

1-1-2006

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### Recommended Citation

CENGİZ, NEJMİ; TARAĞÇI, Ö.; and SALİMOV, A. A. (2006) "A Note on Kaehlerian Manifolds," *Turkish Journal of Mathematics*: Vol. 30: No. 4, Article 7. Available at: <https://journals.tubitak.gov.tr/math/vol30/iss4/7>

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## A Note on Kaehlerian Manifolds

*N. Cengiz, Ö. Tarakcı, A. A. Salimov*

### Abstract

The main purpose of the present paper is to study nearly Kaehlerian manifolds. We give the condition for an almost Hermitian manifold to be nearly Kaehlerian.

**Key Words:** Hybrid tensor, Hermitian manifold, Kaehlerian manifold, Tachibana operator.

### 1. Introduction

Let  $M$  be an almost Hermitian manifold with almost complex structure  $\varphi$  and hybrid Riemannian metric tensor field  $g$ . Then

$$\varphi^2 = -I, \quad g(\varphi X, \varphi Y) = g(X, Y) \quad (1)$$

for any vector field  $X$  and  $Y$  on  $M$ . We denote by  $\nabla$  the operator of covariant differentiation with respect to  $g$  in  $M$ . If the almost complex structure  $\varphi$  of  $M$  satisfies

$$(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0$$

for any vector field  $X$  and  $Y$  on  $M$ , then the manifold  $M$  is called a nearly Kaehlerian manifold (Tachibana spaces). The condition above reduces to

$$(\nabla_X \varphi)X = 0.$$

Let  $N$  be the Nijenhuis tensor field of  $\varphi$  defined by

$$N(X, Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] - [X, Y]$$

any vector field  $X$  and  $Y$  on  $M$ . By a simple computation we have

$$N(X, Y) = -4\varphi(\nabla_X \varphi)Y.$$

**Proposition:** *If the Nijenhuis torsion  $N$  of a nearly Kaehlerian manifold vanishes, then  $M$  is a Kaehlerian manifold.*

We define a Tachibana operator [3] (see also [2, 4])  $\Phi_\varphi \xi$  associated with an almost complex structure  $\varphi$  and an arbitrary  $X \in \mathfrak{S}_0^1(M)$  and applied to a tensor  $\xi \in \mathfrak{S}_2^0(M)$  as

$$\begin{aligned} \Phi_\varphi \xi(X, Z_1, Z_2) &= (L_{\varphi X} \xi)(Z_1, Z_2) - L_X(\xi \circ \varphi)(Z_1, Z_2) \\ &\quad + \xi(Z_1, \varphi(L_X Z_2)) - \xi(\varphi Z_1, L_X Z_2), \end{aligned} \tag{2}$$

where  $L_X$  denotes the operator of Lie derivation with respect to  $X$  and  $(\xi \circ \varphi)(Z_1, Z_2) = \xi(\varphi Z_1, Z_2)$ . Expression (2) defines a tensor field  $\Phi_\varphi \xi \in \mathfrak{S}_2^0(M)$  if and only if  $\xi$  as a pure tensor [4]. When

$$\Phi_\varphi \xi(X, Z_1, Z_2) = (L_{\varphi X} \xi)(Z_1, Z_2) - L_X(\xi \circ \varphi)(Z_1, Z_2) = 0 \tag{3}$$

for a pure tensor  $\xi$  and for any  $X, Z_1, Z_2 \in \mathfrak{S}_0^1(M)$ ,  $M$  being a manifold with almost complex structure  $\varphi$ ,  $\xi$  is said to be almost analytic [3].

## 2. Operator $\Phi$ Applied to a Hybrid Tensor

Let  $g$  be a hybrid Riemannian metric tensor. The following formulas are known (see [1]):

$$(L_X g)(Y_1, Y_2) = X(g(Y_1, Y_2)) - g([X, Y_1], Y_2) - g(Y_1, [X, Y_2]), \tag{4}$$

$$L_X Y = [X, Y] = \nabla_X Y - \nabla_Y X - T(X, Y) = \nabla_X Y - \nabla_Y X, \quad (5)$$

$$(\nabla K)(X_1, X_2, \dots, X_s, X) = (\nabla_X K)(X_1, X_2, \dots, X_s) = \nabla_X (K(X_1, X_2, \dots, X_s)) \quad (6)$$

$$- \sum_{i=1}^s K(X, \dots, \nabla_X X_i, \dots, X_s), \quad K \in \mathfrak{S}_s^1(M),$$

where  $\nabla$  denotes the operator of the Riemannian covariant derivation. By virtue of (1), (4) and (5), from (2) we get

