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S. KARIMZADEH

REZA NEKOOEI

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On Dimension of Modules

S. Karimzadeh, R. Nekooei

Abstract

In this paper we prove the lying over and going down theorems for modules. Finally, we apply the above theorems and prove some results on the dimension of a module and its submodule.

Key Words: Prime Submodule, Multiplication module, Dimension of a module.

Introduction

Throughout this note, all rings are commutative with identity and all modules are unital. For R -modules M and M' , we denote all R -module homomorphisms of M into M' by $Hom_R(M, M')$. For any submodule N of an R -module M , we define

$$(N : M) = \{r \in R : rM \subseteq N\}$$

and denote $(0 : M)$ by $Ann_R(M)$.

A submodule P of M is called prime if $P \neq M$, and whenever $r \in R$, $m \in M$ and $rm \in P$, then $m \in P$ or $r \in (P : M)$ [see 8]. It is easy to show that, if P is a prime submodule of an R -module M , then $(P : M)$ is a prime ideal of R . The sets of all prime submodules and proper maximal submodules of M are respectively denoted by $Spec(M)$ and $Max(M)$. Following [4], we denote the intersection of all prime submodules of an R -module M by $rad_M(0)$ and the intersections of all proper maximal submodules by $Rad(M)$. The radicals of R and an ideal I of R are denoted by $N(R)$ and \sqrt{I} , respectively.

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An R -module M is called a multiplication module if for any submodule N of M there exists an ideal I of R such that $N = IM$. It is easy to check that M is a multiplication module if and only if $N = (N : M)M$ for every submodule N of M (See [7]).

Let R be a principal ideal domain (PID) and m and n be positive integers. Let $A = (a_{ij}) \in M_{m \times n}(R)$ and F be the free R -module $R^{(n)}$. We shall use the notation $\langle A \rangle$ for the submodule N of F generated by the rows of A , and the notation $(r_1, \dots, r_m)A$, $r_i \in R$, for an element of N .

In this paper we shall first prove the lying-over and going-down theorem for modules, and then prove results on the dimension of a module and its submodule.

1. Lying over Theorem for Modules

The following Proposition is used widely in the sequel.

Proposition 1.1 *Let $\varphi \in \text{Hom}_R(M, M')$, N and N' be submodules of M and M' respectively. Then we have:*

(i) *If $\varphi^{-1}(N') \subseteq N$ then there exists a submodule P' of M' containing N' which is maximal with respect to $\varphi^{-1}(P') \subseteq N$. Furthermore, $\varphi^{-1}(P') = N$;*

(ii) *If $N \in \text{Spec}(M)$ and $rm' \in P'$, for $r \in R$ and $m' \in M'$, then $m' \in P'$ or $r \in (N : M)$.*

Proof. (i) Put $T = \{L' \leq M' \mid N' \subseteq L' \text{ and } \varphi^{-1}(L') \subseteq N\}$. Since $N' \in T$, $T \neq \emptyset$. By Zorn's Lemma, T has a maximal element P' . Suppose that $\varphi^{-1}(P') \subset N$. Then there exists $n \in N$ such that $\varphi(n) \notin P'$. Hence $P' \subset P' + \langle \varphi(n) \rangle \notin T$ and so there exists $m \in \varphi^{-1}(P' + \langle \varphi(n) \rangle)$ such that $m \notin N$. Therefore $\varphi(m - rn) \in P'$ for some $r \in R$, and hence $m \in N$, which is a contradiction. Thus $\varphi^{-1}(P') = N$.

(ii) Let $rm' \in P'$ and $m' \notin P'$. Hence $P' \subset P' + Rm' \notin T$ and so there exists $m \in \varphi^{-1}(P' + Rm')$ such that $m \notin N$. Therefore $r\varphi(m) \in P'$ and so $rm \in N$. Since $N \in \text{Spec}(M)$ and $m \notin N$, hence $r \in (N : M)$. \square

Lemma 1.2 *Let $\varphi \in \text{Hom}_R(M, M')$. If $N' \in \text{Spec}(M')$ and $\varphi(M) \not\subseteq N'$, then*

(i) $\varphi^{-1}(N') \in \text{Spec}(M)$;

(ii) $(\varphi^{-1}(N') : M) = (N' : M')$.

Proof. (i) By [6, Proposition 1.2].

(ii) Suppose that $r \in (N' : M')$ and $m \in M$. Hence $r\varphi(m) \in N'$ and so $rm \in \varphi^{-1}(N')$. Therefore $(N' : M') \subseteq (\varphi^{-1}(N') : M)$. Now let $r \in (\varphi^{-1}(N') : M)$ and $m \in M \setminus \varphi^{-1}(N')$. Since $N' \in \text{Spec}(M')$ and $\varphi(m) \notin N'$, $r \in (N' : M')$, hence $(\varphi^{-1}(N') : M) \subseteq (N' : M')$ and the proof is complete. \square

Definition. Let M and M' be R -modules. We said to be lying over (or simply, LO) holds for (M, M') if $M \subseteq M'$ and for any $P \in \text{Spec}(M)$ there exists $P' \in \text{Spec}(M')$ with $P' \cap M = P$.

Example. Let V be a vector space over a field F with $\dim_F V \geq 2$. Let W be a proper subspace of V . Since every proper subspace of a vector space is prime, hence LO holds for (V, W) .

Proposition 1.3 Let $\varphi \in \text{Hom}_R(M, M')$. Suppose that, for every $m' \in M'$ and $P \in \text{Spec}(M)$, there exists $s \in R \setminus (P : M)$ such that $sm' \in \varphi(M)$.

If $\text{Ker}\varphi \subseteq \text{rad}_M(0)$, then for any $P \in \text{Spec}(M)$ there exists $P' \in \text{Spec}(M')$ with $\varphi^{-1}(P') = P$.

