{m+1} - x_0} \leq \frac{g(x) - g(x_0)}{x - x_0} \leq \frac{1}{d} \cdot \frac{g(x_m) - g(x_0)}{x_m - x_0}$$

we get the result. □

Proposition 3.2 *f is differentiable on J in the improper sense ($f'(p) = +\infty$ for all $p \in J$).*

Proof. First we show that the derivative is $+\infty$ at $p = 1/2 \in J$. Using (2.2.3) for $l_m = 1/2 - 1/2 \cdot (1/2)^m$, we get

$$\lim_{m \longrightarrow \infty} \frac{f(l_m) - f(\frac{1}{2})}{l_m - \frac{1}{2}} = \lim_{m \longrightarrow \infty} \frac{3}{5} \cdot (\frac{6}{5})^m \cdot (\gamma - \alpha).$$

As $\gamma > \alpha$ this limit is $+\infty$, and by lemma 3.1 ($x_0 = 1/2$, $d = 1/2$, $a = -1/2$), the left derivative of f at $p = 1/2$ is $+\infty$. Using (2.2.4) and r_m we see that the right derivative of f at $p = 1/2$ is also $+\infty$. Applying this method to subintervals in the form of (1.1.3), we see that the derivative at midpoints exists and is $+\infty$. This shows that $f'(p) = +\infty$ for $p \in J\backslash\{0,1\}$. Similarly from (2.2.1) and (2.2.2) we see that the right derivative at $p = 0$ and the left derivative at $p = 1$ is also $+\infty$. □

4. Zero Derivatives on Another Dense Set

Proposition 4.1 *There exists a dense subset of* $[0,1]$ *where the function* f *is differentiable and the derivative vanishes.*

Proof. First we show that the function has vanishing derivative at $p = 1/3$. To apply lemma 3.1 we will use two sequences approaching $p = 1/3$ from the left and right, geometrically. For this purpose we define the following sequence I_m ($m = 0, 1, ...$) of closed intervals: $I_0 = [0,1]$, I_m = right half of the left half of I_{m-1} (i.e. If $I_{m-1} = [a,b]$, then $I_m = [(3a+b)/4, (a+b)/2]$). Set $I_m = [x_m, z_m]$ and $y_m = (x_m + z_m)/2$. Then one can see that

$$x_m = \frac{1}{4} + (\frac{1}{4})^2 + ... + (\frac{1}{4})^m = \frac{1}{3} - \frac{1}{3}(\frac{1}{4})^m$$

and

$$z_m = x_m + (\frac{1}{4})^m = \frac{1}{3} + \frac{2}{3}(\frac{1}{4})^m.$$

Let $f(x_m) = \alpha_m$, $f(y_m) = \beta_m$ and $f(z_m) = \gamma_m$. We need analytic expressions for computation of the left and right derivatives at $p = 1/3$. By (1.1.2) we can write

$$\begin{aligned} \alpha_m &= \tfrac{8}{25}\alpha_{m-1} + \tfrac{4}{5}\beta_{m-1} - \tfrac{3}{25}\gamma_{m-1} \\ \beta_m &= \tfrac{8}{25}\beta_{m-1} + \tfrac{4}{5}\alpha_m - \tfrac{3}{25}\alpha_{m-1} \\ \gamma_m &= \beta_{m-1}. \end{aligned} \tag{4.4.1}$$

From (4.4.1) we obtain

$$5\alpha_m + 25\beta_m - 3\gamma_m = 5\alpha_{m-1} + 25\beta_{m-1} - 3\gamma_{m-1} \tag{4.4.2}$$

for $m = 1, 2, ...$. From (4.4.2) we get

$$5\alpha_m + 25\beta_m - 3\gamma_m = c, \tag{4.4.3}$$

where $c = 5\alpha + 25\beta - 3\gamma$ (remember that $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\gamma_0 = \gamma$). From (4.4.3) we get

$$f(\frac{1}{3}) = \frac{c}{27}.$$

Using (4.4.1) and (4.4.3) we can write

$$\begin{aligned} \alpha_m &= \tfrac{1}{125}(4c + 20\alpha_{m-1} - 3\gamma_{m-1}) \\ \gamma_m &= \tfrac{1}{25}(c - 5\alpha_{m-1} + 3\gamma_{m-1}). \end{aligned} \quad (4.4.4)$$

Consider the sequence

$$t_m = u\alpha_m + v\gamma_m,$$

where $u = 10$, $v = 1 - \sqrt{13}$. Then we get the recursion

$$t_m = w + s \cdot t_{m-1}, \quad (4.4.5)$$

where $w = \frac{9-\sqrt{13}}{25}c$, $s = \frac{7+\sqrt{13}}{50}$. From (4.4.5) we can compute t_m:

$$t_m = w\frac{s^m - 1}{s - 1} + s^m \cdot t_0, \quad (t_0 = 10\alpha + (1 - \sqrt{13})\gamma). \quad (4.4.6)$$

Using (4.4.4) and (4.4.6) we get

$$\gamma_m = \frac{c}{25} - \frac{1}{50}t_{m-1} + \frac{v+6}{50}\gamma_{m-1},$$

and inserting the value of t_{m-1} we get the recursion

$$\gamma_m = l + k \cdot s^{m-1} + h \cdot \gamma_{m-1},$$

where $l = \frac{c}{25} + \frac{w}{50(s-1)}$, $k = -\frac{1}{50}(\frac{w}{s-1} + t_0)$, $h = \frac{v+6}{50}$. From this recursion we obtain

$$\gamma_m = [\frac{l}{h-1} - \frac{k}{s-h} + \gamma] \cdot h^m + \frac{k}{s-h}s^m - \frac{l}{h-1}. \quad (4.4.7)$$

As $-\frac{l}{h-1} = \frac{c}{27} = f(\frac{1}{3})$, $h < 1/4$ and $s < 1/4$ we obtain finally

$$\lim_{m \longrightarrow \infty} \frac{f(z_m) - f(\frac{1}{3})}{z_m - \frac{1}{3}} = \lim_{m \longrightarrow \infty} \{[\frac{l}{h-1} - \frac{k}{s-h} + \gamma] \cdot (4h)^m + \frac{k}{s-h}(4s)^m\} \cdot \frac{3}{2} = 0$$

By lemma 3.1 (for $x_0 = 1/3$, $d = 1/4$, $a = 2/3$) we see that the right derivative at $p = 1/3$ exists and vanishes. Similarly, using (4.4.6) and (4.4.7) we can compute explicitly α_m as

$$\alpha_m = \frac{1}{u}[\frac{w}{s-1} - \frac{vk}{s-h} + t_0] \cdot s^m - \frac{v}{u}(\frac{l}{h-1} - \frac{k}{s-h} + \gamma) \cdot h^m + \frac{c}{27}$$

and see that the left derivative also exists and vanishes. Thus we get $f'(1/3) = 0$. Applying the same argument to subintervals of the form as in (1.1.3) we see that the derivative at the point dividing this subinterval in the ratio 1 : 3 exists and vanishes. But those points are dense in $[0, 1]$, and the proposition is proved. □

Concluding, we wish to conjecture, that there are no points in $[0, 1]$, where this continuous, monotone function has nonzero finite derivative.

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