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Abstract

It is shown that local Fourier bases are unconditional bases for modulation spaces. We prove first a version of the Schur test for double sequence with mixed norm and then use it to show boundedness of the analysis operator on the modulation space $M^w_{p,q}$

Key Words: local Fourier bases, Schur test, mixed norm space, atomic decomposition

1. Introduction

It is an important theme that "nice" functions are unconditional bases for some function spaces. Results of this type are proven in several articles and in almost every book on wavelets. In all those places, the results depend on the particular space and the bases. In this paper we consider the spaces of modulation spaces $M^w_{p,q}$ and the local Fourier bases to be defined later.

In [16], Feichtinger, Gröchenig and Walnut proved that the Wilson bases of Daubbechies, Jaffard and Journé constructed in [5] are unconditional bases for modulation spaces. Since the Wilson bases are a special case of the local Fourier bases, we proved in [18] during our work in nonlinear approximation problem that local Fourier bases are unconditional bases for the modulation spaces $M^w_{p,p}$. In this paper we generalize this result to the case $M^w_{p,q}$ with $p \neq q$.

Let us start with some definitions and notations. If w is a weight function then the *weighted mixed-norm* space $L^{p,q}_w(\mathbb{R}^2)$ is defined to be the space of all measurable functions on $\mathbb{R}^2$ for which the following norm is finite:

$$\|f\|_{L^{p,q}_w} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|^p w(x,y)^p dx \right)^{q/p} dy \right)^{1/q}, \tag{1}$$

with the obvious modifications if p or $q = \infty$, in which case we use the supremum.

If $p = q$, then $L^{p,q}_w(\mathbb{R}^2)$ coincides with $L^p_w(\mathbb{R}^2)$. It is known [3] that these spaces are Banach spaces for $p, q \geq 1$ and quasi-Banach spaces for $0 < p, q < 1$. If Λ_1 and Λ_2 are two countable index sets, then $\ell^{p,q}(\Lambda_1 \times \Lambda_2)$ is the space of all sequences $c = (c_{ij})_{(i,j)\in\Lambda_1\times\Lambda_2} \subseteq \mathbb{C}$ such that

$$\|c\|_{\ell_{p,q}} = \left(\sum_{i\in\Lambda_1} \left(\sum_{j\in\Lambda_2} |c_{ij}|^p \right)^{q/p} \right)^{1/q} < \infty.$$

If w' is the restriction of w to the set $\Lambda_1 \times \Lambda_2$, then $\ell^{p,q}_{w'}(\Lambda_1 \times \Lambda_2)$ is

$$\{c = (c_{ij})_{i\in\Lambda_1, j\in\Lambda_2} : \|c\|_{\ell^{p,q}_{w'}} = \|cw'\|_{p,q} < \infty\}.$$

The Hölder inequality for double sequences with discrete mixed-norm is given as follows
If $a \in \ell^{p,q}_w$ and $b \in \ell^{p',q'}_{1/w}$, $1 < p, p', q, q' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, then

$$\left| \sum_{\mathbb{Z}^2} a_{\alpha\beta} \overline{b_{\alpha\beta}} \right| \leq ||a||_{\ell^{p,q}_w} ||b||_{\ell^{p',q'}_{1/w}}. \tag{2}$$

The Minkowski's inequality is given by

$$\left(\sum_{\beta} \left| \sum_{\alpha} a_{\alpha\beta} \right|^p \right)^{1/p} \leq \sum_{\alpha} \left(\sum_{\beta} |a_{\alpha\beta}|^p \right)^{1/p} \tag{3}$$

for $1 \leq p < \infty$ These inequalities will be used later.

2. The Modulation Spaces

In this section we collect some facts about the short-time Fourier transform (STFT) such as its definition and a pointwise estimation. We also give some results on modulation spaces needed for the subject. Let $\mathcal{S}$ be the Schwartz space of smooth functions with rapid decay, and let $\mathcal{S}'$ be the space of tempered distributions. Also let

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_y f(t) = e^{2\pi i y t} f(t) \tag{4}$$

be the operators of translation and modulation by $x, y \in \mathbb{R}$. Then the STFT of $f \in \mathcal{S}'$ with respect to the window $g \in \mathcal{S}$ is defined to be

$$S_g f(x,y) = \langle f, M_y T_x g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t-x)} e^{-2\pi i y t} dt \tag{5}$$

for all $x, y \in \mathbb{R}$. In this paper we consider the weighted mixed norm of the function $S_g f(x,y)$ which induces a norm on f and obtain the so-called modulation spaces. These spaces are a mathematical tool to measure the joint time-frequency distribution of tempered distribution, and they are defined by the decay properties of the STFT. To define these spaces we consider moderate weights, i.e. non-negative continuous functions satisfying

