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Decompositions of Continuity

Talal Al-Hawary, Ahmad Al-Omari

Abstract

In 2004, Al-Hawary and Al-Omari introduced and explored the class of ω^o -open sets which is strictly stronger than the class of ω -open sets and weaker than that of open sets. In this paper, we introduce what we call ω^o -continuity and ω_X^o -continuity and we give several characterizations and two decompositions of ω^o -continuity. Finally, new decompositions of continuity are provided.

Key Words: ω^o -open, ω^o -continuity, Continuity.

1. Introduction

Let (X, \mathfrak{T}) be a topological space (or simply, a space). If $A \subseteq X$, then the closure of A and the interior of A will be denoted by $Cl_{\mathfrak{T}}(A)$ and $Int_{\mathfrak{T}}(A)$, respectively. If no ambiguity appears, we use \bar{A} and A^o instead. By X, Y and Z we mean topological spaces with no separation axioms assumed. $\mathfrak{T}_{standard}$, $\mathfrak{T}_{indiscrete}$, $\mathfrak{T}_{left ray}$ and $\mathfrak{T}_{cocountable}$ stand for the standard, indiscrete, left ray and the cocountable topologies, respectively. A space (X, \mathfrak{T}) is *anti locally countable* if all non-empty open subsets are uncountable.

In [3], the concept of ω -closed subsets was explored where a subset A of a space (X, \mathfrak{T}) is ω -closed if it contains all of its condensation points. In [4], several characterizations of ω -continuity were provided where a map $f : X \rightarrow Y$ is ω -continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ω -open subset U in X containing x such that $f(U) \subseteq V$. f is ω -continuous if it is ω -continuous at every $x \in X$. Several properties of ω -continuous mappings were also explored. Analogous to [4, 5, 8, 9], in

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Section 2 we introduce the relatively new notion of ω^o -continuity, which is closely related to continuity and ω -continuity. In fact, properly placed between them. Moreover, we show that ω^o -continuity preserves Lindelof property and a space (X, \mathfrak{T}) is Lindelof if and only if $(X, \mathfrak{T}_{\omega^o})$ is Lindelof, where \mathfrak{T}_{ω^o} is the collection of all ω^o -open subsets of X . Sections 3 is devoted for studying four weaker notions of ω^o -continuity by which we provide two decompositions of ω^o -continuity. Finally, in Section 4 we give several decompositions of continuity which seem to be new.

Next, we recall several necessary definitions and results from [1].

Definition 1 *A subset A of a space (X, \mathfrak{T}) is called ω^o -open if for every $x \in A$, there exists an open subset $U_x \subseteq X$ containing x such that $U_x \setminus \overset{\circ}{A}$ is countable. The complement of an ω^o -open subset is called ω^o -closed.*

Clearly every open set is ω^o -open and every ω^o -open is ω -open.

Theorem 1 *If (X, \mathfrak{T}) is a space, then $(X, \mathfrak{T}_{\omega^o})$ is a space such that $\mathfrak{T} \subseteq \mathfrak{T}_{\omega^o} \subseteq \mathfrak{T}_{\omega}$, where \mathfrak{T}_{ω} is the collection of all ω -open subsets of X .*

Corollary 1 *If (X, \mathfrak{T}) is anti locally countable and A is ω^o -closed, then $Int_{\mathfrak{T}}(A) = Int_{\mathfrak{T}_{\omega^o}}(A)$.*

2. ω^o -Continuous Mappings

We begin this section by introducing the notion of ω^o -continuous mappings. Several characterizations of this class of mappings are also provided.

Definition 2 *A map $f : X \rightarrow Y$ is ω^o -continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ω^o -open subset U in X containing x such that $f(U) \subseteq V$. f is ω^o -continuous if it is ω^o -continuous at every $x \in X$.*

As every open set is ω^o -open and every ω^o -open set is ω -open, every continuous map is ω^o -continuous and every ω^o -continuous map is ω -continuous. The converses need not be true.

Example 1 *Let $X = \{a, b\}$, $\mathfrak{T}_1 = \{\emptyset, X, \{a\}\}$ and $\mathfrak{T}_2 = \{\emptyset, X, \{b\}\}$. Then the identity map $id : (X, \mathfrak{T}_1) \rightarrow (X, \mathfrak{T}_2)$ is ω^o -continuous but not continuous.*

Example 2 Let $Y = \{0, 1\}$ and $\mathfrak{T} = \{\emptyset, Y, \{0\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{\text{standard}}) \rightarrow (Y, \mathfrak{T})$ defined by $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is ω -continuous but not ω^o -continuous.

The proofs of the following three results are similar to those for ω -continuous maps given in [4] and are thus omitted.

