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Abstract

Let R be a ring, M be a right R-module and $S = End_R(M)$. M is called a quasi-dual module if, for every R-submodule N of M, N is a direct summand of $r_M(X)$ where $X \subseteq S$. In this article, we study and provide several characterizations of this module classes. We show that if M is quasi-dual module, then, for all $m \in M$, $r_M \ell_S(m) = mR \oplus K$ for some submodule K of M. We also show that every quasi-dual module is a Kasch module and $Z({}_SM) \subseteq Rad(M_R)$.

Key Words: Quasi-dual module, Kasch module, Ikeda-Nakayama module.

1. Introduction

Throughout this paper, R is an associative ring with identity, modules are right and unitary over it and $S = End_R(M)$ is the ring of R-endomorphisms of M. Submodules of M will be right R-modules unless specified otherwise. Clearly, the module M is a left S and right R-bimodule.

A ring R is called a *right dual ring* if every right ideal of R is an annihilator and R is called *right quasi-dual ring* if every right ideal of R is a direct summand of a right annihilator. Right dual and, as a generalization of right dual rings, right quasi-dual rings were discussed in detail in [4] and [9]. Some of the known results on right quasi-dual rings can be recalled as follows: R is a right quasi-dual ring if and only if $r\ell(I) = I$ for every essential ideal I of R; if R is a right quasi-dual ring then, R is a right Kasch ring and $r\ell(Soc(R_R)) = Soc(R_R)$ and $r\ell(J(R)) = J(R)$.

In this paper, the notion of a quasi-dual module is introduced as a generalization of quasi-dual rings to modules.

2. Preliminaries

Let R and S be rings and ${}_SM_R$ be a bimodule. For any $X \leq M$ and $T \subseteq S$, denote $\ell_S(X) = \{s \in S : sX = 0\}$ and $r_M(T) = \{m \in M : Tm = 0\}$.

Lemma 2.1 *For a right R-module M, let $S = End_R(M)$, $N \leq M$, $I \leq R_R$, $J \leq {}_SS$ and $0 \in S$; we then have*

$$r_M(0) = M$$

$$\ell_S(0) = S$$

$$r_M(S) = \ell_S(S) = \ell_S(M) = 0$$

$$\ell_M(r_R(\ell_M(I))) = \ell_M(I)$$

$$\ell_S(r_M(\ell_S(N))) = \ell_S(N)$$

$$r_R(\ell_M(r_R(N))) = r_R(N)$$

$$r_M(\ell_S(r_M(J))) = r_M(J)$$

$$\ell_S(\oplus_{i \in I} N_i) = \cap_{i \in I} \ell_S(N_i).$$

Proof. See [2, 12]. □

Definition 2.2 A ring R is said to be a *right dual* if every right ideal of R is an annihilator ([4]).
Definition 2.3 A ring R is called a *right quasi-dual* if every right ideal of R is a direct summand of a right annihilator ([9]).

Definition 2.4 A module M is called *Ikeda-Nakayama module* if

$$\ell_S(A \cap B) = \ell_S(A) + \ell_S(B)$$

for any submodules A, B of M_R (see [10]).

Definition 2.5 A module M is called *Kasch module* if $\hat{M}$ is an (injective) cogenerator in $\sigma[M]$, where $\hat{M}$ is injective hull of M in $\sigma[M]$ ([1]).

The notations, "$\leq$" will denote a submodule, "$\leq_e$" an essential submodule, and "$<<$" a small submodule.

We will refer to [2, 3, 4, 8, 9, 11] for all undefined notions used in the text, and also for basic facts concerning (quasi-)dual rings and annihilators.

3. Quasi-Dual Modules

In this paper, we shall introduce the notion of quasi-dual modules and try to give a module theoretic characterizations of quasi-dual ring.

Definition 3.1 (See [5]) Let R be a ring, M be a right R-module and $S = End_R(M)$. M is called *a dual module* if

1. $r_M\ell_S(N) = N$ for every submodule N of M;
2. $\ell_S r_M(I) = I$ for every right ideal I of S.

Definition 3.2 Let R be a ring, M be a right R-module and $S = End_R(M)$. We shall call M *a quasi-dual module* if, for every R-submodule N of M, N is a direct summand of $r_M(X)$, where $X \subseteq S$(compare with [6] and [7]). Trivially,

1. A right quasi-dual ring is a quasi-dual module as right module.
2. Every dual module is a quasi-dual module.
3. Every semisimple module is a quasi-dual module.