$$w(x+y) \leq C(1+|x|)^a w(y) \quad \text{for} \quad x, y \in \mathbb{R}, \tag{6}$$

for some constants $C > 0, a \geq 0$. The modulation space $M^w_{p,q}$ for $0 < p, q \leq \infty$ is the space of all tempered distribution f for which the norm

$$\|f\|_{M^w_{p,q}} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |S_g f(x,y)|^p (w(x,y))^p \, dx \right)^{q/p} dy \right)^{1/q} \tag{7}$$

is finite. For p or $q = \infty$, we use the supremum. If $p = q$ we write M^w_p, and if w is constant then we write $M_{p,q}$. These spaces were defined by Feichtinger and most of their properties were established (see [7, 10, 11, 15]). $M^w_{p,q}$ are Banach spaces for $1 \leq p, q \leq \infty$ and quasi Banach spaces for $0 < p, q < 1$ whose definition is independent of the window g (see [14, 20]). This means that different windows define the same space and yield equivalent norms. Recall that the dual space of $M^w_{p,q}$ is the space $M^{1/w}_{p',q'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$

Examples of modulation spaces:

1. Feichtinger's Segal algebra $S_0(\mathbb{R}) = M_{1,1}$ [8].

2. $M_{2,2}(\mathbb{R}) = L^2(\mathbb{R})$.

3. The Bessel potential space $H^s = M_2^w$, where $w(x,y) = (1+|y|^2)^{s/2}$.

The modulation spaces have nice atomic decomposition. Given $\varphi \in \mathcal{S}$ there exist $\beta > 0$ and $\gamma > 0$ small enough and a dual window $\varphi^\circ \in \mathcal{S}$, independent of $1 \le p, q \le \infty$ and w such that every $f \in \mathcal{S}'$ has an expansion

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, T_{\beta m} M_{\gamma n} \varphi^\circ \rangle T_{\beta m} M_{\gamma n} \varphi. \tag{8}$$

Moreover, $f \in M_{p,q}^w$ if and only if

$$\left(\sum_{m \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} |\langle f, T_{\beta m} M_{\gamma n} \varphi^\circ \rangle|^p w(\beta m, \gamma n)^p \right)^{q/p} \right)^{1/q} < \infty \tag{9}$$

and the sequence space norm in (9) is equivalent to the $M_{p,q}^w$-norm. If $1 \le p, q < \infty$, then the so-called *Gabor expansion* (8) converges unconditionally in the norm of $M_{p,q}^w$. (see [7, 13, 17]).

Next we mention a pointwise estimate of STFT of elements of the set $\mathcal{C}$ defined by

$$\mathcal{C} = \mathcal{C}(M,K,N) = \{g \in C^N(\mathbb{R}) : \text{ supp } g \subseteq [-K,K], \max_{k=0,1,\ldots,N} \|g^{(k)}\|_1 \le M\}, \tag{10}$$

Lemma 1 *[18, 19] Let $\varphi \in C^\infty(\mathbb{R})$, supp $\varphi \subseteq [-L, L]$, and $C = K + L$. Then*

$$\sup_{g \in \mathcal{C}} |S_\varphi g(x,y)| \le C_0 \frac{1}{(1+|y|)^N} \chi_{[-C,C]}(x) \quad \textit{for all } x, y \in \mathbb{R}, \tag{11}$$

with a constant $C_0 > 0$ depending only on M, K, N.

3. Local Fourier Bases

One way to study a signal is to focus on its local properties. In mathematical language this means that if we are given a function f on $\mathbb{R}$, we divide $\mathbb{R}$ into intervals $I_j = [\alpha_j, \alpha_{j+1}]$ with $-\infty < \ldots < \alpha_j < \alpha_{j+1} < \ldots < \infty$ and study $\chi_{I_j} f$. An example of a complete orthonormal system for $L^2[\alpha_j, \alpha_{j+1}]$ is

$$\varphi_{j,k}(x) = \sqrt{\frac{2}{|I_j|}} \chi_{I_j}(x) \sin \frac{2k+1}{2} \frac{\pi}{|I_j|}(x - \alpha_j),$$

where k ranges through the non-negative integers $\mathbb{Z}^+$. If j ranges through the integers $\mathbb{Z}$, then we obtain a basis for $L^2(\mathbb{R})$. Such systems are appropriate for focusing on local properties but not for global properties due to the abrupt "cutoff" effected by multiplication by the characteristic function χ_{I_j}. In [4], Coifman and Meyer introduced orthonormal bases of this type that involve an arbitrary smooth cut off. In [1] Auscher, Weiss and Wickerhauser constructed such bases by introducing the bell function b_{I_j} using projections such that the set

