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A Generalization of Ankeny and Rivlin's Result on the Maximum Modulus of Polynomials not Vanishing in the Interior of the Unit Circle

V. K. Jain

Abstract

For an arbitrary entire function $f(z)$, let

$$M(f, r) = \max_{|z|=r} |f(z)|.$$

For a polynomial $p(z)$ of degree n , it is known that

$$M(p, R) \leq R^n M(p, 1), \quad R > 1.$$

By considering the polynomial $p(z)$ with no zeros in $|z| < 1$, Ankeny and Rivlin obtained the refinement

$$M(p, R) \leq \{(R^n + 1)/2\} M(p, 1), \quad R > 1.$$

By considering the polynomial $p(z)$ with no zeros in $|z| < k$, ($k \geq 1$) and simultaneously thinking of s^{th} derivative ($0 \leq s < n$) of the polynomial, we have obtained the generalization

$$M(p^{(s)}, R) \leq \begin{cases} (1/2) \{ \frac{d^s}{dR^s} (R^n + k^n) \} (2/(1+k))^n M(p, 1), & R \geq k, \\ (1/(R^s + k^s)) \{ \frac{d^s}{dx^s} (1 + x^n) \}_{x=1} \} ((R+k)/(1+k))^n M(p, 1), & 1 \leq R \leq k, \end{cases}$$

of Ankeny and Rivlin's result.

Key words and phrases: Polynomial, maximum modulus principle, not vanishing in the interior of unit circle, generalization, s^{th} derivative.

1. Introduction and statement of results

For an arbitrary entire function $f(z)$, let

$$M(f, r) = \max_{|z|=r} |f(z)|.$$

As a consequence of maximum modulus principle, we have the following result.

Theorem A *If $p(z)$ is a polynomial of degree n , then*

$$M(p, R) \leq R^n M(p, 1), \quad R > 1,$$

with equality only for $p(z) = \lambda z^n$.

Ankeny and Rivlin [1] considered polynomials not vanishing in the interior of the unit circle and obtained the following refinement of Theorem A.

Theorem B *If $p(z)$ is a polynomial of degree n , having no zeros in $|z| < 1$, then*

$$M(p, R) \leq \{(1 + R^n)/2\}M(p, 1), \quad R > 1,$$

with equality only for $p(z) = \lambda + \mu z^n$, with $|\lambda| = |\mu|$.

In this paper, we have obtained a generalization of Theorem B, by considering polynomials with no zeros in $|z| < k, k \geq 1$ and simultaneously thinking of the s^{th} derivative, ($0 \leq s < n$), of the polynomial, instead of the polynomial itself. More precisely we have proved the following theorem.

Theorem *If $p(z)$ is a polynomial of degree n having no zeros in $|z| < k, (k \geq 1)$ then for $0 \leq s < n$*

$$M(p^{(s)}, R) \leq \begin{cases} (1/2)\{\frac{d^s}{dR^s}(R^n + k^n)\}(2/(1+k))^n M(p, 1), & R \geq k, & (1.1) \\ (1/(R^s + k^s))\{\frac{d^s}{dx^s}(1 + x^n)\}_{x=1}\{(R+k)/(1+k)\}^n M(p, 1), & 1 \leq R \leq k. & (1.2) \end{cases}$$

Equality holds in (1.1) (with $k = 1$ & $s = 0$) for $p(z) = z^n + 1$ and equality holds in (1.2) (with $s = 1$) for $p(z) = (z + k)^n$.

2. Lemmas

For the proof of the theorem we require following lemmas.

Lemma 1 *Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$. If $p(z)$ is a polynomial of degree at most n such that*

$$|p(z)| \leq |P(z)|, \quad |z| = 1, \quad (2.1)$$

then for $0 \leq s < n$,

$$|p^{(s)}(z)| \leq |P^{(s)}(z)|, \quad |z| \geq 1. \quad (2.2)$$

Proof of Lemma 1 By using (2.1), we can say that an application of maximum modulus principle to the function $p(z)/P(z)$ will yield

$$|p(z)| \leq |P(z)|, \quad |z| \geq 1. \quad (2.3)$$

Therefore the polynomial

$$p(z) - \lambda P(z)$$

will not vanish in $|z| > 1$ for every λ with $|\lambda| > 1$. Gauss-Lucas theorem will then imply that polynomial

$$p^{(s)}(z) - \lambda P^{(s)}(z), \quad 1 \leq s < n$$

will not vanish in $|z| > 1$ for every λ with $|\lambda| > 1$ and therefore

$$|p^{(s)}(z)| \leq |P^{(s)}(z)|, \quad |z| > 1,$$

leading to

$$|p^{(s)}(z)| \leq |P^{(s)}(z)|, \quad |z| \geq 1, \text{ \& } 1 \leq s < n,$$

which, on being combined with (2.3), completes the proof of Lemma 1. \square

Lemma 2 *If $p(z)$ is a polynomial of degree at most n then for $0 \leq s < n$,*

$$|p^{(s)}(z)| + |q^{(s)}(z)| \leq \left\{ \left| \frac{d^s}{dz^s}(1) \right| + \left| \frac{d^s}{dz^s}(z^n) \right| \right\} M(p, 1), \quad |z| \geq 1, \quad (2.4)$$

