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Note on Generalized Jordan Derivations Associate with Hochschild 2-cocycles of Rings*

Atsushi Nakajima

Abstract

We introduce a new type of generalized derivations associate with Hochschild 2-cocycles and prove that every generalized Jordan derivation of this type is a generalized derivation under certain conditions. This result contains the results of I. N. Herstein [6, Theorem 3.1] and M. Ashraf and N-U. Rehman [1, Theorem].

Key Words: Derivation, Jordan derivation, generalized derivation, generalized Jordan derivation, Hochschild 2-cocycle .

1. Introduction

Let R be a ring and let x, y be arbitrary elements of R . M is called an R -bimodule if M is a left and a right R -module such that $x(my) = (xm)y$ for all $m \in M$. Let $f : R \rightarrow M$ be an additive map. f is called a generalized derivation if there exists a derivation $d : R \rightarrow M$ such that

$$f(xy) = f(x)y + xd(y). \quad (1)$$

and f is called a generalized Jordan derivation if there exists a Jordan derivation $J : R \rightarrow M$ such that

$$f(x^2) = f(x)x + xJ(x), \quad (2)$$

We denote (1) and (2) by (f, d) and (f, J) , respectively. These types of generalized derivations were introduced by M. Brešar [2] and their properties have been discussed in

*Dedicated to Professor Şerif Yenigül on his 60th birthday

many papers. In [7], another type of generalized derivations was defined by the author as follows. f is called a generalized derivation if there exists an element $\omega \in M$ such that

$$f(xy) = f(x)y + xf(y) + x\omega y, \quad (3)$$

and f is called a generalized Jordan derivation if

$$f(x^2) = f(x)x + xf(x) + x\omega x, \quad (4)$$

which denote by (f, ω) . Some categorical properties of these generalized derivations were given in [7]. In the case R has an identity element 1, if (f, d) is a generalized derivation of type (1), then $(f, -f(1))$ is a generalized derivation of type (3); and conversely, if (f, ω) is a generalized derivation of type (3), then $f + \omega_\ell : R \rightarrow M$ is a derivation and $(f, f + \omega_\ell)$ is a generalized derivation of type (1), where $\omega_\ell : R \ni x \mapsto \omega x \in M$.

A Jordan derivation of 2-torsion free prime rings is a derivation. This result was first proved by Herstein [6, Theorem 3.1] and was extended to 2-torsion free semiprime rings by Brešar [4, Theorem 1]. In [8], the author proved that a generalized Jordan derivation (f, ω) of type (4) is also a generalized derivation of type (3), and in [1] they proved that a generalized Jordan derivation (f, d) of type (2) is a generalized derivation of type (1) under a certain commutator condition.

In this note, we introduce a new type of generalized derivations and show that our generalized Jordan derivation is a generalized derivation under certain conditions. This result contains the results of I. N. Herstein [6, Theorem 3.1], M. Ashraf and N-U. Rehman [1, Theorem].

Throughout the following, we assume that R is a ring, M is an R -bimodule and x, y, z are arbitrary elements of R , unless otherwise stated.

2. Definitions and Lemmas

Let $\alpha : R \times R \rightarrow M$ be a biadditive map, that is, an additive map on each components. α is called a *Hochschild 2-cocycle* if

$$x\alpha(y, z) - \alpha(xy, z) + \alpha(x, yz) - \alpha(x, y)z = 0. \quad (5)$$

2-cocycle α is called *symmetric* (resp. *skew symmetric*) if $\alpha(x, y) = \alpha(y, x)$ (resp. $\alpha(x, y) = -\alpha(y, x)$). An additive map $f : R \rightarrow M$ is called a *generalized derivation* if

there exists a 2-cocycle α such that

$$f(xy) = f(x)y + xf(y) + \alpha(x, y), \tag{6}$$

and f is called a *generalized Jordan derivation* if

$$f(x^2) = f(x)x + xf(x) + \alpha(x, x). \tag{7}$$

We denote it by (f, α) . If $\alpha = 0$, then they are the usual derivations and Jordan derivations. We give some examples of our generalized derivations.

