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The Radius of Starlikeness p-Valently Analytic Functions in the Unit Disc

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Abstract

In the present paper we shall give the radius of starlikeness for the classes of p-valent analytic functions in the unit disc $D = \{z \mid |z| < 1\}$.

Key Words: p-valent analytic functions, Radius of starlikeness, Radius of convexity.

1. Introduction

Let A_p the class of $f(z)$ normalized by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad p \in N = \{1, 2, 3, ...\} \tag{1.1}$$

which are analytic and p-valent in D. Further, let Ω be the family of functions $\omega(z)$ which are regular in D and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$. Next, for arbitrary fixed numbers A, B, $-1 \leq B < A \leq 1$, denote by $P(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \tag{1.2}$$

which are regular in D such that $p(z) \in P(A, B)$ if and only if

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$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)} \tag{1.3}$$

for some function $\omega(z) \in \Omega$ and every $z \in D$. This class was introduced by W. Janowski [4].

Morever, let $S^*(A, B, b, p, q)$ denote the family of functions $f(z) \in A_p$ and such that $f(z)$ is in $S^*(A, B, b, p, q)$ if and only if

$$1 + \frac{1}{b}(z.\frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) = p(z) \tag{1.4}$$

for some functions $p(z) \in P(A, B)$ and all $z \in D$, and $q \in N_0 = N \cup \{0\}$, whereas, as usual, $f^{(q)}(z)$ denotes the derivative of $f(z)$ with respect to z of order q, and

$$f^{(0)}(z) = f(z).$$

We note that by giving specific values to A, B, b, p and q, we obtain the subclasses of the class $S^*(A, B, b, p, q)$ which were considered earlier by various authors [1], [2], [5], [6], [9], and [10].

We shall need the following definition and lemma.

Definition 1.1 *The radius for the property $\Im$ in the class F is denoted by $R_{\Im}(F)$ and is the largest R such that every function in the class F has the property $\Im$ in each disc D_r for every $r < R$.*

2. New Results

In this section of this paper, we shall give the radius of starlikeness and the radius of convexity for the class $S^*(A, B, b, p, q)$.

Lemma 2.1 *Let $\omega(z)$ be regular in the unit disc with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, we can write $z_1\omega'(z_1) = k\omega(z_1)$, where k is real and $k \geq 1$.*

This lemma was proved by I. S. Jack [3].

Lemma 2.2 *The function*

$$w=\begin{cases}\frac{1+Az}{1+Bz} & , B\neq 0\\ \\ 1+Az & , B=0\end{cases}$$

maps $|z|=r$ *onto a disc centred at* $C(r)$*, and having the radius* $\rho(r)$*, viz.*

$$\begin{cases}C(r)=(\frac{(1-ABr^2)}{1-B^2r^2},0) & , \rho(r)=\frac{(A-B).r}{1-B^2r^2}\ ,\ B\neq 0\\ \\ C(r)=(1,0) & , \rho(r)=|A|.r\ ,\ B=0.\end{cases}$$

Proof.

$$\begin{cases}\left.\begin{array}{l} w=\frac{1+Az}{1+Bz}\Leftrightarrow z=\frac{w-1}{A-Bw}\Leftrightarrow |z|^2=r^2=\frac{|w-1|^2}{|A-Bw|^2}\\ \Rightarrow u^2+v^2+\frac{(2ABr^2-2)}{1-B^2r^2}u+\frac{(1-A^2r^2)}{1-B^2r^2}=0\end{array}\right\} , B\neq 0\\ \\ \left.\begin{array}{l} w=1+Az\Leftrightarrow z=\frac{w-1}{A}\Leftrightarrow |z|^2=r^2=\frac{|w-1|^2}{|A|^2}\\ \Rightarrow u^2+v^2-2u+(1-A^2r^2)=0\end{array}\right\} , B=0.\end{cases} \tag{2.1}$$