where

$$q(z) = z^n \overline{p(1/\bar{z})}. \quad (2.5)$$

Proof of Lemma 2 We consider the polynomial

$$t(z) = p(z) - \lambda M(p, 1), \quad |\lambda| > 1,$$

of degree at most n . Then the polynomial

$$T(z) = z^n \overline{t(1/\bar{z})} = q(z) - \bar{\lambda} M(p, 1) z^n, \quad (\text{by (2.5)}),$$

of degree n , possesses the characteristic

$$|t(z)| \leq |T(z)|, \quad |z| = 1$$

and has all its zeros in $|z| \leq 1$. Therefore on applying Lemma 1 to polynomials $t(z)$ and $T(z)$ we get for $0 \leq s < n$ and $|\lambda| > 1$

$$|p^{(s)}(z) - \lambda M(p, 1) \frac{d^s}{dz^s}(1)| \leq |q^{(s)}(z) - \bar{\lambda} M(p, 1) \frac{d^s}{dz^s}(z^n)|, \quad |z| \geq 1,$$

which, by choosing $\arg \lambda$ suitably, can be rewritten as

$$|p^{(s)}(z)| - |\lambda| M(p, 1) \left| \frac{d^s}{dz^s}(1) \right| \leq \left| |\lambda| M(p, 1) \left| \frac{d^s}{dz^s}(z^n) \right| - |q^{(s)}(z)| \right|, \quad |z| \geq 1. \quad (2.6)$$

We can apply Lemma 1 to polynomials $q(z)$ and

$$z^n M(p, 1)$$

also, and obtain for $0 \leq s < n$

$$|q^{(s)}(z)| \leq M(p, 1) \left| \frac{d^s}{dz^s}(z^n) \right|, \quad |z| \geq 1,$$

which helps us to rewrite(2.6) as

$$|p^{(s)}(z)| - |\lambda| M(p, 1) \left| \frac{d^s}{dz^s}(1) \right| \leq \left| |\lambda| M(p, 1) \left| \frac{d^s}{dz^s}(z^n) \right| - |q^{(s)}(z)| \right|, \quad |z| \geq 1, |\lambda| > 1, 0 \leq s < n.$$

Now on letting

$$|\lambda| \rightarrow 1,$$

(2.4) follows. □

Lemma 3 *If $P(z)$ is a polynomial of degree n , having no zeros in $|z| < k$, ($k \geq 1$), with*

$$M(P, 1) = 1$$

then for $1 \leq R \leq k^2$

$$M(P, R) \leq ((R + k)/(1 + k))^n.$$

This lemma is due to Aziz and Mohammad [2].

Lemma 4 *Let $P(z)$ be a polynomial of degree n , having no zeros in $|z| < k$, ($k \geq 1$). Then*

$$|P(z)| \leq 1 \text{ for } |z| \leq 1$$

implies

$$|P^{(s)}(z)| \leq n(n-1) \dots (n-s+1)/(1+k^s) \text{ for } |z| \leq 1 \text{ and } s \geq 1.$$

This lemma is due to Govil and Rahman [3].

From Lemma 4 we easily get

Lemma 5 *If $P(z)$ is a polynomial of degree n , having no zeros in $|z| < k$, ($k \geq 1$) then for $0 \leq s < n$*

$$M(P^{(s)}, 1) \leq (1/(1+k^s))M(P, 1)[\{\frac{d^s}{dx^s}(1+x^n)\}_{x=1}].$$

3. Proof of Theorem 1

We consider the polynomial

$$P(z) = p(kz). \tag{3.1}$$

Then the polynomial

$$Q(z) = z^n \overline{P(1/\bar{z})}$$

possesses the characteristic

$$|P(z)| \leq |Q(z)|, \quad |z| = 1$$

and has all its zeros in $|z| \leq 1$. Therefore on applying Lemma 1 to the polynomials $P(z)$ and $Q(z)$ we get for $0 \leq s < n$ and $t \geq 1$

$$|P^{(s)}(te^{i\theta})| \leq |Q^{(s)}(te^{i\theta})|, \quad 0 \leq \theta \leq 2\pi. \quad (3.2)$$

Further, by Lemma 2 we have for $t \geq 1$ and $0 \leq s < n$

$$|P^{(s)}(te^{i\theta})| + |Q^{(s)}(te^{i\theta})| \leq \left\{ \frac{d^s}{dt^s}(t^n + 1) \right\} M(P, 1), \quad 0 \leq \theta \leq 2\pi,$$

which, by (3.2), implies that

$$|P^{(s)}(te^{i\theta})| \leq (1/2) \left\{ \frac{d^s}{dt^s}(1 + t^n) \right\} M(p, 1),$$

i.e.

$$\begin{aligned} |p^{(s)}(kte^{i\theta})| &\leq (1/(2k^s)) \left\{ \frac{d^s}{dt^s}(1 + t^n) \right\} M(p, k), \quad (\text{by (3.1)}), \\ &\leq (1/(2k^s))(2k/(1+k))^n M(p, 1) \left\{ \frac{d^s}{dt^s}(1 + t^n) \right\}, \quad (\text{by Lemma 3}), \end{aligned}$$

thereby leading to inequality (1.1).

Now by applying Lemma 5 to the polynomial $p(Rz)$, ($1 \leq R \leq k$), having no zeros in $|z| < k/R$, we have for $0 \leq s < n$

$$M(p^{(s)}, R) \leq (1/(R^s + k^s)) M(p, R) \left[\left\{ \frac{d^s}{dx^s}(1 + x^n) \right\}_{x=1} \right],$$

and the inequality (1.2) follows by using Lemma 3. This completes the proof of Theorem 1. \square

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