$$\begin{aligned} (\Phi_\varphi g)(X, Z_1, Z_2) &= \varphi(X)(g(Z_1, Z_2)) - g(\nabla_{\varphi X} Z_1 - \nabla_{Z_1} \varphi(X), Z_2) \quad (7) \\ &\quad - g(Z_1, \nabla_{\varphi X} Z_2 - \nabla_{Z_2} \varphi(X)) - X(g(\varphi Z_1, Z_2)) + (g \circ \varphi)(\nabla_X Z_1 - \nabla_{Z_1} X, Z_2) \\ &\quad + (g \circ \varphi)(Z_1, \nabla_X Z_2 - \nabla_{Z_2} X) + g(Z_1, \varphi(\nabla_X Z_2 - \nabla_{Z_2} X)) - g(\varphi Z_1, \nabla_X Z_2 - \nabla_{Z_2} X) \\ &= \varphi(X)(g(Z_1, Z_2)) - X(g(\varphi Z_1, Z_2)) \\ &\quad - g(\nabla_{\varphi X} Z_1, Z_2) + g(\nabla_{Z_1} \varphi(X), Z_2) - g(Z_1, \nabla_{\varphi X} Z_2) + g(Z_1, \nabla_{Z_2} \varphi(X)) \\ &\quad + g(\varphi(\nabla_X Z_1), Z_2) - g(\varphi(\nabla_{Z_1} X), Z_2) + g(\varphi Z_1, \nabla_X Z_2) - g(\varphi Z_1, \nabla_{Z_2} X) \\ &\quad + g(Z_1, \varphi(\nabla_X Z_2)) - g(Z_1, \varphi(\nabla_{Z_2} X)) - g(\varphi Z_1, \nabla_X Z_2) + g(\varphi Z_1, \nabla_{Z_2} X) \\ &= \varphi(X)(g(Z_1, Z_2)) - X(g(\varphi Z_1, Z_2)) - g(\nabla_{\varphi X} Z_1, Z_2) + g(\nabla_{Z_1} \varphi(X), Z_2) \\ &\quad - g(Z_1, \nabla_{\varphi X} Z_2) + g(Z_1, \nabla_{Z_2} \varphi(X)) + g(\varphi(\nabla_X Z_1), Z_2) - g(\varphi(\nabla_{Z_1} X), Z_2) \\ &\quad + g(\varphi Z_1, \nabla_X Z_2) + g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2)) - g(Z_1, \varphi(\nabla_{Z_2} X)) \end{aligned}$$

and making use of (6), we have

$$\begin{aligned} g(\nabla_{Z_1} \varphi(X), Z_2) - g(\varphi(\nabla_{Z_1} X), Z_2) + g(Z_1, \nabla_{Z_2} \varphi(X)) - g(Z_1, \varphi(\nabla_{Z_2} X)) \quad (8) \\ = g((\nabla \varphi)(X, Z_1), Z_2) + g(Z_1, (\nabla \varphi)(X, Z_2)). \end{aligned}$$

Substitution (8) into (7) may be written as

$$\begin{aligned}
 (\Phi_{\varphi}g)(X, Z_1, Z_2) &= \varphi(X)(g(Z_1, Z_2)) - X(g(\varphi Z_1, Z_2)) + g((\nabla\varphi)(X, Z_1), Z_2) \quad (9) \\
 &+ g(Z_1, (\nabla\varphi)(X, Z_2)) - g(\nabla_{\varphi X}Z_1, Z_2) - g(Z_1, \nabla_{\varphi X}Z_2) \\
 &+ g(\varphi(\nabla_X Z_1), Z_2) + g(\varphi Z_1, \nabla_X Z_2) + g(Z_1, \varphi(\nabla_X Z_2)) \\
 &+ g(Z_1, \varphi(\nabla_X, Z_2)).
 \end{aligned}$$

On the other hand, with respect to the Riemannian connection, we have

$$\varphi(X)(g(Z_1, Z_2)) - g(\nabla_{\varphi X}Z_1, Z_2) - g(Z_1, \nabla_{\varphi X}Z_2) = (\nabla_{\varphi X}g)(Z_1, Z_2) = 0 \quad (10)$$

and

$$\begin{aligned}
 X(g(\varphi Z_1, Z_2)) - g(\nabla_X\varphi(Z_1), Z_2) - g(\varphi Z_1, \nabla_X Z_2) &= (\nabla_X g)(\varphi Z_1, Z_2) = 0 \quad (11) \\
 \Rightarrow -X(g(\varphi Z_1, Z_2)) + g(\varphi Z_1, \nabla_X Z_2) &= -g(\nabla_X\varphi(Z_1), Z_2).
 \end{aligned}$$

By virtue of (6), (9), (10) and (11) reduces to

$$\begin{aligned}
 (\Phi_{\varphi}g)(X, Z_1, Z_2) &= -g(\nabla_X\varphi(Z_1), Z_2) + g(\varphi(\nabla_X Z_1), Z_2) + g((\nabla_{Z_1}\varphi)(X), Z_2) \quad (12) \\
 &+ g(Z_1, (\nabla_{Z_2}\varphi)(X)) + g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2)) \\
 &= g(\varphi(\nabla_X Z_1) - \nabla_X\varphi(Z_1), Z_2) + g(Z_1\varphi(X), Z_2) \\
 &+ g(Z_1, (\nabla_{Z_2}\varphi)(X)) + g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2)) \\
 &= -g(\nabla_X\varphi(Z_1), Z_2) + g((\nabla_{Z_1}\varphi)(X), Z_2) + g(Z_1, (\nabla_{Z_2}\varphi)(X)) \\
 &+ g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2)).
 \end{aligned}$$