Lemma follows from (2.1). □

Lemma 2.3 *The function*

$$w=\begin{cases}\frac{(A-B)z}{1+Bz} & , B\neq 0\\ \\ Az & , B=0\end{cases}$$

maps $|z|=r$ *onto the disc centred at* $C(r)$*, and having radius* $\rho(r)$

$$\begin{cases}C(r)=(-\frac{B(A-B)r^2}{1-B^2r^2},0) & , \rho(r)=\frac{(A-B).r^2}{1-B^2r^2}\ ,\ B\neq 0\\ \\ C(r)=(0,0) & , \rho(r)=|A|.r\ ,\ B=0.\end{cases}$$

Proof.

$$\begin{cases} \left.\begin{array}{l} w = \frac{(A-B)z}{1+Bz} \Leftrightarrow z = \frac{w}{(A-B)-Bw} \Leftrightarrow |z|^2 = r^2 = \frac{|w|^2}{|(A-B)-Bw|^2} \\ \Rightarrow u^2 + v^2 + \frac{(2B(A-B)r^2)}{1-B^2r^2}u + \frac{(A-B)^2r^2}{1-B^2r^2} = 0 \end{array}\right\} , \ B \neq 0 \\ \\ \left.\begin{array}{l} w = Az \Leftrightarrow z = \frac{w}{A} \Leftrightarrow |z|^2 = r^2 = \frac{|w|^2}{|A|^2} \\ \Rightarrow u^2 + v^2 - r^2A^2 = 0 \end{array}\right\} , \ B = 0. \end{cases} \tag{2.2}$$

□

Lemma follows from (2.2).

Theorem 2.1 *Let $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + ...$ be an analytic function in the unit disc D. If $f(z)$ satisfies*

$$\frac{1}{b}(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) \prec \begin{cases} \frac{(A-B)z}{1+Bz} = F_1(z) \ , \ B \neq 0 \\ \\ A.z = F_2(z) \quad , \ B = 0, \end{cases} \tag{2.3}$$

then $f(z) \in S^*(A, B, b, p, q)$, and this result is as sharp as the function $(\frac{1+Az}{1+Bz})$.

Proof. We define the function $w(z)$ by

$$\frac{f^{(q)}(z)}{z^{p-q}} = \begin{cases} (1 + Bw(z))^{\frac{b(A-B)}{B}} \ , \ B \neq 0 \\ \\ e^{Abw(z)} \quad , \ B = 0, \end{cases} \tag{2.4}$$

where $(1 + Bw(z))^{\frac{b(A-B)}{B}}$ and $e^{bAw(z)}$ have the values 1 at the origin.

Then $w(z)$ is analytic in D and $w(0) = 0$. If we take the logarithmic derivative from the equality (2.4) and after the brief calculations we get

$$\frac{1}{b}(z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) \prec \begin{cases} \frac{(A-B)zw'(z)}{1+Bw(z)} \quad , \ B \neq 0 \\ \\ A.z.w'(z) \quad , \ B = 0. \end{cases} \tag{2.5}$$

Now it is easy to realize that subordination (2.3) is equivalent to $|w(z)| < 1$ for all $z \in D$. Indeed, assume the contrary: there exists a $z_1 \in D$ such that $|w(z_1)| = 1$. Then by the Lemma of I. S. Jack, $z_1 w'(z_1) = kw(z_1)$ and $k \geq 1$ for such $z_1 \in D$ (using Lemma 2.3), and we have

$$\frac{1}{b}(z_1 \frac{f^{(q+1)}(z_1)}{f^{(q)}(z_1)} - p + q) \prec \begin{cases} \frac{(A-B)kw(z_1)}{1+Bw(z_1)} = F_1(w(z_1)) \notin F_1(D) & , \ B \neq 0 \\ A.k.w(z_1) = F_2(w(z_1)) \notin F_2(D) & , \ B = 0. \end{cases} \tag{2.6}$$

But this is a contradiction of (2.3) of this theorem; so our assumption is wrong, i.e., $|w(z)| < 1$ for all $z \in D$. By using condition (2.5), we get

$$1 + \frac{1}{b}(z \frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) = \begin{cases} \frac{1+Aw(z)}{1+Bw(z)} & , \ B \neq 0 \\ 1 + Aw(z) & , \ B = 0. \end{cases} \tag{2.7}$$