$$\left\{\sqrt{\frac{2}{\alpha_{j+1}-\alpha_j}}\, b_{I_j}(x)\sin\frac{2k+1}{2}\frac{\pi}{\alpha_{j+1}-\alpha_j}(x-\alpha_j), j\in\mathbb{Z},\ \ k\in\mathbb{N}\right\}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Other forms are also given in [1]. The main property of such bases is for any $N\in\mathbb{N}\cup\{\infty\}$ there exists a bell function $b_j\in C^N(\mathbb{R})$ with support in $[\alpha_j-\epsilon_j,\alpha_{j+1}+\epsilon_{j+1}]$ such that these sets are orthonormal bases for $L^2(\mathbb{R})$, where $\epsilon_k\geq\epsilon>0$. If $\Delta_k=\alpha_{k+1}-\alpha_k$ and $\{\epsilon_k : k\in\mathbb{Z}\}$ is the accompanying sequence, then these formes are written [18] as

$$\frac{1}{2}\left(\frac{2}{\Delta_k}\right)^{1/2} T_{\alpha_k}\left(M_{\frac{l}{2\Delta_k}}\pm M_{\frac{-l}{2\Delta_k}}\right)g_k, \tag{12}$$

or

$$\frac{1}{2}\left(\frac{2}{\Delta_k}\right)^{1/2} T_{\alpha_k}\left(M_{\frac{l-\frac{1}{2}}{2\Delta_k}}\pm M_{-\frac{l-\frac{1}{2}}{2\Delta_k}}\right)g_k, \tag{13}$$

where M_y and T_x are as in (4) and supp $g_k\subseteq[-\epsilon_k,\alpha_{k+1}-\alpha_k+\epsilon_{k+1}]$.

In what follows, we work with the structure (12) since the other is similar, and it requires only replacing l by $l-\frac{1}{2}$.

4. Unconditional bases For Modulation Spaces

In this section we prove that local Fourier bases are unconditional bases for modulation spaces $M^w_{p,q}$.

Definition 1 . *A set $\{e_i, i\in I\}$ of vectors in a Banach space B is an unconditional basis, if*

(1) the finite linear combinations of the e_i's are dense in B, and if

(2) there exists a constant $C \geq 1$, such that

$$\|\sum_{i\in F} c_i\lambda_i e_i\| \leq C \sup_i |\lambda_i| \, \|\sum_{i\in F} c_i e_i\|$$

holds for any finite subset $F \subseteq I$ and any sequence $(\lambda_i) \subseteq \mathbb{C}$.

Theorem 1 *Suppose that $\{\psi_{kl}, (k,l) \in \mathbb{Z} \times \mathbb{N}\} \subseteq C^N(\mathbb{R})$ is a local Fourier base whose underlying partition satisfies $\frac{1}{A} \leq \alpha_{k+1} - \alpha_k \leq A,\ A > 1$, and $\inf_k \epsilon_k = \epsilon > 0$. If w is a weight function on $\mathbb{R}^2$ for $N > \frac{q}{p} + a,\ \ 0 < p \leq q < \infty$, then $\{\psi_{kl}\}$ is an unconditional basis for $M^w_{p,q}$. Every distribution $f \in M^w_{p,q}$ has a unique expansion*

$$f = \sum_{(k,l)\in\mathbb{Z}\times\mathbb{N}} \langle f, \psi_{kl}\rangle \psi_{kl} \tag{14}$$

with unconditional convergence in the norm of $M^w_{p,q}$. Moreover,

$$\frac{1}{C}\|f\|_{M^w_{p,q}} \leq \left(\sum_{k\in\mathbb{Z}}\left(\sum_{l\in\mathbb{N}} |\langle f, \psi_{kl}\rangle|^p w(\alpha_k, \frac{l}{2\Delta_k})^p\right)^{q/p}\right)^{1/q} \leq C\|f\|_{M^w_{p,q}} \tag{15}$$

for some constant $C > 0$. If $p = q = \infty$, then $\{\psi_{kl}\}$ is a weak basis, that is, the expansion (14) converges only in the weak-topology with respect to the predual $M^{1/w}_{1,1}$.*

Since the Wilson bases are a special case of local Fourier bases [1], then this theorem covers the results given in [16]. the following two corollaries.

Corollary 1 *The Wilson basis of exponential decay constructed in [5] is an unconditional basis for $M^w_{p,q}$ for $0 < p \leq q < \infty$.*

Corollary 2 *(see [18]) Any local Fourier basis in $C^N(\mathbb{R})$ is an unconditional bases for S_0, if $N > 1$ and H^s if $s < N - 1$.*

To prove Theorem 1 we use the pointwise estimate for the STFT given in Lemma 1. We consider

the set of windows given by

$$\mathcal{C} = \mathcal{C}(M, K, N) = \{g \in C^N(\mathbb{R}) : \sup\ g \subseteq [-K, K], \max_{k=0,1,\ldots,N} \|g^{(k)}\|_1 \leq M\}.$$

The explicit construction of the bell functions of a local Fourier basis leads to the following consequence.