Proof. Suppose that $P \in \text{Spec}(M)$. Since $\varphi^{-1}(\{0\}) = \text{Ker}\varphi \subseteq P$, by Proposition 1.1 (i), there exists a submodule P' of M' which is maximal with respect to $\varphi^{-1}(P') = P$. Now we show that $P' \in \text{Spec}(M')$. It is clear that P' is a proper submodule of M' . Suppose that $r \in R$, $m' \in M'$ and $rm' \in P'$. If $m' \notin P'$, then by Proposition 1.1 (ii), $r \in (P : M)$. Assume that $rM' \not\subseteq P'$, hence there exists $m'_1 \in M'$ such that $rm'_1 \notin P'$. By assumption, there exists $s \in R \setminus (P : M)$ such that $rs m'_1 \in \varphi((P : M)M)$. Hence $srm'_1 \in P'$. Again by Proposition 1.1 (ii), $s \in (P : M)$, which is a contradiction. Therefore $rM' \subseteq P'$ and $P' \in \text{Spec}(M')$. \square

Example. Let R be a commutative ring with identity. Let M be a flat R -module which is not faithfully flat (for example, \mathbf{Q} as \mathbf{Z} -module).

By [9, Proposition 2.11.24], $mM = M$ for some maximal ideal m of R . Define the monomorphism $\varphi \in \text{Hom}_R(M, M \oplus R/m)$ by $\varphi(x) = (x, 0)$, $x \in M$. Let P be a prime submodule of M . Since $mM = M$ and $P \neq M$, hence $(P : M) \neq m$. Therefore there exists $s \in (R \setminus (P : M)) \cap m$. Now for any $(x, r + m) \in (M \oplus R/m)$, we have

$s(x, r+m) = (sx, 0) \in \varphi(M)$. By Proposition 1.3, we conclude that for any $P \in \text{Spec}(M)$ there exists $P' \in \text{Spec}(M \oplus R/m)$ with $\varphi^{-1}(P') = P$.

Theorem 1.4 (*Lying Over*) *Let $M \subseteq M'$ be R -modules.*

If for each $m' \in M'$, $P \in \text{Spec}(M)$, there exists $s \in R \setminus (P : M)$ such that $sm' \in M$, then LO holds for (M, M') .

Proof. This follows by Proposition 1.3. □

A ring R is called Von Neumann regular ring if for every $a \in R$ there exists an element $b \in R$ such that $a = aba$.

Theorem 1.5 *Let $M \subseteq M'$ be R -modules and $P \in \text{Spec}(M)$. Then there exists $P' \in \text{Spec}(M')$ with $P' \cap M = P$, if one of the following conditions hold.*

- (i) *For each $m' \in M'$ and $r \in (P : M)$ there exists $s \in R \setminus (P : M)$ such that $rs m' \in P$;*
- (ii) *R is Von Neumann regular ring and $(P : M) \subseteq \sqrt{P : M'}$.*

Proof. By Proposition 1.1, there exists a submodule P' of M' that is maximal with respect to $P' \cap M = P$. Now we show that $P' \in \text{Spec}(M')$. Let $re \in P'$ and $e \notin P'$, where $r \in R$ and $e \in M'$. By Proposition 1.1 (ii), $r \in (P : M)$. Suppose that (i) holds. If $rM' \not\subseteq P'$ then there exists $m' \in M'$ such that $rm' \notin P'$. By assumption there is $s \in R \setminus (P : M)$ such that $rs m' \in P$. Thus $rs m' \in P'$ and so by Proposition 1.1, $s \in (P : M)$, which is a contradiction. Now suppose that (ii) holds. Since $r \in (P : M)$, hence there is $n \in \mathbb{N}$ such that $r^n \in (P : M') \subseteq (P' : M')$. Therefore P' is a primary submodule of M' and so $P' \in \text{Spec}(M')$. □

Proposition 1.6 *Let $\varphi \in \text{Hom}_R(M, M')$ and M' be a multiplication module. If $\text{Ker}\varphi \subseteq \text{rad}_M(0)$, then for every $P \in \text{Spec}(M)$, there exists $P' \in \text{Spec}(M')$ with $\varphi^{-1}(P') = P$ if and only if $(\varphi(M) : M') \not\subseteq (P : M)$.*

Proof. Suppose that there exists $P' \in \text{Spec}(M')$ such that $\varphi^{-1}(P') = P$. By Lemma 1.2, $(P' : M') = (P : M)$. If $(\varphi(M) : M') \subseteq (P : M)$ then $\varphi(M) \subseteq P'$, because M' is a multiplication module. Hence $P = M$, which is a contradiction. Conversely, suppose that $(\varphi(M) : M') \not\subseteq (P : M)$. Then there exists $s \in (\varphi(M) : M') \setminus (P : M)$. Let $m' \in M'$ and $r \in (P : M)$. Hence $rs m' \in \varphi((P : M)M)$ and so by Proposition 1.3, there exists

$P' \in \text{Spec}(M')$ such that $\varphi^{-1}(P') = P$. □

Corollary 1.7 *Let $M \subseteq M'$ be R -modules. If $\sqrt{(M : M')} + \text{Ann}_R(M) = R$ then LO holds for (M, M') .*

Proof. Suppose that $P \in \text{Spec}(M)$. We have $\text{Ann}_R(M) \subseteq (P : M)$. Since $\sqrt{(M : M')} + \text{Ann}_R(M) = R$, hence $(M : M') \not\subseteq (P : M)$. Therefore LO holds. □

Corollary 1.8 *Let $M \subseteq M'$ be R -modules and M' be a multiplication module. LO holds for (M, M') if and only if for every $P \in \text{Spec}(M)$, $(M : M') \not\subseteq (P : M)$.*

Definition. (See [2]) *Let R be a principal ideal domain (PID). Let $J = \{j_1, \dots, j_\alpha\}$ be a subset of the integer between 1 and n and let $p \in R$ be a prime element. A matrix $A \in M_n(R)$, $A = (a_{ij})$, is said to be a prime matrix (or simply prime), if A satisfies the following conditions:*

- (i) A is a upper triangular;
 - (ii) For all i , $1 \leq i \leq n$, $a_{ii} = p$ if $i \in J$ and $a_{ii} = 1$ if $i \notin J$;
 - (iii) For all i , $1 \leq i \leq j \leq n$, $a_{ij} = 0$ except possibly when $i \notin J$ and $j \in J$.
- Sometimes we call J the set of integers associated with A and denote it by J_A .
By (i) and (ii) it's clear that $\det(A) = p^\alpha$.