Examples (1) If (f, d) and (f, ω) are generalized derivations of types (1) and (3), respectively, then the maps

$$\alpha_1 : R \times R \ni (x, y) \mapsto x(d - f)(y) \in M \quad \text{and} \quad \alpha_2 : R \times R \ni (x, y) \mapsto x\omega y \in M$$

are biadditive and satisfy the 2-cocycle condition (5). Since $(f, d) = (f, \alpha_1)$ and $(f, \omega) = (f, \alpha_2)$, the usual generalized derivations are generalized derivations in our sense.

(2) If $f : R \rightarrow M$ is a left multiplier, that is, f is additive and $f(xy) = f(x)y$, then by $f(xy) = f(x)y + xf(y) + x(-f)(y)$, we have a 2-cocycle $\alpha_3 : R \times R \ni (x, y) \mapsto x(-f)(y) \in M$ and $f = (f, \alpha_3)$. Thus a left multiplier is also a generalized derivation.

(3) Let f be a (σ, τ) -derivation, that is, σ and τ are ring homomorphisms of R and $f(xy) = f(x)\sigma(y) + \tau(x)f(y)$. Then the map

$$\alpha_4 : R \times R \ni (x, y) \mapsto f(x)(\sigma(y) - y) + (\tau(x) - x)f(y) \in M$$

is biadditive and satisfies the 2-cocycle condition. Since $f(xy) = f(x)y + xf(y) + \alpha_4(x, y)$, (σ, τ) -derivation f is also a generalized derivation (f, α_4) .

(4) In general, we have the following. Let $f : R \rightarrow M$ be an additive map and let $\alpha : R \times R \rightarrow M$ be a biadditive map. If $f(xy) = f(x)y + xf(y) + \alpha(x, y)$ holds, then by the associativity $f((xy)z) = f(x(yz))$, α satisfies the 2-cocycle condition. Thus (f, α) is a generalized derivation in our sense.

Now the following lemma is elementary and can be found everywhere such as in [3, Proposition 2] or [1, Lemma 2.1].

Lemma 1 *Let $(f, d) : R \rightarrow M$ be a generalized Jordan derivation and M a 2-torsion free module, where $d : R \rightarrow M$ is a derivation. Then the following relations hold:*

- (1) $f(xy + yx) = f(x)y + xd(y) + f(y)x + yd(x);$
- (2) $f(xyx) = f(x)yx + xd(y)x + xyd(x);$
- (3) $f(xyz + zyx) = f(x)yz + xd(y)z + xyd(z) + f(z)yx + zd(y)x + zyd(x).$

In our case, the above relations are generalized as follows.

Lemma 2 *Let $(f, \alpha) : R \rightarrow M$ be a generalized Jordan derivation associate with Hochschild 2-cocycle α and M a 2-torsion free module. Then the following relations hold:*

- (1) $f(xy + yx) = f(x)y + xf(y) + \alpha(x, y) + f(y)x + yf(x) + \alpha(y, x),$
- (2) $f(xyx) = f(x)yx + xf(y)x + xyf(x) + x\alpha(y, x) + \alpha(x, yx),$
- (3) $f(xyz + zyx) = f(x)yz + xf(y)z + xyf(z) + x\alpha(y, z) + \alpha(x, yz)$
 $+ f(z)yx + zf(y)x + zyf(x) + z\alpha(y, x) + \alpha(z, yx).$

Proof. (1) Since $f(x^2) = f(x)x + xf(x) + \alpha(x, x)$, (1) is easily obtained by $f(xy + yx) = f((x + y)^2) - f(x^2) - f(y^2).$