The analogue to (12) is

$$\begin{aligned}
 (\Phi_{\varphi}g)(Z_2, Z_1, X) &= -g(\nabla_{Z_2}\varphi(Z_1), X) + g((\nabla_{Z_1}\varphi)(Z_2), X) + g(Z_1, (\nabla_X\varphi)(Z_2)) \quad (13) \\
 &+ g(Z_1, \varphi(\nabla_{Z_2}X)) + g(Z_1, \varphi(\nabla_{Z_2}X)).
 \end{aligned}$$

**Lemma:** *If a Riemannian metric tensor  $g$  is hybrid, then we have*

$$g((\nabla_Y \varphi)(Z), X) = -g(Z, (\nabla_Y \varphi)(X)), \quad (14)$$

where  $\nabla$  denotes the operator of the Riemannian covariant derivative with respect to  $g$ .

**Proof.** By virtue of (1) and

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

we have

$$Yg(\varphi Z, X) = -Yg(Z, \varphi X),$$

$$g(\nabla_Y \varphi(Z), X) + g(\varphi Z, \nabla_Y X) = -g(\nabla_Y Z, \varphi X) - g(Z, \nabla_Y \varphi(X))$$

or

$$-g(\nabla_Y Z, \varphi X) - g(\nabla_Y \varphi(Z), X) = g(\varphi Z, \nabla_Y X) + g(Z, \nabla_Y \varphi(X))$$

$$g(\varphi(\nabla_Y Z) - \nabla_Y \varphi(Z), X) = -g(Z, \varphi(\nabla_Y X) + \nabla_Y \varphi(X))$$

and therefore, by (6), the proof is completed.  $\square$

We have

$$\begin{aligned} (\Phi_\varphi g)(X, Z_1, Z_2) - (\Phi_\varphi g)(Z_2, Z_1, X) &= g(Z_1, (\nabla_X \varphi)(Z_2)) - g(Z_1, (\nabla_X \varphi)(Z_2)) \\ &\quad -g(X, (\nabla_{Z_1} \varphi)(Z_2)) - g(X, (\nabla_{Z_1} \varphi)(Z_2)) \\ &\quad +g(Z_1, (\nabla_{Z_2} \varphi)(X)) - g(Z_1, (\nabla_{Z_2} \varphi)(X)) \\ &\quad +g(Z_1, \varphi(\nabla_X Z_2)) + g(Z_1, \varphi(\nabla_X Z_2)) \\ &\quad -g(Z_1, \varphi(\nabla_{Z_2} X)) - g(Z_1, \varphi(\nabla_{Z_2} X)) \end{aligned}$$

$$\begin{aligned}
 &= -2g(X, (\nabla_{Z_1}\varphi)(Z_2)) + 2g(Z_1, \varphi(\nabla_X Z_2)) \\
 &\quad - 2g(Z_1, \varphi(\nabla_{Z_2}X)) \\
 &= -2g(X, (\nabla_{Z_1}\varphi)(Z_2)) + 2g(Z_1, \varphi(L_X Z_2))
 \end{aligned}$$

or

$$(\psi_\varphi g)(X, Z_1, Z_2) - (\psi_\varphi g)(Z_2, Z_1, X) = -2g(X, (\nabla_{Z_1}\varphi)(Z_2)) \quad (15)$$

where

$$\begin{aligned}
 (\psi_\varphi g)(X, Z_1, Z_2) &= (\Phi_\varphi g)(X, Z_1, Z_2) - g(Z_1, \varphi(L_X Z_2)) \\
 &= (L_{\varphi X} g)(Z_1, Z_2) - (L_X(g \circ \varphi))(Z_1, Z_2) - g(\varphi Z_1, L_X Z_2),
 \end{aligned}$$

$$\begin{aligned}
 (\psi_\varphi g)(Z_2, Z_1, X) &= (\Phi_\varphi g)(Z_2, Z_1, X) - g(Z_1, \varphi(L_{Z_2} X)) \\
 &= (L_{\varphi Z_2} g)(Z_1, X) - (L_{Z_2}(g \circ \varphi))(Z_1, X) - g(\varphi Z_1, L_{Z_2} X).
 \end{aligned}$$

From (15) we have

$$\begin{aligned}
 &(\psi_\varphi g)(X, Z_1, Z_2) - (\psi_\varphi g)(Z_2, Z_1, X) + (\psi_\varphi g)(X, Z_2, Z_1) - (\psi_\varphi g)(Z_1, Z_2, X) \\
 &= -2g(X, (\nabla_{Z_1}\varphi)(Z_2)) + (\nabla_{Z_2}\varphi)(Z_1).
 \end{aligned}$$

Thus we have the following theorem.

**Theorem** A necessary and sufficient condition that an almost Hermitian manifold to be nearly Kahlerian is that

$$\mathop{Alt}_{X, Z_2}(\psi_\varphi g)(X, Z_1, Z_2) + \mathop{Alt}_{X, Z_1}(\psi_\varphi g)(X, Z_2, Z_1) = 0.$$

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Received 03.05.2005