Theorem 1.9 *Let R be a PID.*

(i) *Let M be a free R -module of rank m such that $M = \langle A_{m \times n} \rangle$ ($m \leq n$) and $N = \langle CA \rangle$ for some $C \in M_{\ell \times m}(R)$. Then N is a prime submodule of M if and only if $\langle C \rangle$ is a prime submodule of R^m .*

(ii) *Let M be a free R -module such that $M = \langle A_{n \times n} \rangle$ ($\det A \neq 0$) and $N = \langle CA \rangle$ for some $C \in M_n(R)$ then $\det C \in (N : M)$.*

(iii) *Let M be a free R -module such that $M = \langle A_{n \times n} \rangle$ ($\det A \neq 0$). If N is a prime submodule of M and $\text{rank} N = n$, then there exists a prime matrix $C_{n \times n}$ such that $N = \langle CA \rangle$.*

Proof. (i) Suppose that $N \in \text{Spec}(M)$. If $r(x_1, \dots, x_m) \in \langle C \rangle$ for $r, x_i \in R$, $1 \leq i \leq m$, then there exists $(d_1, \dots, d_\ell) \in R^\ell$ such that $r(x_1, \dots, x_m) = (d_1, \dots, d_\ell)C$ and

so $r(x_1, \dots, x_m)A = (d_1, \dots, d_\ell)CA$. Since $r((x_1, \dots, x_m)A) \in N$, hence $r \in (N : M)$ or $(x_1, \dots, x_m)A \in N$.

Case 1. Let $(x_1, \dots, x_m)A \in N$. There exists $(b_1, \dots, b_\ell) \in R^\ell$ such that $(x_1, \dots, x_m)A = (b_1, \dots, b_\ell)CA$. Put $(c_1, \dots, c_m) = (b_1, \dots, b_\ell)C$. Suppose that a_1, \dots, a_m are rows of A , then $x_1a_1 + \dots + x_ma_m = c_1a_1 + \dots + c_ma_m$. Since $\text{rank}M = m$, we have $(x_1, \dots, x_m) = (c_1, \dots, c_m)$. This implies that $(x_1, \dots, x_m) = (b_1, \dots, b_\ell)C$ and therefore $(x_1, \dots, x_m) \in \langle C \rangle$.

Case 2. Now let $r \in (N : M)$. Suppose that j , $1 \leq j \leq n$, is fixed. Since $(0, \dots, 0, r, 0, \dots, 0)A \in N$, there exists $(d_1, \dots, d_\ell) \in R^\ell$ such that $(0, \dots, 0, r, 0, \dots, 0)A = (d_1, \dots, d_\ell)CA$. Put $(c_1, \dots, c_m) = (d_1, \dots, d_\ell)C$. Suppose that a_1, \dots, a_m are rows of A . We have $(0, \dots, 0, r, 0, \dots, 0)A = (c_1, \dots, c_m)A$ and so $ra_j = c_1a_1 + \dots + c_ma_m$. Therefore $c_j = r$, $c_i = 0$ ($i \neq j$). Then $(c_1, \dots, c_m) = (0, \dots, 0, r, 0, \dots, 0) = (d_1, \dots, d_\ell)C$ and hence $r \in (\langle C \rangle : R^m)$.

Conversely, suppose that $\langle C \rangle \in \text{Spec}(R^m)$. If $r(x_1, \dots, x_n) \in N$, for $r \in R$ and $(x_1, \dots, x_n) \in M$, there exist $(y_1, \dots, y_m) \in R^m$ and $(d_1, \dots, d_\ell) \in R^\ell$ such that $r(x_1, \dots, x_n) = (d_1, \dots, d_\ell)CA$ and $(x_1, \dots, x_n) = (y_1, \dots, y_m)A$. Suppose that $(b_1, \dots, b_m) = (d_1, \dots, d_\ell)C$ such that $(b_1, \dots, b_m) \in R^m$ and a_1, \dots, a_m are rows of A . Then $ry_1a_1 + \dots + ry_ma_m = b_1a_1 + \dots + b_ma_m$ and therefore $r(y_1, \dots, y_m) = (b_1, \dots, b_m)$. We have $r(y_1, \dots, y_m) = (b_1, \dots, b_m) = (d_1, \dots, d_\ell)C$. Hence $r(y_1, \dots, y_m) \in \langle C \rangle$ and $\langle C \rangle \in \text{Spec}(R^m)$, and this implies that $r \in (\langle C \rangle : R^m)$ or $(y_1, \dots, y_m) \in \langle C \rangle$.

Case 1. Suppose that $r \in (\langle C \rangle : R^m)$ and j , $1 \leq j \leq m$. We have $(0, \dots, 0, r, 0, \dots, 0) \in \langle C \rangle$. There exists $(d_1, \dots, d_\ell) \in R^\ell$ such that $(0, \dots, 0, r, 0, \dots, 0) = (d_1, \dots, d_\ell)C$. Also $(0, \dots, 0, r, 0, \dots, 0)A = (d_1, \dots, d_\ell)CA$; hence $r \in (N : M)$.

Case 2. Suppose that $(y_1, \dots, y_m) \in \langle C \rangle$. There exists $(d_1, \dots, d_\ell) \in R^\ell$ such that $(y_1, \dots, y_m) = (d_1, \dots, d_\ell)C$. Hence $(x_1, \dots, x_n) = (y_1, \dots, y_m)A = (d_1, \dots, d_\ell)CA$, therefore $(x_1, \dots, x_n) \in N$.