Lemma 2 *[18] If $\frac{1}{A} \leq \alpha_{k+1} - \alpha_k \leq A$ and if $\epsilon_k \geq \epsilon > 0$ for all $k \in \mathbb{Z}$, then*

$$\{g_k = T_{-\alpha_k} b_k,\ k \in \mathbb{Z}\} \subseteq \mathcal{C}(M, K, N)$$

for some $M, K,$ and N.

To prove Theorem 1 so we shall extend the orthonormal expansion (14) from $L^2(\mathbb{R})$ to the modulation spaces $M^w_{p,q}$. For this purpose we study the action of certain associated operators.

The *analysis operator* τ is defined by

$$\tau f = (\langle f, \psi_{kl} \rangle)_{(k,l)\in I}. \tag{16}$$

Since $\{\psi_{kl}, (k,l) \in I\}$ is an orthonormal basis, τ is well defined and maps $L^2(\mathbb{R})$ onto $\ell^2(I)$. The formal adjoint is the *synthesis operator* τ^* which acts on "sequences" $c = (c_{kl})_{(k,l)\in I}$ as

$$\tau^*((c_{kl})_{k,l\in I}) = \sum_{(k,l)\in I} c_{kl} \psi_{kl}. \tag{17}$$

The next two propositions show that both operators extend to other function or sequence spaces. We write

$$\eta_{kl} = (\alpha_k, \frac{l}{2\Delta_k}), (k,l) \in \mathbb{Z}^2,$$

for the points in the time-frequency plane associated to ψ_{kl}, and for a given weight function w, w' denotes its restriction $w'(k,l) = w(\eta_{kl})$ to the discrete set $\{\eta_{kl}\}$. Then $\ell^{p,q}_{w'}(I)$ consists of all sequences on I for which

$$\|c\|_{p,q,w'} = \left(\sum_k \left(\sum_l |c_{kl}|^p w(\eta_{kl})^p \right)^{q/p} \right)^{1/q} < \infty\,.$$

We first prove an estimate for the STFT of a local Fourier bases

Lemma 3 *[18, 19] Using the notation of Lemma 1, set*

$$G(x,y) = \chi_{[-C,C]}(x)\frac{1}{(1+|y|)^N}.$$

If $\{\psi_{kl}, (k,l) \in I\} \subseteq C^N(\mathbb{R})$ *is a local Fourier bases which satisfies the assumptions of Theorem 1, then there exists* $C_1 > 0$*, such that*

$$|S_\varphi \psi_{kl}(x,y)| \leq C_1(T_{\eta_{kl}}G(x,y) + T_{\eta_{k,-l}}G(x,y)) \quad \text{for all } x, y \in \mathbb{R}\,. \tag{18}$$

To prove that τ is bounded, we prove first a weighted version of Schur's Test.

5. A Schur Test For Discrete Weighted Mixed-Norm Spaces

In this section, we prove a version of Schur's test for discrete weighted mixed-norm spaces $\ell_w^{p;q}$. Precisely, we prove that if $A = (a_{(i,j),(k,l)})_{i\in I, j\in J, k\in K, l\in L}$ is an infinite matrix, then under some certain conditions the map A from $\ell_{w_1^{p,q}}$ to $\ell_{w_2^{p,q}}$ is bounded.

Lemma 4 *Suppose that* $w_1(k,l)$ *and* $w_2(i,j)$ *are two weight functions on the index sets* $K \times L$ *and* $I \times J$ *respectively, and let* $A = (a_{(i,j),(k,l)})_{i\in I, j\in J, k\in K, l\in L}$ *be an infinite matrix such that*

$$\sum_{k\in K}\sum_{l\in L} |a_{(i,j),(k,l)}| w_1(k,l)^{-1} \leq C_\circ w_2(i,j)^{-1} < \infty \quad \text{for all } \ i \in I, j \in J \tag{19}$$

and

$$\sum_{i\in I}\left(\sum_{J\in J} |a_{(i,j),(k,l)}|^{q/p} w_2(i,j)^{q/p}\right)^{p/q} \leq C_1 w_1(k,l)\varphi(k) < \infty \quad \text{for all } \ k \in K, l \in L \tag{20}$$

for some constants $C_\circ, C_1 > 0$ *and some sequence* $\varphi(k) \in \ell_u$*. If* $1 \leq p \leq q < \infty$ *and* $u = \frac{q}{q-p}$*, then the map* A *is bounded from* $\ell_{w_1}^{p;q}(K \times L)$ *into* $\ell_{w_2}^{p;q}(I \times J)$*.*

Proof. The case $p = q$ is treated in [18], so we study the case $p \neq q$. Let $c = (c_{kl})$ be an element in $\ell_{w_1}^{p;q}$, and let $1 < p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, then we estimate $\|Ac\|_{\ell_{w_2}^{p,q}}^q$ using