(2) Replacing y by $xy + yx$ in (1) and using the 2-cocycle condition (5), we have

$$\begin{aligned}
 2f(xyx) &= f(x(xy + yx) + (xy + yx)x) - f(x^2y + yx^2) \\
 &= 2\{f(x)yx + xf(y)x + xyf(x)\} \\
 &\quad + x\{\alpha(x, y) + \alpha(y, x)\} + \alpha(x, xy) + \alpha(x, yx) \\
 &\quad + \{\alpha(x, y) + \alpha(y, x)\}x + \alpha(xy, x) + \alpha(yx, x) \\
 &\quad - \{\alpha(x, x)y + \alpha(x^2, y) + y\alpha(x, x) + \alpha(y, x^2)\} \\
 &= 2\{f(x)yx + xf(y)x + xyf(x)\} \\
 &\quad + \{x\alpha(x, y) - \alpha(x^2, y) + \alpha(x, xy) - \alpha(x, x)y\} \\
 &\quad - \{y\alpha(x, x) - \alpha(yx, x) + \alpha(y, x^2) - \alpha(y, x)x\} \\
 &\quad + x\alpha(y, x) + \alpha(x, yx) + \alpha(x, y)x + \alpha(xy, x) \\
 &= 2\{f(x)yx + xf(y)x + xyf(x)\} \\
 &\quad + x\alpha(y, x) + \alpha(x, yx) + \alpha(x, y)x + \alpha(xy, x).
 \end{aligned}$$

Since $x\alpha(y, x) + \alpha(x, yx) = \alpha(xy, x) + \alpha(x, y)x$ and M is 2-torsion free, we have the relation (2).

(3) Replace x by $x + z$ in (2), then (3) is easily seen. □

The following lemma is useful in the calculations of 2-torsion free semiprime rings which can be found in [5, Lemmas 1.1 and 1.2].

Lemma 3 (1) *Let R be a 2-torsion free semiprime ring and $a, b \in R$. If $axb + bxa = 0$ for all $x \in R$, then $axb = bxa = 0$ for all $x \in R$. Especially, $ab = ba = 0$.*

(2) *Let G_1, G_2, \dots, G_n be additive groups and R a semiprime ring. Suppose that mappings $S : G_1 \times G_2 \times \dots \times G_n \rightarrow R$ and $T : G_1 \times G_2 \times \dots \times G_n \rightarrow R$ are additive in each argument. If $S(a_1, a_2, \dots, a_n)xT(a_1, a_2, \dots, a_n) = 0$ for all $x \in R$, $a_i \in G_i$, $i = 1, 2, \dots, n$, then $S(a_1, a_2, \dots, a_n)xT(b_1, b_2, \dots, b_n) = 0$ for all $x \in R$, $a_i, b_i \in G_i$, $i = 1, 2, \dots, n$.*

Now for all $x, y \in R$, we set

$$F(x, y) = f(xy) - f(x)y - xf(y), \tag{8}$$

and

$$\delta(x, y) = F(x, y) - \alpha(x, y). \tag{9}$$

Then $F(x, y)$ and $\delta(x, y)$ are biadditive and by Lemma 2 (1), we have

$$\delta(x, y) + \delta(y, x) = 0. \tag{10}$$

Lemma 4 *Let $(f, \alpha) : R \rightarrow M$ be a generalized Jordan derivation and M a 2-torsion free module. Then the following relations hold:*

- (1) $\delta(x, y)z[x, y] + [x, y]z\delta(x, y) = 0$ where $[x, y] = xy - yx$,
- (2) $\delta(x, y)[x, y] = 0$.

Proof. (1) By (2) and (3) of Lemma 2, we have

$$\begin{aligned} 0 &= f((xy)z(yx) + (yx)z(xy)) - f(x(yzy)x + y(xzx)y) \\ &= F(x, y)zyx + F(y, x)zxy + xyzF(y, x) + yxzF(x, y) \\ &\quad + xy\{\alpha(z, yx) - \alpha(z, y)x\} + \{\alpha(xy, zyx) - \alpha(x, yzyx)\} \\ &\quad + yx\{\alpha(z, xy) - \alpha(z, x)y\} + \{\alpha(yx, zxy) - \alpha(y, xzxy)\} \\ &\quad - \{x\alpha(y, zy)x + x\alpha(yzy, x) + y\alpha(x, zx)y + y\alpha(xzx, y)\}(*). \end{aligned}$$

Since α is a 2-cocycle, we have the following relations:

- (1) $xy\{\alpha(z, yx) - \alpha(z, y)x\} = xy\{\alpha(zx, y) - z\alpha(y, x)\},$
 - (2) $\alpha(xy, zyx) - \alpha(x, yzyx) = x\alpha(y, zyx) - \alpha(x, y)zyx,$
 - (3) $yx\{\alpha(z, xy) - \alpha(z, x)y\} = yx\{\alpha(zx, y) - z\alpha(x, y)\},$
 - (4) $\alpha(yx, zxy) - \alpha(y, xzxy) = y\alpha(x, zxy) - \alpha(y, x)zxy.$
-

Substituting from (1) to (4) in the above relation (*) and using the 2-cocycle condition (5) we get

$$\begin{aligned} 0 &= \delta(x, y)zyx + \delta(y, x)zxy + xyz\delta(y, x) + yxz\delta(x, y) \\ &\quad + x\{y\alpha(zx, y) - \alpha(xzx, y) + \alpha(x, zxy) - \alpha(x, zx)y\} \\ &\quad + y\{x\alpha(zx, y) - \alpha(xzx, y) + \alpha(x, zxy) - \alpha(x, zx)y\} \\ &= \delta(x, y)zyx + \delta(y, x)zxy + xyz\delta(y, x) + yxz\delta(x, y). \end{aligned}$$

Since $\delta(x, y) = -\delta(y, x)$ by (10), we have

$$\delta(x, y)z[x, y] + [x, y]z\delta(x, y) = 0.$$

(2) Similarly by Lemma 2 (2) and (3), we have

$$\begin{aligned} 0 &= f((xy)^2 + xy^2x) - f(xy(xy) + (xy)yx) \\ &= F(x, y)[x, y] + \{\alpha(xy, xy) - x\alpha(y, xy) - \alpha(x, yxy)\} \\ &\quad + x\{\alpha(y^2, x) + \alpha(y, y)x - y\alpha(y, x)\} + \alpha(x, y^2x) - \alpha(xy, yx)(**). \end{aligned}$$

Substituting

$$\begin{aligned} \alpha(xy, xy) &= x\alpha(y, xy) + \alpha(x, yxy) - \alpha(x, y)xy \quad \text{and} \\ \alpha(y^2, x) &= y\alpha(y, x) + \alpha(y, yx) - \alpha(y, y)x \end{aligned}$$

in the relation (**), we get

$$F(x, y)[x, y] - \alpha(x, y)xy + x\alpha(y, yx) + \alpha(x, y^2x) - \alpha(xy, yx) = 0.$$

Since $x\alpha(y, yx) + \alpha(x, y^2x) - \alpha(xy, yx) = \alpha(x, y)yx$, we have

$$\delta(x, y)[x, y] = 0.$$

Lemma 5 *Let R be a 2-torsion free ring and G_1, G_2 additive groups. Let $S, T : G_1 \times G_2 \rightarrow R$ be biadditive maps. Assume that $S(x_1, x_2)T(x_1, x_2) = 0$ for all $x_i \in G_i, i = 1, 2$. If there exists a non-zero divisor $T(a_1, a_2)$ for some $a_i \in G_i, i = 1, 2$, then $S(x_1, x_2) = 0$ for all $x_i \in G_i, i = 1, 2$.*

Proof. We may assume that $T(a_1, a_2)$ is a non-zero divisor for some $a_i \in G_i, i = 1, 2$ and so $S(a_1, a_2) = 0$. Then by $S(x_1 + a_1, x_2)T(x_1 + a_1, x_2) = 0$, we have

$$S(x_1, x_2)T(a_1, x_2) + S(a_1, x_2)T(x_1, x_2) = 0 \tag{11}$$

for all $x_i \in G_i, i = 1, 2$. Replacing x_2 by $x_2 + a_2$ in (11), and using (11) again, we get

$$S(x_1, x_2)T(a_1, a_2) + S(x_1, a_2)T(a_1, x_2) + S(x_1, a_2)T(a_1, a_2) + S(a_1, x_2)T(x_1, a_2) = 0$$

for all $x_i \in G_i, i = 1, 2$. Take $x_1 = a_1$ in the above relation, we have $2S(a_1, x_2)T(a_1, a_2) = 0$. Since $T(a_1, a_2)$ is non-zero divisor and R is 2-torsion free, we see $S(a_1, x_2) = 0$ for all $x_2 \in G_2$. Then by (11), we get $S(x_1, x_2)T(a_1, x_2) = 0$ and so $S(x_1, a_2) = 0$. Thus we have $S(x_1, x_2) = 0$ for all $x_i \in G_i, i = 1, 2$. \square