Lemma 1 Let X, Y and Z be spaces. Then

- (1) If $f : X \rightarrow Y$ is ω^o -continuous surjection and $g : Y \rightarrow Z$ is continuous surjection, then $g \circ f$ is ω^o -continuous.
- (2) If $f : X \rightarrow Y$ is ω^o -continuous surjection and $A \subseteq X$, then $f|_A$ is ω^o -continuous.
- (3) If $f : X \rightarrow Y$ is a map such that $X = X_1 \cup X_2$ where X_1 and X_2 are closed and both $f|_{X_1}$ and $f|_{X_2}$ are ω^o -continuous, then f is ω^o -continuous.
- (4) If $f_1 : X \rightarrow X_1$ and $f_2 : X \rightarrow X_2$ are maps and $g : X \rightarrow X_1 \times X_2$ is the map defined by $g(x) = (f_1(x), f_2(x))$ for all $x \in X$, then g is ω^o -continuous if and only if f_1 and f_2 are ω^o -continuous.

Lemma 2 For a map $f : X \rightarrow Y$, the following are equivalent:

- (1) f is ω^o -continuous.
- (2) The inverse image of every open subset of Y is ω^o -open in X .
- (3) The inverse image of every closed subset of Y is ω^o -closed in X .
- (4) The inverse image of every basic open subset of Y is ω^o -open in X .
- (5) The inverse image of every subbasic open subset of Y is ω^o -open in X .

Lemma 3 A space (X, \mathfrak{T}_X) is Lindelof if and only if $(X, \mathfrak{T}_{\omega^o})$ is Lindelof.

Next we show that being Lindelof is preserved under ω^o -continuity.

Theorem 2 If $f : (X, \mathfrak{T}_X) \rightarrow (Y, \mathfrak{T}_Y)$ is ω^o -continuous and X is Lindelof, then Y is Lindelof.

Proof. Let $\mathfrak{B} = \{V_\alpha : \alpha \in \nabla\}$ be an open cover of Y . Since f is ω^o -continuous, $\mathfrak{A} = \{f^{-1}(V_\alpha) : \alpha \in \nabla\}$ is a cover of X by ω^o -open subsets and as X is Lindelof, by Lemma 3, \mathfrak{A} has a countable subcover $\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N}\}$. Now $Y = f(X) = f(\cup\{f^{-1}(V_{\alpha_n}) : n \in \mathbb{N}\}) \subseteq \cup\{V_{\alpha_n} : n \in \mathbb{N}\}$. Therefore Y is Lindelof. \square

If X is a countable space, then every subset of X is ω^o -open and hence every map $f : X \rightarrow Y$ is ω^o -continuous. Next, we show that if X is uncountable such that every ω^o -continuous map $f : X \rightarrow Y$ is a constant map, then X has to be connected.

Theorem 3 *If X is uncountable space such that every ω^o -continuous map $f : X \rightarrow Y$ is a constant map, then X is connected.*

Proof. If X is disconnected, then there exists a non-empty proper subset A of X which is both open and closed. Let $Y = \{a, b\}$ and $\mathfrak{T}_Y = \{\emptyset, Y, \{b\}\}$ and $f : X \rightarrow Y$ defined by $f(A) = \{a\}$ and $f(X \setminus A) = \{b\}$. Then f is a non-constant ω^o -continuous map. \square

The converse of the preceding result need not be true even when X is uncountable.

Example 3 *The identity map $id : (\mathbb{R}, \mathfrak{T}_{left}) \rightarrow (\mathbb{R}, \mathfrak{T}_{indiscrete})$ is a non-constant ω^o -continuous.*

3. Decompositions of ω^o -Continuity

We begin by recalling the following well-known two definitions.

Definition 3 *A map $f : X \rightarrow Y$ is weakly continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an open subset U in X containing x such that $f(U) \subseteq \overline{V}$. f is weakly continuous if it is weakly continuous at every $x \in X$.*

Definition 4 *A map $f : X \rightarrow Y$ is W^* -continuous if for every open subset V in Y , $f^{-1}(Fr(V))$ is closed in X , where $Fr(V) = \overline{V} \setminus \overset{\circ}{V}$.*

Weakly continuity and W^* -continuity are independent notions that are weaker than continuity and the two together characterize continuity (see for example [7]). Next we give two relatively new such definitions.

Definition 5 *A map $f : X \rightarrow Y$ is weakly ω^o -continuous at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ω^o -open subset U in X containing x such that $f(U) \subseteq \overline{V}$. f is weakly ω^o -continuous if it is weakly ω^o -continuous at every $x \in X$.*

Clearly, every ω^o -continuous and every weakly continuous map is weakly ω^o -continuous. Non of the converses need be true as shown next.

Example 4 Let $Y = \{a, b, c\}$ and $\mathfrak{T} = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{\text{cocountable}}) \rightarrow (Y, \mathfrak{T})$ defined by $f(x) = a$ for all $x \in \mathbb{R}$. Then f is weakly ω^o -continuous but not ω^o -continuous.