Lemma 3.3 *The following conditions are equivalent for a right R-module M.*

1. *M is a quasi-dual module.*
2. *For every essential submodule K of M, $r_M\ell_S(K) = K$*
3. *For every submodule L of M, L is a direct summand of $r_M\ell_S(L)$.*

Proof. (1) $\Rightarrow$ (2) Let M be a quasi-dual module and K be an essential submodule of M. Then K is a direct summand of $r_M(Y)$ for some $Y \subseteq S$. Let $r_M(Y) = K \oplus K'$ for some K'. Then $K = r_M(Y)$. Note that $\ell_S(K) = \ell_S r_M(Y)$ implies $r_M\ell_S(K) = r_M\ell_S r_M(Y) = r_M(Y) = K$.
(2) $\Rightarrow$ (3) Let L be a submodule of M. If L is essential in M, $r_M\ell_S(L) = L$ by (2). Hence L is a direct summand of $r_M\ell_S(L)$. Assume that L is not essential in M. Then

$L \oplus L'$ is an essential for some submodule L' of M. By (2), $r_M \ell_S(L \oplus L') = L \oplus L'$. Since $L \subseteq r_M \ell_S(L) \subseteq r_M \ell_S(L \oplus L')$, L is a direct summand of $r_M \ell_S(L)$ by modularity.
(3) $\Rightarrow$ (1) clear. □

Following [10], M is called *almost principally injective* (AP-injective for short) if, for any $m \in M$, there exists an S-submodule K of M such that $r_M \ell_S(m) = mR \oplus K$.

Theorem 3.4 *Every quasi-dual right R-module is an AP-injective module.*

Proof. Clear. □

Let N be any module. N is said to be $M-$*cyclic module* if N is isomorphic to M/X for some $X \leq M$, and in case $N \leq M$ and N is M-cyclic module then it is called $M-$*cyclic submodule* of M and N is called M-*singular* if $N \cong M/K$ with $K \leq_e M$.

Proposition 3.5 *Let M be an R-module. Then*

1. *If, for every essential submodule K of M, $r_M \ell_S(K) = K$ then, every M-cyclic singular R-module is cogenerated by M.*

2. *If every singular factor submodule (i.e. M-cyclic submodule) of M is cogenerated by M, then $r_M \ell_S(K) = K$ for every essential submodule K of M.*

Proof. (1) Let N be a singular R-module with $N \cong M/K$ and $K \leq_e M$. Since K is essential in M, $r_M \ell_S(K) = K$ by assumption. Let $I = \ell_S(K)$. We define $\phi : M/K \longrightarrow \Pi_{\alpha \in I} M_\alpha$ by $m + K \rightarrow \phi(m + K) = (\alpha m)_{\alpha \in I}$. Let $(\alpha m)_{\alpha \in I} = 0$. Then $\alpha m = 0$ for all $\alpha \in I$. Hence $\alpha \in \ell_S(K)$ and so $m \in r_M \ell_S(K) = K$. Therefore ϕ is a monomorphism.
(2) Let M/K be a singular module for some $K \leq_e M$. By hypothesis, there exists a monomorphism $\sigma : M/K \longrightarrow \Pi_{\alpha \in I} M_\alpha$ for some index set I with $M_\alpha = M$ for all $\alpha \in I$. We consider the natural epimorphism $\pi : M \longrightarrow M/K$ and canonical projection $p_\alpha : \Pi_{\alpha \in I} M_\alpha \longrightarrow M_\alpha$. Then $p_\alpha \sigma \pi \in \ell_S(K)$. Let $m \in r_M \ell_S(K)$. Then $p_\alpha \sigma \pi(m) = 0$ for all $\alpha \in I$. Therefore $\sigma\pi(m) \in Ker(p_\alpha)$ for all $\alpha \in I$ and so $\sigma\pi(m) \in \cap_{\alpha \in I} Ker(p_\alpha)$. Since $\cap_{\alpha \in I} Ker(p_\alpha) = 0$, $\sigma\pi(m) = 0$. But σ is a monomorphism, so $\pi(m) = 0$. Therefore $m \in K$. Other side is obvious. Hence $r_M \ell_S(K) = K$. □

$\sigma[M]$ will denote the full subcategory of left R-modules whose objects are the submodules of $M-$generated modules. Hence

$$\sigma[M] = \{N \in R - Mod : N \cong K/L \leq M^{(\Lambda)}/L \ for \ \ some \ \Lambda\}.$$

Following [1], a module M is called *Kasch module* if $\hat{M}$ is an (injective) cogenerator in $\sigma[M]$, where $\hat{M}$ is injective hull of M in $\sigma[M]$.

