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Slant submanifolds of Kaehler Product Manifolds

Bayram Şahin and Sadık Keleş

Abstract

In this paper, we study slant submanifolds of a Kaehler product manifold. We show that an F -invariant slant submanifold of Kaehler product manifold is a product manifold. We also obtain some curvature inequalities in terms of scalar curvature and Ricci tensor.

Key Words: Slant Submanifold, Kaehler Manifold, Ricci tensor.

1. Introduction

Submanifolds of a Kaehler manifold are defined with respect to the behaviour of complex structure J . More precisely, a real submanifold M of a Kaehler manifold is called invariant if $J(TM) = TM$, where TM denotes the tangent bundle of M . M is called totally real if $J(TM) \subset TM^\perp$ and M is called CR-submanifold [1] if there are orthogonal complement two distributions D^\perp , D such that D is invariant and D^\perp is totally real. Recently, B. Y. Chen introduced slant submanifolds as follows: Let M be a submanifold of a Kaehler manifold \bar{M} , for each non zero vector $X \in T_pM$, we denote the angle between JX and T_pM by $\theta(X)$. Then M is said to be slant ([2]) if the angle $\theta(X)$ is constant, i.e., it is independent of the choice of $p \in M$ and $X \in T_pM$. The angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A proper slant immersion is neither invariant nor anti-invariant.

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The geometry of submanifolds of a Kaehler manifold has been investigated by many authors. In [6], K.Yano and M. Kon studied the geometry of F -invariant and F -anti-invariant submanifolds of Kaehler product manifolds and showed that an F -invariant, invariant submanifold of a Kaehler product manifold is also a product manifold.

Same result was obtained for anti-invariant submanifold [6]. On the other hand, CR-submanifolds of Kaehler product manifolds were studied in [5] by M. H. Shahid.

In this paper, we consider slant submanifolds of Kaehler product manifolds. We show that an F -invariant, slant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$ is a product manifold $M_1 \times M_2$ and M_1 (resp. M_2) is also a slant submanifold of \bar{M}^m (resp. \bar{M}^n). Also we obtain, if $M = M_1 \times M_2$ is a Kaehler slant submanifold of \bar{M} , then M_1 is a Kaehler slant submanifold of \bar{M}^m and M_2 is a Kaehler slant submanifold of \bar{M}^n . In the last section we study scalar curvature and Ricci tensor of various submanifolds of a Kaehler product manifold $\bar{M} = \bar{M}^m(c_1) \times \bar{M}^n(c_2)$ and obtain several inequalities for slant, invariant and anti-invariant submanifolds of \bar{M} .

2. Preliminaries

Let (\bar{M}, g) be a $2k$ -dimensional Riemannian manifold with Riemannian metric g . An almost complex structure on \bar{M} is a tensor field J of type $(1,1)$ such that at every $p \in \bar{M}$ we have $J^2 = -I$, where I denotes the identity transformation of $T_p\bar{M}$. Then, \bar{M} is called an almost complex manifold. The Nijenhuis torsion tensor N_J , of J , is defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y],$$

$\forall X, Y \in \Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ is the module of differentiable sections of the tangent bundle $T\bar{M}$. If the torsion tensor N_J vanishes identically on \bar{M} then J is complex structure on \bar{M} which becomes a complex manifold. A Hermitian metric on \bar{M} is a Riemannian metric g satisfying

$$g(X, Y) = g(JX, JY), \quad \forall X, Y \in \Gamma(T\bar{M}). \quad (2.1)$$

An almost complex manifold endowed with a Hermitian metric is called an almost Hermitian manifold, denoted by (\bar{M}, g, J) . Denote the Levi-Civita connection on \bar{M} with respect to g by $\bar{\nabla}$. Then, \bar{M} is called an Kaehler manifold if J is parallel with

respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X J)Y = 0, \quad \forall X, Y \in \Gamma(T\bar{M}). \quad (2.2)$$

The Riemann curvature tensor field, denoted by \bar{R} , satisfies

$$\bar{R}(X, Y)J = J\bar{R}(X, Y) \quad \bar{R}(JX, JY) = \bar{R}(X, Y). \quad (2.3)$$

A complex space form is a connected Kaehler manifold of constant holomorphic sectional curvature c , denoted by $\bar{M}(c)$. The curvature tensor field of $\bar{M}(c)$ is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ &- g(JX, Z)JY + 2g(X, JY)JZ\}, \quad \forall X, Y \in \Gamma(T\bar{M}). \end{aligned} \quad (2.4)$$