Example 5 Let $Y = \{a, b, c\}$ and $\mathfrak{T} = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{\text{cocountable}}) \rightarrow (Y, \mathfrak{T})$ defined by $f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ for all $x \in \mathbb{R}$. Then f is weakly continuous and hence weakly ω^o -continuous but not ω^o -continuous.

Definition 6 A map $f : X \rightarrow Y$ is coweakly ω^o -continuous if for every open subset V in Y , $f^{-1}(Fr(V))$ is ω^o -closed in X , where $Fr(V) = \overline{V} \setminus V^o$.

Clearly, every ω^o -continuous is coweakly ω^o -continuous. The converse need not be true.

Example 6 Let $X = Y = \{a, b\}$, $\mathfrak{T}_X = \{\emptyset, X\}$ and $\mathfrak{T}_Y = \{\emptyset, Y, \{a\}, \{b\}\}$ Then the identity map $id : X \rightarrow Y$ is coweakly ω^o -continuous but not ω^o -continuous.

Our first characterization of ω^o -continuity in terms of the preceding two notions of continuity is given next.

Theorem 4 The following are equivalent for a map $f : (X, \mathfrak{T}_X) \rightarrow (Y, \mathfrak{T}_Y)$:

- (1) f is ω^o -continuous.
- (2) $f : (X, \mathfrak{T}_{\omega^o}) \rightarrow (Y, \mathfrak{T}_Y)$ is continuous.
- (3) $f : (X, \mathfrak{T}_{\omega^o}) \rightarrow (Y, \mathfrak{T}_Y)$ is weakly continuous and W^* -continuous.

Proof. (1) \Rightarrow (2) : Obvious.

(2) \Rightarrow (3) : Follows from Theorem 1.

(3) \Rightarrow (1) : Since $f : (X, \mathfrak{T}_{\omega^o}) \rightarrow (Y, \mathfrak{T}_Y)$ is W^* -continuous, it is coweakly ω^o -continuous and as it is weakly-continuous, it is weakly ω^o -continuous. Thus by Theorem 1, $f : (X, \mathfrak{T}_X) \rightarrow (Y, \mathfrak{T}_Y)$ is ω^o -continuous. \square

We show that weakly ω^o -continuity and coweakly ω^o -continuity are independent notions, but together they characterize ω^o -continuity. This will be our first decomposition of ω^o -continuity which is analogous to the result that can be found in [2] for ω -continuity.

Example 7 The map id in Example 6 is coweakly ω^o -continuous but not weakly ω^o -continuous.

Example 8 Let $Y = \{a, b\}$ and $\mathfrak{T} = \{\emptyset, Y, \{a\}\}$. Then the map $f : (\mathbb{R}, \mathfrak{T}_{cocountable}) \rightarrow (Y, \mathfrak{T})$ defined by $f(x) = \begin{cases} a & x \in \mathbb{Q} \\ b & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ for all $x \in \mathbb{R}$. Then f is weakly ω^o -continuous but not coweakly ω^o -continuous.

Theorem 5 A map $f : X \rightarrow Y$ is ω^o -continuous if and only if f is both weakly and coweakly ω^o -continuous.

Proof. ω^o -continuity implies weakly and coweakly ω^o -continuity is obvious. Conversely, suppose $f : X \rightarrow Y$ is both weakly and coweakly ω^o -continuous and let $x \in X$ and V be an open subset of Y such that $f(x) \in V$. Then as f is weakly ω^o -continuous, there exists an ω^o -open subset U of X containing x such that $f(U) \subseteq \overline{V}$. Now $Fr(V) = \overline{V} \setminus V$ and hence $f(x) \notin Fr(V)$. So $x \in U \setminus f^{-1}(Fr(V))$ which is ω^o -open in X since f is coweakly ω^o -continuous. For every $y \in f(U \setminus f^{-1}(Fr(V)))$, $y = f(a)$ for some $a \in U \setminus f^{-1}(Fr(V))$ and hence $f(a) = y \in f(U) \subseteq \overline{V}$ and $y \notin Fr(V)$. Thus $f(a) = y \notin Fr(V)$ and thus $f(a) \in V$. Therefore, $f(U \setminus f^{-1}(Fr(V))) \subseteq V$ and hence f is ω^o -continuous. \square

Next, we define a new class of open sets that is independent of ω -open class, but together they characterize ω^o -open.

Definition 7 For a space (X, \mathfrak{T}) , let $\omega_\omega^o = \{A \subseteq X : Int_{\mathfrak{T}_{\omega^o}}(A) = Int_{\mathfrak{T}_\omega}(A)\}$. A is ω_ω^o -set if $A \in \omega_\omega^o$.