Proposition 3.6 *For a module M, the following are equivalent;*

1. *M is a Kasch module;*
2. *Any simple module in $\sigma[M]$ can be embedded in M;*
3. *Any simple module in $\sigma[M]$ is cogenerated by M;*
4. *$Hom(C, M) \neq 0$ for any nonzero (cyclic) R-module C from $\sigma[M]$;*
5. *$\ell_S(N) \neq 0$ for every proper submodule N of M;*
6. *$r_M\ell_S(N) = N$ for every maximal submodule N of M.*

Proof. $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ by [1, Proposition 2.6], the other equivalences follows from Lemma 3.3 and Proposotion 3.5. □

Theorem 3.7 *Let M be a quasi dual module.*

1. *$r_M\ell_S(Soc(M)) = Soc(M)$.*
2. *For every maximal submodule N of M, $r_M\ell_S(N) = N$. Therefore, M is a Kasch module and $r_M\ell_S(Rad(M)) = Rad(M)$.*
3. *If L is a submodule of M, then $r_M\ell_S(L) = L \oplus L'$ for a submodule L' with $\ell_S(L) \leq \ell_S(L')$.*

Proof. (1) Let M be a quasi dual module. Then, for each essential submodule K of M, $r_M\ell_S(K) = K$ by Lemma 3.3. By Proposition 3.5, M/K is cogenerated by M. Since $Soc(M)$ is the intersection of all essential submodules, $M/Soc(M)$ is cogenerated by M. Since $Soc(M)$ is an essential submodule of M and $M/Soc(M)$ is singular factor module, so $r_M\ell_S(Soc(M)) = Soc(M)$ by Lemma 3.3.
(2) Let N be a maximal submodule of M. Assume that $r_M\ell_S(N) \neq N$. By maximality

of N, $r_M\ell_S(N) = M$. Note that, for $x \in \ell_S(N)$, $xN = 0$ implies $xM = 0$. Since M is a quasi-dual module, N is a direct summand of $r_M\ell_S(N)$ by Lemma 3.3, and so of M. Let $M = N \oplus N'$ for some submodule N' of M. We consider the canonical projection π on N'. Since $\pi(N) = 0$ implies $\pi(M) = 0$, we have $M = N$. It is a contradiction by maximality of N. Hence $r_M\ell_S(N) = N$. So, M is a Kasch module by Proposition 3.6. Let $x \in r_M\ell_S(Rad(M))$. Then $\ell_S(Rad(M))x = 0$. Note that $M/Rad(M) = M/\cap_{N\leq_{max}M} N$. We consider

$$M \xrightarrow{\pi} M/Rad(M) = M/\cap_{N\leq_{max}M} N \xrightarrow{\sigma} \Pi_{N\leq_{max}M} M/N \xrightarrow{\beta} \Pi_{\alpha\in I} M_\alpha \xrightarrow{p_\alpha} M_\alpha = M.$$

We know that σ and β are one to one. Since $p_\alpha\beta\sigma\pi \in \ell_S(Rad(M))$, we have $(p_\alpha\beta\sigma\pi)(x) = 0$ for all $\alpha \in I$. Then $\beta\sigma\pi(x) = 0$ and so $\pi(x) = 0$. This implies that $x \in Rad(M)$. Other side is obvious.

(3) Let L be a submodule of M. Then $r_M\ell_S(L) = L \oplus L'$ for a submodule L' by Lemma 3.3. Note that $\ell_S(r_M\ell_S(L)) = \ell_S(L \oplus L') = \ell_S(L) \cap \ell_S(L')$ by Lemma 2.1. Hence $\ell_S(L) \leq \ell_S(L')$, as required. □

Recall that;

($C1$) Every complement submodule is a direct summand.

($C2$) If every submodule isomorphic to a direct summand of M is itself a direct summand.

($C3$) If N and K are direct summands of M and $N \cap K = 0$, then $N \oplus K$ is a direct summand of M.

M is called a *continous* (or a *quasi-continous*) module if M has $C1$ and $C2$ (or $C1$ and $C3$).

Theorem 3.8 *Let M be a finitely generated Kasch module such that, any complement submodule N of M, $r_M\ell_S(N) = N$. Then M is quasi-continuous.*

Proof. Let N_1 and N_2 be submodules of M such that they are complements of each other in M. Then $N_1 \cap N_2 = 0$. So $0 = N_1 \cap N_2 = r_M\ell_S(N_1) \cap r_M\ell_S(N_2) = r_M(\ell_S(N_1) + \ell_S(N_2))$. Since M is a Kasch module, by Proposition 3.6, $\ell_S(N_1) + \ell_S(N_2) = M$. Hence M is a quasi-continuous by [11, Theorem 8]. □

Question : When M is a semiperfect module with essential socle in $\sigma[M]$ under the conditions of Theorem 3.8 ?