(ii) Suppose that $\det C \neq 0$ and $A' = (a'_{ij})$, $C' = (c'_{ij})$ are the adjoint matrices of A and C respectively. If $(x_1, \dots, x_n) \in M$, then by [2, Lemma 1.2] $\det A | \sum_{i=1}^n x_i a'_{ij}$, for every j , $1 \leq j \leq n$. But $(\det C)(\det A) = \det(CA) | \sum_{i=1}^n (\det C)x_i (\sum_{k=1}^n a'_{ik} c'_{kj})$, hence by [2, Lemma 1.2] we have $((\det C)x_1, \dots, (\det C)x_n) \in N$.

(iii) By (i), we have $\langle C \rangle \in \text{Spec}(R^n)$. Now there exists a prime matrix $B_{n \times n}$ such that $\langle C \rangle = \langle B \rangle$ and by [2, Theorem 2.5], we have $\langle CA \rangle = \langle BA \rangle$. \square

Theorem 1.10 *Let R be a PID and M be a free R -module such that $M = \langle A \rangle$, $A \in M_n(R)$ ($\det A \neq 0$).*

(i) *Let $N \in \text{Spec}(M)$, $N = \langle BA \rangle$ for some $B \in M_n(R)$. If $(\det B, \det A) = 1$ then there exists $N' \in \text{Spec}(R^n)$ such that $N' \cap M = N$.*

(ii) *Let $N \in \text{Spec}(M)$ and $N = \langle BA \rangle$ such that A is a diagonal matrix and B is a prime matrix. If there exists a prime element $p \in R$ such that $p | (\det B, \det A)$ then there exists $N' \in \text{Spec}(R^n)$ such that $N' \cap M = N$ if and only if for*

$I = \{i : p | a_{ii}, 1 \leq i \leq n\}$, $A = (a_{ij})$ and $B = (b_{ij})$ we have (1) for all $i \in I$, $b_{ii} = 1$ and (2) if $i_0 = \min I$, $j > i_0$, $b_{jj} = p$ then $p | b_{ij}, \forall i \in I$.

(iii) *If $N \in \text{Spec}(M)$ and $\text{rank} N < \text{rank} M$ then there exists $N' \in \text{Spec}(R^n)$ such that $N' \cap M = N$.*

Proof. (i) Suppose that $N \in \text{Spec}(M)$. By [2], $\det B = up^\alpha$ ($\alpha \geq 1$) such that p is prime and u is a unit element of R . By theorem 1.9(ii), $up^\alpha \in (N : M)$ and therefore $p \in (N : M)$. Also, $\det A \in (M : R^n)$. Since $(\det B, \det A) = 1$, $\det A \notin (N : M)$. By theorem 1.4, there exists $N' \in \text{Spec}(R^n)$ such that $N' \cap M = N$.

(ii) Let $P = (p_{ij})$ be a diagonal matrix such that $p_{ii} = p$, $1 \leq i \leq n$. We show that $\langle P \rangle \cap M \subseteq N$.

If $m \in \langle P \rangle \cap M$ then $m = (d_1 a_{11}, \dots, d_n a_{nn})$ such that $d_j \in R$ and $p | d_j a_{jj}$ for all $j, 1 \leq j \leq n$. Let i be the smallest integer such that $p \nmid d_i$ and therefore $p | a_{ii}$ and $i \in I$. Now $(d_1 a_{11}, \dots, d_{i-1} a_{i-1, i-1}, 0, \dots, 0) \in N$, since by Theorem 1.9.(ii), $\det B \in (N : M)$. But,

$$BA = \begin{pmatrix} b_{11}a_{11} & \dots & \dots & b_{1n}a_{nn} \\ 0 & b_{22}a_{22} & \dots & b_{2n}a_{nn} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & b_{nn}a_{nn} \end{pmatrix}.$$

It is enough to show that $(0, \dots, 0, d_i a_{ii}, \dots, d_n a_{nn}) = (r_1, \dots, r_n)BA$, for some $r_j \in R$.

Put $r_1 = \cdots = r_{i-1} = 0$, thus $d_i a_{ii} = r_i b_{ii} a_{ii}$, hence $d_i = r_i b_{ii}$. By (1), $b_{ii} = 1$. So $r_i = d_i$. We will show that the equation $d_i b_{i+1} a_{i+1} + r_{i+1} b_{i+1} a_{i+1} = d_{i+1} a_{i+1}$ or equivalently, $d_i b_{i+1} + r_{i+1} b_{i+1} = d_{i+1}$ has a solution.

Case 1: If $b_{i+1} = 1$ then $r_{i+1} = d_{i+1} - d_i b_{i+1}$.

Case 2: If $b_{i+1} = p$ then $i+1 \notin I$.

Since $i+1 \notin I$ by hypothesis $p \nmid a_{i+1}$, which implies that $p \mid d_{i+1}$.

Now $d_{i+1} = p d'_{i+1}$ and by (2), $b_{i+1} = p b'_{i+1}$ and hence $r_{i+1} = d'_{i+1} - d_i b'_{i+1}$.

Suppose that for every $j < k$ the equation $d_i b_{ij} + r_{i+1} b_{i+1j} + \cdots + r_{j-1} b_{j-1j} + r_j b_{jj} = d_j$ has a solution, we shall find $r_k \in R$ such that $d_i b_{ik} + r_{i+1} b_{i+1k} + \cdots + r_k b_{kk} = d_k$.

For $b_{kk} = 1$,

$$r_k = d_k - (d_i b_{ik} + \cdots + r_{k-1} b_{k-1k}).$$

If $b_{kk} = p$ then $k \notin I$. Hence $p \nmid a_{kk}$ and therefore $p \mid d_k$, by (2). If $1 \leq j < k$ and $j \in I$ by hypothesis we have $p \mid b_{jk}$. It follows that $p \mid r_j b_{jk}$.