Hölder inequality (2), viz

$$\begin{aligned}
\|Ac\|^q_{\ell^{p,q}_{w_2}} &= \sum_j \left(\sum_i \left|\sum_k \sum_l a_{(i,j),(k,l)} c_{kl}\right|^p w_2(i,j)^p\right)^{q/p} \\
&\leq \sum_j \left\{\sum_i w_2(i,j)^p \left(\sum_k \sum_l \left|a_{(i,j),(k,l)} c_{kl}\right|\right)^p\right\}^{q/p} \\
&= \sum_j \left\{\sum_i w_2(i,j)^p \left(\sum_k \sum_l \left|a_{(i,j),(k,l)}\right|^{\frac{1}{p}+\frac{1}{p'}} |c_{kl}| w_1(k,l)^{\frac{1}{p'}-\frac{1}{p'}}\right)^p\right\}^{q/p} \\
&\leq \sum_j \left\{\sum_i w_2(i,j)^p \left(\sum_k \sum_l \left|a_{(i,j),(k,l)}\right| |c_{kl}|^p w_1(k,l)^{\frac{p}{p'}}\right)\right. \\
&\quad \left. \cdot \left(\sum_k \sum_l \left|a_{(i,j),(k,l)}\right| w_1(k,l)^{-1}\right)^{\frac{p}{p'}}\right\}^{q/p}
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_j \left\{\sum_i w_2(i,j)^p C_\circ^{\frac{p}{p'}} w_2(i,j)^{-\frac{p}{p'}} \sum_k \sum_l \left|a_{(i,j),(k,l)}\right| |c_{kl}|^p w_1(k,l)^{\frac{p}{p'}}\right\}^{\frac{q}{p}} \\
&= C_\circ^{\frac{q}{p'}} \sum_j \left\{\sum_i w_2(i,j) \sum_k \sum_l \left|a_{(i,j),(k,l)}\right| |c_{kl}|^p w_1(k,l)^{\frac{p}{p'}}\right\}^{q/p}
\end{aligned}$$

where we used Hölder inequality (2) in the third inequality and condition (19) in the fourth inequality. Now we write $\|Ac\|^q_{\ell^{p,q}_{w_2}}$ in the form

$$\left(\|Ac\|^q_{\ell^{p,q}_{w_2}}\right)^{p/q} \leq C_\circ^{p/p'} \left(\sum_j \left\{\sum_k \sum_l \sum_i \left|a_{(i,j),(k,l)}\right| w_2(i,j) |c_{kl}|^p w_1(k,l)^{\frac{p}{p'}}\right\}^{q/p}\right)^{p/q}.$$

Using Minkowski's inequality (3) twice, we get

$$\begin{aligned}
\|Ac\|^p_{\ell^{p,q}_{w_2}} &\leq C_\circ^{p/p'} \sum_k \left(\sum_j \left\{ \sum_l \sum_i \left| a_{(i,j),(k,l)} \right| w_2(i,j) |c_{kl}|^p w_1(k,l)^{\frac{p}{p'}} \right\}^{q/p} \right)^{p/q} \\
&\leq C_\circ^{p/p'} \sum_k \sum_l \sum_i \left(\sum_j \left\{ \left| a_{(i,j),(k,l)} \right| w_2(i,j) |c_{kl}|^p w_1(k,l)^{\frac{p}{p'}} \right\}^{q/p} \right)^{p/q} \\
&= C_\circ^{p/p'} \sum_k \sum_l |c_{kl}|^p w_1(k,l)^{\frac{p}{p'}} \sum_i \left(\sum_j \left| a_{(i,j),(k,l)} \right|^{q/p} w_2(i,j)^{q/p} \right)^{p/q}.
\end{aligned}$$

Now, using condition (20), we have

$$\begin{aligned}
\|Ac\|^p_{\ell^{p,q}_{w_2}} &\leq C_\circ^{p/p'} C_1 \sum_k \sum_l |c_{kl}|^p w_1(k,l)^{\frac{p}{p'}+1} \varphi(k) \\
&= C_\circ^{p/p'} C_1 \sum_k \sum_l |c_{kl}|^p w_1(k,l)^p \varphi(k).
\end{aligned}$$

Since $\frac{p}{q} + \frac{q-p}{q} = 1$, we apply Hölder inequality and get

$$\|Ac\|^p_{\ell^{p,q}_{w_2}} \leq C_\circ^{p/p'} C_1 \left(\sum_k \left(\sum_l |c_{(k,l)}|^p w_1(k,l)^p \right)^{q/p} \right)^{p/q} \cdot \left(\sum_k \varphi(k)^{\frac{q}{q-p}} \right)^{\frac{q-p}{q}}.$$

Therefore

$$\|Ac\|_{\ell^{p,q}_{w_2}} \leq C_\circ^{1/p'} C_1^{1/p} \|c\|_{\ell^{p,q}_{w_1}} \|\varphi(k)\|^{1/p}_{\ell_u} < \infty$$

□

The boundedness of the analysis operator is the content of the following proposition.