Proposition 3.9 *The following conditions are equivalent for a right R-module M.*

1. *M is a quasi-dual module and, for every right ideal I of S, I is a direct summand of $\ell_S(K)$ where $K \leq M$.*
2. (a) *For every essential submodule K of M, $r_M\ell_S(K) = K$*

   (b) *For every essential right ideal I of S, $\ell_S r_M(I) = I$*
3. (a) *For every submodule L of M, L is a direct summand of $r_M\ell_S(L)$*

   (b) *For every essential right ideal I of S, I is a direct summand of $\ell_S r_M(I)$.*

Proof. Similar to Lemma 3.3. □

Definition 3.10 We shall call M *a strongly quasi-dual module* if, for every R-submodule N of M and for every right ideal I of S, N is a direct summand of $r_M(X)$ and I is a direct summand of $\ell_S(K)$ where $X \subseteq S$ and $K \leq M$.

Let R and S be any rings and M be an $S-R$-bimodule. Following [6,7], if M is strongly quasi-dual module, then M is called *quasi-dual bimodule*

Proposition 3.11

1. *Let M be a quasi-dual module and A be a submodule of M. Then we have:*

   (i) *If $\ell_S(A) = 0$, then $A = M$.*

   (ii) *If M is an IN-module and $\ell_S(A) << S$, then $A \leq_e M$.*
2. *Let M be a strongly quasi dual module and I be a right ideal of S. Then we have:*

   (i) *If $r_M(I) = 0$, then $I = S$.*

   (ii) *If $\ell_S(A) \leq_e S$, then $A << M$.*

   (iii) *If M is indecomposable and $A \leq_e M$, then $\ell_S(A) << S$.*

   (iv) *If $r_M(I) \leq_e M$, then $I << S$.*

Proof. **1.(i)** Assume that A is an essential submodule of M. By Lemma 3.3, $r_M\ell_S(A) = A$. But $\ell_S(A) = 0$ and M is a quasi-dual module, we have $M = A$. If A is not essential submodule of M, then there exists a submodule B of M such that $A \oplus B$ is essential. So $M = A \oplus B$. Let π_B projection on B. Then $\pi_B(A) = 0$, and so $\pi_B \in \ell_S(A)$. Therefore $B = 0$.

(ii) Assume that A is not essential in M. Then there exist a non-zero submodule K of M such that $A \cap K = 0$. Hence $\ell_S(A \cap K) = S$. Since M is an IN-module, $\ell_S(A \cap K) = \ell_S(A) + \ell_S(K) = S$. Then $\ell_S(K) = S$. Therefore, $K = 0$.
2. **(i)** Similar to 1.(i).

(ii) Let $A+B = M$ for some submodule B of M. Then $\ell_S(A+B) = \ell_S(A) \cap \ell_S(B) = 0$. By assumption, $\ell_S(B) = 0$. By 1.(i), we have $B = M$.

(iii) Let $\ell_S(A) + X = S$ for $X \subseteq S$. Then $r_M(\ell_S(A) + X) = r_M(S) = 0$. But $r_M(\ell_S(A)+X) = r_M\ell_S(A) \cap r_M(X) = A \cap r_M(X) = 0$. Since A is an essential submodule of M, $r_M(X) = 0$. Then $X = S$ by 2.(i).

(iv) Let $I + J = S$ for some $J \subseteq S$. Then $0 = r_M(I + J) = r_M(I) \cap r_M(J)$. Since $r_M(I)$ is essential in M, $r_M(J) = 0$ and so $J = S$ by 2.(i). □

In Theorem 3.4, shown that every quasi-dual module is AP-injective. Following [9], we have $Z(R_R) = J(R)$, where $J(R)$ and $Z(M_R)$ denote Jacobson radical of R and the singular submodule of an R-module M, respectively. Therefore,

Theorem 3.12 *Let M be a quasi-dual module. Then $Z(_SM) \subseteq Rad(M_R)$.*

Proof. If $x \in Z(_SM)$, then xR is small in M by Proposition 3.12 and hence $x \in Rad(M_R)$. □

Question : Let M be a quasi-dual module. When $Z(_SM) \subseteq Rad(M_R)$?

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