If $j \notin I$ we have two cases for b_{jj} :

If $b_{jj} = p$, since B is a prime matrix, $b_{jk} = 0$ ($k < j$), which implies that $p \mid r_j b_{jk}$ ($j < k$).

If $b_{jj} = 1$, since B is a prime matrix, $b_{\ell j} = 0$ for $1 \leq \ell < j$ and so $r_j = d_j$, $p \mid d_j$ since $j \notin I$. Hence in any case we have $p \mid r_j b_{jk}$, $1 \leq j < k$, and so the equation has a solution.

Therefore $\langle P \rangle \cap M \subseteq N$. Put $T = \{L \leq R^n \mid \langle P \rangle \subseteq L \text{ and } L \cap M \subseteq N\}$. Since $\langle P \rangle \in T$, $T \neq \emptyset$. By Zorn's Lemma T has a maximal element N' . It is clear that $N' \cap M = N$. Since $\langle p \rangle = (\langle P \rangle : R^n) \subseteq (N' : R^n) \subseteq (N : M) = \langle p \rangle$, we have $(N' : R^n) = (N : M)$. Therefore $N' \in \text{Spec}(R^n)$.

Conversely, let there exist $N' \in \text{Spec}(R^n)$ such that $N' \cap M = N$. By Lemma 1.2, $\langle p \rangle = (N : M) = (N' : R^n)$. Let P be as above. Since $\langle P \rangle \subseteq N'$, hence $\langle P \rangle \cap M \subseteq N$. If $i \in I$ then $p \mid a_{ii}$ and hence $(0, \dots, 0, a_{ii}, 0, \dots, 0) \in \langle P \rangle$. But $(0, \dots, 0, a_{ii}, 0, \dots, 0) \in M$, implies that $(0, \dots, 0, a_{ii}, 0, \dots, 0) \in N$. Thus $(0, \dots, 0, a_{ii}, 0, \dots, 0) = (r_1, \dots, r_n)BA$, for some $r_j \in R$. Hence $r_1 = \cdots = r_{i-1} = 0$ and therefore $r_i b_{ii} a_{ii} = a_{ii}$. So $r_i b_{ii} = 1$ and hence $b_{ii} = 1$. Suppose that $k > i_0$ and $b_{kk} = p$, so $p \nmid a_{kk}$. If $k-1 \in I$ we have $p \mid a_{k-1k-1}$. But

$$(0, \dots, 0, a_{k-1k-1}, p a_{kk}, 0, \dots, 0) \in \langle P \rangle \cap M$$

and hence $b_{k-1k} + r_k p = p$, for some $r_k \in R$, so $p \mid b_{k-1k}$. In general, if $i \in I$ then $p \mid a_{ii}$ and

$$(0, \dots, 0, a_{ii}, 0, \dots, 0, p a_{kk}, 0, \dots, 0) \in \langle P \rangle \cap M.$$

Thus

$$(0, \dots, 0, a_{ii}, 0, \dots, 0, pa_{kk}, 0, \dots, 0) = (r_1, \dots, r_n)BA$$

for some $r_j \in R$. Hence $r_1 = r_2 = \dots = r_{i-1} = 0$. Now $r_i b_{ii} a_{ii} = a_{ii}$, $i \in I$. It follows that $b_{ii} = 1$, which implies that $r_i = 1$. Since $(b_{ik} + r_{i+1} b_{i+1 k} + \dots + r_k b_{kk}) a_{kk} = p a_{kk}$, so $b_{ik} + r_{i+1} b_{i+1 k} + \dots + r_k b_{kk} = p$. We now show that $r_j b_{jk} = 0$, $i + 1 \leq j < k$.

If $b_{jj} = p$, since B is a prime matrix, hence $b_{jk} = 0$.

If $b_{jj} = 1$. Since $b_{ij} + r_{i+1} b_{i+1 j} + \dots + r_j b_{jj} = 0$, and B is a prime matrix, hence $b_{\ell j} = 0$ for every $i \leq \ell \leq j - 1$. It follows that $r_j = 0$. Hence $r_j b_{jk} = 0$. We have $b_{ik} + r p = p$, for some $r \in R$. Thus $p | b_{ik}$.

(iii) Put $T = \{N' \leq R^n | N' \cap M = N\}$. Since $N \in T$, $T \neq \emptyset$. By Zorn's Lemma T has a maximal element N' . Hence $N' \cap M = N$ and $(N' : R^n) = (N : M)$, we have $N' \in \text{Spec}(R^n)$. \square

2. Going down and incomparability theorems for modules

Definition. Let M and M' be R -modules.

(i) We say *to be going down* (or simply GD) holds for (M, M') , if $M \subseteq M'$, and given any $P'_1 \in \text{Spec}(M')$ and $P_0 \subseteq P_1$ in $\text{Spec}(M)$ with $P'_1 \cap M = P_1$, we have $P'_0 \in \text{Spec}(M')$ such that $P'_0 \subseteq P'_1$ and $P'_0 \cap M = P_0$.

(ii) INC=“Incomparability” means that, if $M \subseteq M'$ and given P'_1 and P'_2 in $\text{Spec}(M')$ with $P'_1 \cap M = P'_2 \cap M \neq M$, we have $P'_1 = P'_2$.