Proposition 1 . *Under the hypotheses of Theorem 1, τ is a bounded operator from $M^w_{p,q}$ into $\ell^{p;q}_{w'}(I)$ for $1 \leq p, q \leq \infty$.*

The proof of this proposition is based the Schur test as well as the following lemma

Lemma 5 *Let* $\phi(k) = (\sum_l (1 + C + |\gamma n - \frac{\pm l}{2\Delta_k}|)^{aq/p} (1 + |\frac{\pm l}{2\Delta_k} - \gamma n|)^{-Nq/p})^{p/q}$, *and* $N > \frac{q}{p} + a,\ \frac{1}{A} < \Delta_k < A,\ A \geq 1$.*if* $u = \frac{q}{q-p}$ *for* $1 \leq p \leq q$ *Then* $\phi(k) \in \ell_u$.

Proof. Since $u = \frac{q}{q-p}$ then $u' = \frac{q}{p}$ where $\frac{1}{u} + \frac{1}{u'} = 1$.
Now

$$\begin{aligned}
\|\phi\|_{\ell_u} &= \left(\sum_k \left(\sum_l (1 + C + |\gamma n - \frac{\pm l}{2\Delta_k}|)^{aq/p} (1 + |\frac{\pm l}{2\Delta_k} - \gamma n|)^{-Nq/p} \right)^{\frac{p}{q-p}} \right)^{\frac{q-p}{q}} \\
&= \left\| (1 + C + |\gamma n - \frac{\pm l}{2\Delta_k}|)^{a} (1 + |\frac{\pm l}{2\Delta_k} - \gamma n|)^{-N} \right\|_{u',u} \\
&\leq \left\| (1 + C + |\gamma n - \frac{\pm l}{2\Delta_k}|)^{a} (1 + |\frac{\pm l}{2\Delta_k} - \gamma n|)^{-N} \right\|_{1,u} \\
&= \left(\sum_k \left(\sum_l (1 + C + |\gamma n - \frac{\pm l}{2\Delta_k}|)^{a} (1 + |\frac{\pm l}{2\Delta_k} - \gamma n|)^{-N} \right)^{\frac{q}{q-p}} \right)^{\frac{q-p}{q}} \\
&= \left\| \sum_l (1 + C + |\gamma n - \frac{\pm l}{2\Delta_k}|)^{a} (1 + |\frac{\pm l}{2\Delta_k} - \gamma n|)^{-N} \right\|_{\ell_u} \\
&\leq \left\| \sum_l (1 + C + |\gamma n - \frac{\pm l}{2\Delta_k}|)^{a} (1 + |\frac{\pm l}{2\Delta_k} - \gamma n|)^{-N} \right\|_{\ell_1} \\
&= \sum_k \sum_l (1 + C + |\gamma n - \frac{\pm l}{2\Delta_k}|)^{a} (1 + |\frac{\pm l}{2\Delta_k} - \gamma n|)^{-N}.
\end{aligned}$$

The last equality is less than

$$\sup_k \sum_l (1 + C + |\gamma n - \frac{\pm l}{2\Delta_k}|)^{a} (1 + |\frac{\pm l}{2\Delta_k} - \gamma n|)^{-N},$$

(see [18]) and since $N > \frac{q}{p} + a > 1 + a,\ \frac{1}{A} < \Delta_k < A,\ A \geq 1$. Then

$$\|\phi\|_{\ell_u} < \infty,$$

which implies that. $\phi \in \ell_u$ □

Now we prove proposition 1.

Proof. The cases $p = q = 1$ and $p = q = \infty$ are treated in [18, 19]. For $1 \le p \le q < \infty$, we use lemma 4.

Given $\varphi \in C^\infty$ with compact support and β, γ small enough, there exists [17] a dual window $\varphi^\circ \in \mathcal{S}$, such that every $f \in \mathcal{S}'$ has an atomic decomposition

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, T_{\beta m} M_{\gamma n} \varphi^\circ \rangle T_{\beta m} M_{\gamma n} \varphi \,,$$

with $f \in M^w_{p,q}$, if and only if

$$\left(\sum_{m\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} |\langle f, T_{\beta m} M_{\gamma n} \varphi^\circ \rangle|^p w(\beta m, \gamma n)^p \right)^{q/p} \right)^{1/q} < \infty.$$

Then

$$(\tau f)_{kl} = \sum_{m,n} \langle f, T_{\beta m} M_{\gamma n} \varphi^\circ \rangle \langle T_{\beta m} M_{\gamma n} \varphi, \psi_{kl} \rangle \,.$$