Then we obtain from equality (2.7)

$$1 + \frac{1}{b}(z \frac{f^{(q+1)}(z)}{f^{(q)}(z)} - p + q) \prec \begin{cases} \frac{1+A.z}{1+B.z} & , \ B \neq 0 \\ 1 + A.z & , \ B = 0. \end{cases} \tag{2.8}$$

From equality (2.8), we get $f(z) \in S^*(A, B, b, p, q)$. □

Corollary 2.1 *Let $f(z) \in S^*(A, B, b, p, q)$. Then $f(z)$ can be written in the form*

$$f_*^{(q)}(z) = \begin{cases} z^{p-q}(1 + Bw(z))^{\frac{b(A-B)}{B}} & , \ B \neq 0 \\ z^{p-q}.e^{Abw(z)} & , \ B = 0, \end{cases}$$

Theorem 2.2 *The radius of starlikeness and the radius of convexity of the class $S^*(A, B, b, p, q)$ is*

$$R_{sc} = \frac{2(p-q)}{|b|(A-B) + \sqrt{|b|^2 (A-B)^2 - 4(p-q)\left[(B^2-AB)Reb + (q-p)B^2\right]}}. \tag{2.9}$$

This radius is sharp because the extremal function is

$$f_*^{(q)}(z) = \begin{cases} z^{p-q}(1+Bw(z))^{\frac{b(A-B)}{B}} & , \ B \neq 0 \\ z^{p-q}.e^{Abw(z)} & , \ B = 0 \end{cases}.$$

Proof. By using Lemma 2.2. set of values $(z \cdot \frac{f^{(q+1)}(z)}{f^{(q)}(z)})$ is obtained which comprises the closed disc with centre $C(r)$ and the radius $\rho(r)$, where

$$C(r) = \frac{(p-q) - \left[(AB-B^2)b + (p-q)B^2\right].r^2}{1-B^2r^2},$$

$$\rho(r) = \frac{|b|(A-B)r}{1-B^2r^2}.$$

Therefore, by using the definition of the class $S^*(A, B, b, p, q)$, we have

$$\left| z\frac{f^{(q+1)}(z)}{f^{(q)}(z)} - C(r) \right| \leq \rho(r).$$

This gives

$$\mathrm{Re}(z.\frac{f^{(q+1)}(z)}{f^{(q)}(z)}) \geq \frac{(p-q) - |b|(A-B)r + \left[(B^2-AB)Reb + (q-p)B^2\right].r^2}{1-B^2.r^2}. \tag{2.10}$$

Hence for $r < R_{sc}$ the first hand side of the preceeding inequality is positive, implying that

$$R_{sc} = \frac{2(p-q)}{|b|(A-B) + \sqrt{|b|^2 (A-B)^2 - 4(p-q)\left[(B^2-AB)Reb + (q-p)B^2\right]}} \tag{2.11}$$

Also note that inequality (2.9) becomes an equality for the function $f_*^{(q)}(z)$; it follows that

$$R_{sc} = \frac{2(p-q)}{|b|\,(A-B) + \sqrt{|b|^2\,(A-B)^2 - 4(p-q)\,[(B^2-AB)Reb + (q-p)B^2]}}.$$

□

Remark 2.3 (i) By taking $q=0,\ p=1,\ A=1,$ *and* $B=-1$ in (2.9), we obtain

$$R_s = \frac{1}{|b| + \sqrt{|b|^2 - 2Reb + 1}}.$$

This is the radius of starlikeness for the class of starlike functions of complex order which was obtained by M. A. Nasr and M. K. Aouf [6].

(ii) By setting $, q=0$ in (2.9), then we obtain the radius of starlikeness for the class $S^*(A,B,b,p,0)$

$$R_s = \frac{2p}{|b|\,(A-B) + \sqrt{|b|^2\,(A-B)^2 - 4p\,[(B^2-AB)Reb + -pB^2]}}.$$

(iii) *By letting* $q=1$ in (2.9), we also obtain the radius of convexity for the class $S^*(A,B,b,p,1)$

$$R_c = \frac{2(p-1)}{|b|\,(A-B) + \sqrt{|b|^2\,(A-B)^2 - 4(p-1)\,[(B^2-AB)Reb + (1-p)B^2]}}.$$

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