Examples. Let $M = 2\mathbf{Z}$ and $M' = \mathbf{Z}$ be \mathbf{Z} -modules. Let p_1 and p'_1 be prime numbers. Put $P'_1 = p'_1 \mathbf{Z} \in \text{Spec}(M')$ and $P_0 = \{0\} \subseteq P_1 = 2p_1 \mathbf{Z} \in \text{Spec}(M)$. Since $p'_1 \mathbf{Z} \cap 2\mathbf{Z} = 2p_1 \mathbf{Z}$, hence $2 \neq p'_1 = p_1$. We choose $P'_0 = \{0\} \in \text{Spec}(M')$. Thus we have $P'_0 \cap 2\mathbf{Z} = P_0$ and therefore GD holds for (M, M') . It is easy to see that $4\mathbf{Z} \in \text{Spec}(M)$ has no lying over and so LO does not hold for (M, M') . In this example, we show that INC holds for (M, M') . Let $p'_1 \neq 2$ and $p'_2 \neq 2$ be prime numbers. If $P'_1 = p'_1 \mathbf{Z}$ and $P'_2 = p'_2 \mathbf{Z}$, then $P'_1 \cap 2\mathbf{Z} = 2p'_1 \mathbf{Z}$ and $P'_2 \cap 2\mathbf{Z} = 2p'_2 \mathbf{Z}$. We must have $2p'_1 \mathbf{Z} = 2p'_2 \mathbf{Z} \neq 2\mathbf{Z}$ and so $p'_1 = p'_2$. Therefore $P'_1 = P'_2$ and INC holds for (M, M') . Also it is clear that GD holds for (V, W) , where V is a vector space with $\dim_F V \geq 2$ and proper subspace W .

Theorem 2.1 Let $\varphi \in \text{Hom}_R(M, M')$ and $\text{Ker} \varphi \subseteq \text{rad}_M(0)$. If for each $m' \in M'$,

$P \in \text{Spec}(M)$ and $r \in (P : M)$ there exists $s \in R \setminus (P : M)$ such that $rs m' \in \varphi(\text{rad}_M(0))$, then GD holds for (M, M') .

Proof. Suppose $P_0 \subseteq P_1$ in $\text{Spec}(M)$ and $Q_1 \in \text{Spec}(M')$ such that $\varphi^{-1}(Q_1) = P_1$. Put $T = \{L' \leq M' \mid L' \subseteq Q_1 \text{ and } \varphi^{-1}(L') \subseteq P_0\}$. Since $\{0\} \in T$, $T \neq \emptyset$, it follows by Zorn's Lemma that T has a maximal element Q_0 . Now we show that $\varphi^{-1}(Q_0) = P_0$. Assume that $\varphi^{-1}(Q_0) \subset P_0$. Hence there exists $p_0 \in P_0$ such that $\varphi(p_0) \notin Q_0$. Since $Q_0 \subset Q_0 + \langle \varphi(p_0) \rangle \subseteq Q_1$, there exists $m \in \varphi^{-1}(Q_0 + \langle \varphi(p_0) \rangle)$ such that $m \notin P_0$. Hence there exists $q \in Q_0$ and $r \in R$ such that $\varphi(m) - r\varphi(p_0) = q$, and so $m \in P_0$, which is a contradiction. Therefore $\varphi^{-1}(Q_0) = P_0$.

Now we show that $Q_0 \in \text{Spec}(M')$. Let $rm' \in Q_0$, $r \in R$, $m' \in M'$ and $m' \notin Q_0$. Suppose that $Q_0 + Rm' \subseteq Q_1$ and hence there exists $m \in \varphi^{-1}(Q_0 + Rm')$ such that $m \notin P_0$. Therefore $\varphi(m) = q + tm'$, where $q \in Q_0$ and $t \in R$. Thus $r\varphi(m) = rq + rtm'$ and so $rm \in P_0$. Since $P_0 \in \text{Spec}(M)$, and $m \notin P_0$, hence $r \in (P_0 : M)$. By Lemma 1.2 (ii), $r \in (P_1 : M) = (Q_1 : M')$.

Now if $Q_0 + Rm' \not\subseteq Q_1$. Then there exists $x \in (Q_0 + Rm') \setminus Q_1$. Hence there exists $q \in Q_0$ and $s \in R$ such that $rx = rq + rsm'$, and so $rx \in Q_1$. Since $Q_1 \in \text{Spec}(M')$ and $x \notin Q_1$, hence $r \in (Q_1 : M')$. Assume that $r \notin (Q_0 : M')$. Hence there exists $m'_1 \in M'$ such that $rm'_1 \notin Q_0$. By assumption, there exists $s \in R \setminus (P_1 : M)$ such that $rs m'_1 \in \varphi(\text{rad}_M(0))$. Since $\text{rad}_M(0) \subseteq P_0$, hence $\varphi(\text{rad}_M(0)) \subseteq \varphi(P_0) \subseteq Q_0$ and so $rs m'_1 \in Q_0$. Since $rm'_1 \notin Q_0$, hence $Q_0 \subset Q_0 + \langle rm'_1 \rangle \subseteq Q_1$ and by a proof similar to the above, $s \in (P_0 : M)$ and so $s \in (P_1 : M)$; which is a contradiction. We conclude that $Q_0 \in \text{Spec}(M')$ and the proof is complete. \square

Example. Let M be an R -module such that $\text{Ann}_R(M) = m$, where $m \in \text{Max}(R)$ (for example, vector spaces). Define the monomorphism $\varphi \in \text{Hom}_R(M, M \oplus R/m)$ by $\varphi(x) = (x, 0)$, $x \in M$. Let P be a prime submodule of M . It is clear that $(P : M) = m$. For any $s \in R \setminus (P : M)$, $r \in (P : M)$ and $(x, t+m) \in (M \oplus R/m)$, we have $sr(x, t+m) = s(rx, rt+m) = (0, 0) \in \varphi(\text{rad}_M(0))$. Therefore by theorem 2.1, GD holds for (M, M') .

Lemma 2.2 *Let $M \subset M'$ be R -modules. The INC holds if one of the following conditions holds.*

- (i) *For each $m' \in M'$ and $P \in \text{Spec}(M)$ there exists $s \in R \setminus (P : M)$ such that $sm' \in M$;*
- (ii) *M' is a multiplication R -module.*

(iii) $\sqrt{M : M'} + \text{Ann}_R(M) = R$.