Therefore the proposition is proved if we can show that the map defined by the matrix

$$A_{(k,l),(m,n)} = \langle T_{\beta m} M_{\gamma n} \varphi, \psi_{kl} \rangle$$

maps the sequence

$$c_{mn} = \langle f, T_{\beta m} M_{\gamma n} \varphi^\circ \rangle \in \ell^{p;q}_{w_1}(\mathbb{Z}^2), \;\; w_1(m,n) = w(\alpha m, \gamma n)$$

into $\ell^{p;q}_{w'}(I), w'(k,l) = w(\eta_{kl})$. For this it is enough to verify the conditions of Schur's test. Since the first condition is similar to the first condition in [18], it is enough to verify the second condition of Schur's test. We write (6) in the form

$$w(\alpha_k, \frac{\pm l}{2\Delta_k}) \le w(\beta m, \gamma n) \left(1 + |\alpha_k - \beta m| + |\frac{\pm l}{2\Delta_k} - \gamma n| \right)^a \,. \tag{21}$$

Now, using this inequality, we estimate the sum

$$\Sigma = \sum_{k\in\mathbb{N}} \left(\sum_{l\in\mathbb{Z}} |\langle T_{\beta m} M_{\gamma n} \varphi, \psi_{kl} \rangle|^{q/p} w(\alpha_k, \pm \frac{l}{2\Delta_k})^{q/p} \right)^{p/q}$$

as

$$\Sigma\leq\sum_{k\in\mathbb{Z}}\left(\sum_{l\in\mathbb{Z}}|\langle T_{\beta m}M_{\gamma n}\varphi,\psi_{kl}\rangle|^{q/p}w(\beta m,\gamma n)^{q/p}\right.$$

$$\left.\cdot\left(1+|\alpha_k-\beta m|+|\frac{\pm l}{2\Delta_k}-\gamma n|\right)^{aq/p}\right)^{p/q}$$

$$\leq C_1 w(\beta m,\gamma n)\sum_{k\in\mathbb{Z}}\left(\sum_{l\in\mathbb{Z}}\left(G(\beta m-\alpha_k,\gamma n-\frac{l}{2\Delta_k})+G(\beta m-\alpha_k,\gamma n+\frac{l}{2\Delta_k})\right)^{q/p}\right.$$

$$\left.\cdot\left(1+|\alpha_k-\beta m|+|\frac{\pm l}{2\Delta_k}-\gamma n|\right)^{aq/p}\right)^{p/q}$$

$$\leq C_1 w(\beta m,\gamma n)\sum_{k\in\mathbb{Z}}\left(\sum_{l\in\mathbb{Z}}\left(2\sup_k\{G(\beta m-\alpha_k,\gamma n-\frac{l}{2\Delta_k}),G(\beta m-\alpha_k,\gamma n+\frac{l}{2\Delta_k})\}\right)^{aq/p}\right.$$

$$\left.\cdot\left(1+|\alpha_k-\beta m|+|\frac{\pm l}{2\Delta_k}-\gamma n|\right)^{aq/p}\right)^{p/q}.$$

Since $|\beta m-\alpha_k|\leq C$ and

$$G(\beta m-\alpha_k,\gamma n-\frac{\pm l}{2\Delta_k})\leq\chi_{[-C,C]}(\beta m-\alpha_k)\cdot(1+|\frac{\pm l}{2\Delta_k}-\gamma n|)^{-N},$$

we have

$$\Sigma\leq 2^{a+1}C_1C^a w(\beta m,\gamma n)\sum_{k\in\mathbb{Z}}\left(\sum_{l\in\mathbb{Z}}(1+C+|\gamma n-\frac{\pm l}{2\Delta_k}|)^{aq/p}(1+|\frac{\pm l}{2\Delta_k}-\gamma n|)^{-Nq/p}\right)^{p/q}.$$

Since $\frac{1}{A}\leq\alpha_{k+1}-\alpha_k\leq A,\ A>1$, we have

$$\Sigma\leq 2^{a+1}C_1C^a w(\beta m,\gamma n)\sum_{k}\left(\sum_{l}(1+C+|\gamma n-\frac{\pm l}{2\Delta_k}|)^{aq/p}(1+|\frac{\pm l}{2\Delta_k}-\gamma n|)^{-Nq/p}\right)^{p/q}.$$

Let $\phi(k)=(\sum_l(1+C+|\gamma n-\frac{\pm l}{2\Delta_k}|)^{aq/p}(1+|\frac{\pm l}{2\Delta_k}-\gamma n|)^{-Nq/p})^{p/q}$. Then by using lemma 5 we conclude that $(\phi(k)))$ is in ℓ_u. and we have the second condition of lemma 4. □

The following proposition is proved in [18] for $M^w_{p,q},\ p=q$, We prove it for $M^w_{p,q},\ p\neq q$ for convenience.