Clearly every ω^o -open set is ω_ω^o -set, but the converse need not be true.

Example 9 Consider \mathbb{R} with the standard topology $\mathfrak{T}_{standard}$. Then \mathbb{Q} is an ω_ω^o -set which is neither ω^o -open nor ω -open.

Even an ω -open subset need not be an ω_ω^o -set.

Example 10 Consider \mathbb{R} with the standard topology $\mathfrak{T}_{standard}$. Then $\mathbb{R} \setminus \mathbb{Q}$ is an ω -open which is not an ω_ω^o -set.

Theorem 6 A subset A of a space X is ω^o -open if and only if A is ω -open and an ω_ω^o -set.

Proof. Trivially every ω^o -open is ω -open and an ω_ω^o -set. Conversely, let A be an ω -open set that is ω_ω^o -set. Then $A = \text{Int}_{\mathfrak{T}_\omega}(A) = \text{Int}_{\mathfrak{T}_{\omega^o}}(A)$ and therefore A is ω^o -open. \square

Definition 8 A map $f : X \rightarrow Y$ is ω_ω^o -continuous if the inverse image of every open subset of Y is an ω_ω^o -set.

Clearly every ω^o -continuous map is ω_ω^o -continuous, but the converse need not be true as not every ω_ω^o -set is ω^o -open. An immediate consequence of Theorem 6 is the following decomposition of ω^o -continuity.

Theorem 7 A map $f : X \rightarrow Y$ is ω^o -continuous if and only if f is ω -continuous and ω_ω^o -continuous.

4. Decompositions of Continuity

We begin this section by introducing the notion of an ω_X^o -set. We then introduce the notion of ω_X^o -continuity which gives an immediate decomposition of continuity.

Definition 9 For a space (X, \mathfrak{T}) , let $\omega_X^o =: \{A \subseteq X : \text{Int}_{\mathfrak{T}_{\omega^o}}(A) = \text{Int}_{\mathfrak{T}}(A)\}$. A is an ω_X^o -set if $A \in \omega_X^o$.

The proof of the following result follows immediately from Corollary 1.

Corollary 2 If (X, \mathfrak{T}) is anti locally countable, then ω_X^o contains all ω^o -closed subsets of X .

We remark that, in general, an ω^o -closed set need not be an ω_X^o -set as shown in the next example.

Example 11 Let $X = \{a, b\}$ and $\mathfrak{T} = \{\emptyset, X, \{a\}\}$. Set $A = \{b\}$. Then A is ω^o -closed but not an ω_X^o -set.

As every open set is ω^o -open, every open set is an ω_X^o -set but the converse need not be true.

Example 12 Consider \mathbb{R} with the standard topology $\mathfrak{T}_{\text{standard}}$. Then \mathbb{Q} is an ω_X^o -set which is not open.

Next, we show that the notions of ω_X^o -set and ω^o -open are independent, but together they characterize open sets.

Example 13 In Example 11, A is ω^o -open but not an ω_X^o -set.

Example 14 In Example 12, \mathbb{Q} is an ω_X^o -set which is not ω^o -open.

Theorem 8 A subset A of a space X is open if and only if A is ω^o -open and an ω_X^o -set.

Proof. Trivially every open set is ω^o -open and an ω_X^o -set.. Conversely, let A be an ω^o -open set that is ω_X^o -set. Then $A = \text{Int}_{\tau_{\omega^o}}(A) = \text{Int}_{\tau}(A)$ and therefore A is open. \square

In a similar manner, for a space (X, \mathfrak{T}) let $\omega_X =: \{A \subseteq X : \text{Int}_{\tau_{\omega}}(A) = \text{Int}_{\mathfrak{T}}(A)\}$ and call a subset A is ω_X -set if $A \in \omega_X$. Then we have the following result.

Theorem 9 A subset A of a space X is open if and only if A is ω -open and an ω_X -set.

Definition 10 A map $f : X \rightarrow Y$ is ω_X^o -continuous (respectively, ω_X -continuous) if the inverse image of every open subset of Y is an ω_X^o -set (respectively, ω_X -set).

Clearly every continuous map is ω_X^o -continuous, but the converse need not be true as not every ω_X^o -set is open. An immediate consequence of Theorems 5, 7, 8 and 9 are the following decompositions of continuity, which seem to be new.

Theorem 10 For a map $f : X \rightarrow Y$, the following are equivalent:

- (1) f is continuous.
- (2) f is ω^o -continuous and ω_X^o -continuous.
- (3) f is ω -continuous and ω_X^o -continuous.
- (4) f is both weakly ω^o -continuous, coweakly ω^o -continuous and ω_X^o -continuous.
- (5) f is ω -continuous, ω_{ω}^o -continuous and ω_X^o -continuous.

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