Proof. Let P'_0 and P'_1 in $\text{Spec}(M')$ and $P'_0 \cap M = P'_1 \cap M \neq M$. Assume that (i) holds and there exists $p'_0 \in P'_0 \setminus P'_1$. By assumption and Lemma 1.2 (ii), there exists $s \in R \setminus (P'_1 : M')$ such that $sp'_0 \in M$ and hence $sp'_0 \in P'_1$. Since $P'_1 \in \text{Spec}(M')$ and $p'_0 \notin P'_1$, so $s \in (P'_1 : M')$, which is a contradiction. Now assume that (ii) holds. By Lemma 1.2 (ii), we have

$$(P'_1 : M') = (P'_1 \cap M : M) = (P_0 \cap M : M) = (P'_0 : M).$$

Hence $P'_0 = (P'_0 : M)M' = (P'_1 : M')M' = P'_1$.

(iii) The proof is obvious by part (i). □

In the following we will show that LO and GD are local properties.

Lemma 2.3 *Let $M \subseteq M'$ be R -modules then the following conditions are equivalent.*

- (i) *LO holds for $M \subseteq M'$.*
- (ii) *LO holds for $M_P \subseteq M'_P$, for all $P \in \text{Spec}(R)$.*
- (iii) *LO holds for $M_Q \subseteq M'_Q$, for all $Q \in \text{Max}(R)$.*

Proof. (i) \rightarrow (ii) Let $P \in \text{Spec}(R)$, $S = R \setminus P$ and $N'_1 \in \text{Spec}(M_P)$. There exists $N_1 \in \text{Spec}(M)$ such that $S^{-1}N_1 = N'_1$, by [5, Proposition 1]. By (i), there exists $N_2 \in \text{Spec}(M')$ such that $N_2 \cap M = N_1$. Thus $S^{-1}(N_2) \cap M_P = S^{-1}(N_2 \cap M) = S^{-1}N_1 = N'_1$.

(ii) \rightarrow (iii) The proof is obvious.

(iii) \rightarrow (i) Let $N \in \text{Spec}(M)$ there exists $Q \in \text{Max}(R)$ such that $(N : M) \subseteq Q$. Put $S = R \setminus Q$, and so $S^{-1}N \in \text{Spec}(M_Q)$, by [5, Corollary 3]. By (iii), there exists $S^{-1}N' \in \text{Spec}(M_Q)$ such that $S^{-1}N' \cap M_Q = S^{-1}N$ and hence $N = N' \cap M$. □

Lemma 2.4 *Let $M \subseteq M'$ be R -modules. Then the following conditions are equivalent.*

- (i) *GD holds for $M \subseteq M'$,*
- (ii) *GD holds for $M_P \subseteq M'_P$, for all $P \in \text{Spec}(R)$.*
- (iii) *GD holds for $M_Q \subseteq M'_Q$, for all $Q \in \text{Max}(R)$.*

Proof. The proof is similar to the Lemma 2.3. □

3. On the dimension of a module

Let R be a ring and M be an R -module. Let N be a prime submodule of M . Then we define the height of N to be the maximal positive integer k , if it exists, such that there exists a chain of prime submodules of M as follows:

$$N = N_0 \supset N_1 \supset \cdots \supset N_k.$$

We shall denote the height of N in M by $ht_M(N)$ (see [6]). Suppose that M is an R -module and P be a prime ideal of R . Put $S = R \setminus P$ and define the distinguished submodule $PM(S_P) = \{x \in M : sx \in PM, \text{ for some } s \in S\}$ of M . We define [see 1] the dimension of M to be the maximal positive integer k , if such exists, such that there exists a chain of prime distinguished submodules of M as

$$N_0 \subset N_1 \subset \cdots \subset N_k.$$

Lemma 3.1 *Let $M \subseteq M'$ be R -modules. Assume that for every $m' \in M'$ and $P \in \text{Spec}(M)$ there exists $s \in R \setminus (P : M)$ such that $sm' \in M$. If $PM(S_P) \neq M$ then $PM'(S_P) \cap M = PM(S_P)$.*

Proof. Since $PM(S_P) \neq M$, by [1, Proposition 1.1], $PM(S_P) \in \text{Spec}(M)$. It is clear that $PM(S_P) \subseteq M \cap PM'(S_P)$. Suppose that $PM'(S_P) \cap M \not\subseteq PM(S_P)$. Hence there exists $m \in PM'(S_P) \cap M$ such that $m \notin PM(S_P)$. Thus there exists $s \in R \setminus P$ such that

$$sm = \sum_{i=1}^n p_i m'_i, \text{ where } p_i \in P \text{ and } m'_i \in M'. \text{ By assumption there exists } s_i \in R \setminus P \text{ such}$$

$$\text{that } s_i m'_i \in M. \text{ Since } \left(\prod_{j=1}^n s_j\right) sm = \sum_{i=1}^n p_i \left(\prod_{j=1}^n s_j\right) m'_i, \text{ hence } m \in PM(S_P), \text{ which is a}$$

contradiction. □

Lemma 3.2 *Let $M \subseteq M'$ be R -modules and assume that GD holds for (M, M') . Suppose $P \in \text{Spec}(M)$ and $ht_M(P) = k$. If $P' \in \text{Spec}(M')$ such that $P' \cap M = P$ then $ht_{M'}(P') \geq k$.*

Proof. Since $ht_M(P) = k$, there exists a chain of prime submodules of M as follows

$$P = P_0 \supset P_1 \supset \cdots \supset P_k.$$

Since GD holds for (M, M') and so there exists a chain of prime submodules of M' as follows

$$P' = P'_0 \supset P'_1 \supset \cdots \supset P'_k.$$

Hence $ht_{M'}(P') \geq k$. □

Corollary 3.3 *Let $M \subseteq M'$ be R -modules. Let M' be multiplication module and for each $m' \in M'$, $P \in \text{Spec}(M)$ and $r \in (P : M)$, there exists $s \in R \setminus (P : M)$ such that $rs m' \in \text{rad}_M(0)$. Suppose that $ht_M(P) = k$. Then there exists $P' \in \text{Spec}(M')$ such that $P' \cap M = P$ and $ht_{M'}(P') = k$.*