Proposition 2 *The map τ^* is a bounded map from $\ell^{p;q}_{w'}$ into $M^w_{p,q}$ for $1 \le p \le q \le \infty$.*

Proof. Assume that $p, q < \infty$, and let $c = (c_{kl})_{(k,l)\in I}$ be finitely supported. If $h \in M^{1/w}_{p',q'}$, the dual of $M^w_{p,q}$, then proposition 1 implies

$$\left|\sum_{k,l}\langle c_{kl}\psi_{kl}, h\rangle\right| = |\sum_{k,l} c_{kl}\overline{(\tau h)_{kl}}| \le \|c\|_{\ell^{p,q}_{w'}}\|(\tau h)\|_{\ell^{p',q'}_{1/w'}} \le \|c\|_{\ell^{p,q}_{w'}}\|\tau\|_{op}\|h\|_{M^{1/w}_{p',q'}}$$

Hence the estimates

$$\|\tau^* c\|_{M^w_{p,q}} = \|_{M^w_{p,q}} \sup_{\|h\|_{M^{1/w}_{p',q'}}\le 1} |\sum_{k,l}\langle c_{kl}\psi_{kl}, h\rangle| \le \|c\|_{\ell^{p,q}_{w'}}\|\tau\|_{op}$$

show that τ^* is bounded on $\ell^{p;q}_{w'}$ where $\|.\|_{op}$ is the operator norm.

Furthermore, for any $\epsilon > 0$ there exists a finite subset $F_\epsilon \subseteq I$, such that

$$\|\sum_{(k,l)} c_{kl}\psi_{kl}\|_{M^w_{p,q}} \le \|\tau\|_{op}\left(\sum_k\left(\sum_l |c_{kl}|^p w(\eta_{kl})^p\right)^{q/p}\right)^{1/q} < \epsilon$$

where $(k,l) \notin F$, for all finite subsets $F \supseteq F_\epsilon$. This means that $\tau^* c$ converges unconditionally.

If $p = \infty$ or $q = \infty$, then taking the supremum over $M^{1/w}_1$ shows that τ^* is bounded on $M^w_{\infty,\infty}$ and that sum is w^*-convergent. □

Proof of Theorem 1:
Since τ and τ^* are bounded on $M^w_{p,q}$ and $\ell^{p,q}_{w'}$, the identity

$$f = \sum_{k,l}\langle f, \psi_{kl}\rangle\psi_{kl} = \tau^*\tau f \tag{22}$$

extends from $L^2(I\!R)$ to $M^w_{p,q}$, $1 < p, q < \infty$, with unconditional convergence of the series (22). Thus finite linear combinations are dense in $M^w_{p,q}$. The norm equivalence (15) follows from

$$\|f\|_{M^w_{p,q}} \le \|\tau^*\|_{op}\,\|(\langle f, \psi_{kl}\rangle)_{(k,l)\in I}\|_{\ell^{p,q}_{w'}} \le \|\tau^*\|_{op}\|\tau\|_{op}\,\|f\|_{M^w_{p,q}}.$$

Furthermore, since in a (finite) linear combination $f = \sum_{k,l} c_{kl}\psi_{kl}$ the coefficients are uniquely determined as

$$c_{kl} = \langle f, \psi_{kl}\rangle = (\tau f)_{kl},$$

we can estimate

$$\begin{aligned}\|\sum_{k,l}\lambda_{kl}c_{kl}\psi_{kl}\|_{M^w_{p,q}} &= \|\tau^*(\lambda_{kl}c_{kl})\|_{M^w_{p,q}}\\ &\leq \|\tau^*\|_{op}\,\|(\lambda_{kl}c_{kl})_{(k,l)}\|_{\ell^{p,q}_{w'}}\\ &\leq \|\tau^*\|_{op}\|\lambda\|_\infty\|c\|_{\ell^{p,q}_{w'}}\\ &\leq \|\tau^*\|_{op}\,\|\lambda\|_\infty\|\,\|\tau\|_{op}\,\|f\|_{M^w_{p,q}}\,,\end{aligned}$$

where $\lambda = (\lambda_{k,l})_{(k,l)\in I}$ and $c = (c_{kl})_{(k,l)\in I}$. This shows that $\{\psi_{kl},(k,l) \in I\}$ is an unconditional basis for $M^w_{p,q}$. Therefore, Theorem 1 is proved in a general case. (See [18]) □

Remark 1 *We can get all result in [18] if we take $p = q$, and $\phi(k) = 1$, Note that the conditions in lemma 4 becomes exact conditions in Schur test in [18]; and proposition (1, 2) becomes proposition* 1 , 2 *in [18].*

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