3. Generalized Jordan Derivations

In this section, we show that a generalized Jordan derivation (f, α) is a generalized derivation under certain conditions.

Theorem 6 *Let R be a 2-torsion free ring and let $(f, \alpha) : R \rightarrow R$ be a generalized Jordan derivation associate with Hochschild 2-cocycle α . If R satisfies one of the following conditions, then (f, α) is a generalized derivation.*

- (1) R is a non-commutative prime ring.
- (2) There exist $a, b \in R$ such that $[a, b]$ is a non-zero divisor.
- (3) R is commutative and α is symmetric.

Proof. (1) By Lemma 4 (1) and Lemma 3 (1), we have $\delta(x, y)z[x, y] = 0$. Since R is non-commutative, there exist $a, b \in R$ such that $[a, b] \neq 0$. Then by Lemma 3 (2), we have $\delta(x, y)z[a, b] = 0$ and by the primeness of $R, \delta(x, y) = 0$ for all $x, y \in R$. Thus (f, α) is a generalized derivation.

(2) Suppose that $[a, b]$ is a nonzero divisor and so $\delta(a, b) = 0$ for some $a, b \in R$. Since $\delta(x, y)$ and $[x, y]$ are biadditive maps, then by Lemma 5, we have $\delta(x, y) = 0$.

(3) Since R is commutative and α is symmetric, then by Lemma 2 (1), (f, α) is a generalized derivation.

Let $\xi : R \rightarrow M$ be a left Jordan multiplier, that is, ξ is additive and $\xi(x^2) = \xi(x)x$, then by the similar calculations as in the proof of Lemma 2 and Lemma 4 (2), we have the following relations:

$$\begin{aligned} \xi(xy + yx) &= \xi(x)y + \xi(y)x, & 2\xi(xyx) &= 2\xi(x)yx, \\ 2\xi(xyz + zyx) &= 2(\xi(x)yz + \xi(z)yx), & 2(\xi(xy) - \xi(x)y)[x, y] &= 0. \end{aligned}$$

If R is 2-torsion free and has a non-zero divisor $[a, b]$ for some $a, b \in R$, then by the above relations, we have $\xi(ab) = \xi(a)b$ and thus by Lemma 5, $\xi(xy) = \xi(x)y$ for all $x, y \in R$. Therefore a left Jordan multiplier is a left multiplier. If (f, J) is a generalized Jordan derivation of type (2), then

$$f(x^2) = f(x)x + xf(x) + x(J - f)(x)$$

and $\xi = J - f$ is a left Jordan multiplier. In this case, it is easy to see that $\alpha : R \times R \ni (x, y) \mapsto x\xi(y) \in M$ is 2-cocycle. Thus by Theorem 6 (2), we have the following which is a slight generalization of [1, Theorem]. \square

Corollary 7 *Let R be a 2-torsion free ring with non-zero divisor $[a, b]$ for some $a, b \in R$. Then a generalized Jordan derivation (f, J) is a generalized derivation.*

Finally, we note the following. Let (f, α) be a generalized derivation. Then by (6), α is symmetric if and only if $f([x, y]) = [f(x), y] + [x, f(y)]$, that is, f is a Lie derivation. And α is skew symmetric if and only if $f\{x, y\} = \{f(x), y\} + \{x, f(y)\}$, where $\{x, y\} = xy + yx$. Thus in case of R is 2-torsion free, this means that α is skew symmetric if and only if f is a Jordan derivation. Therefore the notion of our generalized derivations has many common properties of the notion of several types of derivations defined until now.

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