Proof. By Theorem 1.5, there exists $P' \in \text{Spec}(M')$ such that $P' \cap M = P$ and by Lemma 3.2, $ht_{M'}(P') \geq k$. Suppose that $ht_{M'}(P') = n$, so there exists a chain of prime submodules of M' as follows:

$$P' = P'_0 \supset P'_1 \supset \cdots \supset P'_n.$$

By Theorem 2.1 and Lemma 2.2, since GD and INC hold, we have the following chain of prime submodules of M :

$$P = P' \cap M \supset P'_1 \cap M \supset \cdots \supset P'_n \cap M.$$

Therefore $ht_M(P) \geq n$ and so $ht_{M'}(P') = k$. □

Proposition 3.4 *Let $M \subseteq M'$ be R -modules and M' be a finitely generated module such that $\text{Ann}_R(M') \subseteq N(R)$. If for all P_1, P_2 in $\text{Spec}(M')$ we have $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ then $\dim M \leq \dim M'$.*

Proof. Let $\dim M = n$. Hence there exists a chain of distinguished submodules of M as follows:

$$N_0 \subset N_1 \subset \cdots \subset N_n.$$

Put $(N_i : M) = P_i$, for all i . Since M' is finitely generated and $\text{Ann}_R(M') \subseteq N(R)$, hence $P_i M'(S_{P_i}) \neq M'$ by [1, Corollary 1.2]. By assumption $P_i M'(S_{P_i}) \subseteq P_{i+1}(M')(S_{P_{i+1}})$ or

$P_{i+1}M'(S_{P_{i+1}}) \subseteq P_iM'(S_{P_i})$. Since $P_i \subseteq P_{i+1}$ implies that $P_iM'(S_{P_i}) \subseteq P_{i+1}M'(S_{P_{i+1}})$, we have the chain

$$P_0M'(S_{P_0}) \subset P_1M'(S_{P_1}) \subset \cdots \subset P_nM'(S_{P_n}).$$

Therefore $\dim M \leq \dim M'$. □

Proposition 3.5 *Let $M \subseteq M'$ be R -modules and M be a finitely generated module such that $\text{Ann}_R(M) \subseteq N(R)$. If for all P_1, P_2 in $\text{Spec}(M)$ we have $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$, then $\dim M' \leq \dim M$.*

Proof. It is similar to the proof of Proposition 3.4. □

Proposition 3.6 *Let $M \subseteq M'$ be R -modules. Suppose that for every $m' \in M'$ and $P \in \text{Spec}(M)$ there exists $s \in R \setminus (P : M)$ such that $sm' \in M$. If $\dim M = n$ then $\dim M = \dim M'$.*

Proof. Since $\dim M = n$, there exists a chain of distinguished submodules of M as follows

$$N_0 \subset N_1 \subset \cdots \subset N_n.$$

Let $(N_i : M) = P_i$, for all i . Then by Lemma 3.1, we have the following chain of distinguished submodules of M'

$$P_0M'(S_{P_0}) \subset P_1M'(S_{P_1}) \subset \cdots \subset P_nM'(S_{P_n}).$$

This implies $\dim M \leq \dim M'$. Now let there exist a chain of distinguished submodules of M' as follows

$$P_0M'(S_{P_0}) \subset P_1M'(S_{P_1}) \subset \cdots \subset P_kM'(S_{P_k}).$$

We show that $M \not\subseteq P_kM'(S_{P_k})$. Suppose that $M \subset P_kM'(S_{P_k})$. Since $P_kM'(S_{P_k}) \neq M'$, there exists $m' \in M' \setminus P_kM'(S_{P_k})$. By assumption there exists $s \in R \setminus P_k$ such that $sm' \in M \subset P_kM'(S_{P_k})$. Since $m' \notin P_kM'(S_{P_k})$, hence $s \in P_k$, which is a contradiction. Therefore $P_iM(S_{P_i}) \neq M$, for all i , and so we have the following chain of distinguished submodules of M

$$P_0M(S_{P_0}) \subset \cdots \subset P_kM(S_{P_k})$$

hence $\dim M \geq k$. □

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References

- [1] Abu-Saymeh, S.: On dimensions of finitely generated modules, *Comm. Algebra* 23, 1131-1144 (1995).
- [2] Hedayat, S., Nekooei, R.: Prime and radical submodules of free modules over a PID, *Houston Journal of Mathematics*, 32, 2,355-367 (2006).
- [3] Lu, C.P.: Prime submodules of modules, *Comment. Math. Univ. Sancti Pauli*, 33, 1, 61-69 (1984).
- [4] Lu, C.P.: M-Radicals of submodules in modules, *Math. Japonica* 34, 2, 211-219 (1989).
- [5] Lu, C.P.: Spectra of modules, *Comm. Alg.* 23, 3471-3752 (1995).
- [6] McCasland, R.L., Smith, P.F.: Prime submodules of Noetherian modules, *Rocky mountain J. Math.* 23, 1041-1062 (1993).
- [7] Nekooei, R.: On finitely generated multiplication modules, *Czechoslovak Mathematical Journal*, 55, 130, 503-510 (2005).
- [8] Pusat-Yilmaz, D.: On prime submodules of finitely generated free modules, *Turkish Journal of Mathematics*, 27, 329- 342 (2003).
- [9] Rowen, L.: *Ring Theory*, Vol. I, Academic Press, San Diego, 1988.
- [10] Sharp, D.W., Vámos, P.: *Injective Modules*, Cambridge University Press, London, 1972.

S. KARIMZADEH and R. NEKOOEI
Department of Mathematics,
Shahid Bahonar University of Kerman
Kerman-IRAN
e-mail: rnekooui@mail.uk.ac.ir

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