We consider a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. We denote by \bar{P} and \bar{Q} the projection operators of the tangent space of \bar{M} to the tangent space of \bar{M}^m and \bar{M}^n , respectively. Then we have

$$\bar{P}^2 = \bar{P}, \quad \bar{Q}^2 = \bar{Q}, \quad \bar{P}\bar{Q} = \bar{Q}\bar{P} = 0.$$

Putting $F = \bar{P} - \bar{Q}$, we have $F^2 = I$. Thus F is an almost product structure on \bar{M} . Then we can define a Riemannian metric g on \bar{M} by

$$g(X, Y) = g_m(\bar{P}X, \bar{P}Y) + g_n(\bar{Q}X, \bar{Q}Y)$$

for any vector field X, Y on \bar{M} . Thus it follows

$$g(FX, Y) = g(FY, X).$$

Now, consider $JX = J_m\bar{P}X + J_n\bar{Q}X$ for any vector field X of \bar{M} . Then it can be verified that the following are satisfied:

$$J_m\bar{P} = \bar{P}J, \quad J_n\bar{Q} = \bar{Q}J, \quad FJ = JF, \quad J^2 = -I, \quad (2.5)$$

$$g(JX, JY) = g(X, Y), \quad \bar{\nabla}_X J = 0, \quad (2.6)$$

where $\bar{\nabla}$ is the metric connection on \bar{M} . Thus \bar{M} is a Kaehler manifold. If $\bar{M}^m(c_1)$ and $\bar{M}^n(c_2)$ are complex space forms with constant holomorphic sectional curvatures c_1 and

c_2 , respectively, then the Riemannian curvature tensor \bar{R} of a Kaehler product manifold \bar{M} is given by [6]

$$\begin{aligned}
 g(\bar{R}(X, Y)Z, W) &= \frac{1}{16}(c_1 + c_2)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
 &+ g(JY, Z)g(JX, W) - g(JX, Z)g(JY, W) \\
 &+ 2g(X, JY)g(JZ, W) + 2g(FY, Z)g(FX, W) \\
 &- g(FX, Z)g(FY, W) + g(JY, FZ)g(JX, FW) \\
 &- g(JX, FZ)g(JY, FW) + 2g(FX, JY)g(JZ, FW)] \\
 &+ \frac{1}{16}(c_1 - c_2)[g(FY, Z)g(X, W) - g(FX, Z)g(Y, W)] \\
 &+ g(Y, Z)g(FX, W) - g(X, Z)g(FY, W) \\
 &+ g(JY, FZ)g(JX, W) - g(JX, FZ)g(JY, W) \\
 &+ g(JY, Z)g(JX, FW) - g(JX, Z)g(JY, FW) \\
 &+ 2g(FX, JY)g(JZ, W) - 2g(X, JY)g(FZ, JW) \quad (2.7)
 \end{aligned}$$

for any vector fields X, Y, Z and W of \bar{M} . \square

Let \bar{M} be a Riemannian manifold and M be a Riemannian manifold isometrically immersed in \bar{M} . Then the formulas of Gauss and Weingarten for M in \bar{M} are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.8)$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.9)$$

for any vector fields X, Y tangent to M and N normal to M , where $\bar{\nabla}$ denotes the Riemannian connection on \bar{M} , h is the second fundamental form, ∇^\perp is the normal connection and A is the shape operator of M in \bar{M} . Moreover, the second fundamental form and the shape operator are related by

$$g(A_N X, Y) = g(h(X, Y), N), \quad (2.10)$$

where g denotes the Riemannian metric on M as well as on \bar{M} .

The equations of Gauss and Codazzi are given by

$$\begin{aligned} \bar{R}(X, Y, Z, W) &= R(X, Y, Z, W) + g(h(X, Z), h(Y, W)) \\ &\quad - g(h(X, W), h(Y, Z)) \end{aligned} \quad (2.11)$$

$$(\bar{R}(X, Y)Z)^\perp = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \quad (2.12)$$

for any X, Y, Z and W tangent to M , where \bar{R}, R denote the curvature tensors of \bar{M}, M , respectively, and $(\bar{R}(X, Y)Z)^\perp$ denotes the normal component of $\bar{R}(X, Y)Z$.

3. Slant Submanifolds of a Kaehler Product Manifold

Let M be an F -invariant submanifold of a Kaehler product manifold $\bar{M}^m \times \bar{M}^n$. Then, it is known that M is a locally decomposable Riemannian manifold $M = M_1 \times M_2$, where M_1 is a submanifold of \bar{M}^m and M_2 is a submanifold of \bar{M}^n . Moreover, if M is Kaehler submanifold of $\bar{M} = \bar{M}^m \times \bar{M}^n$, then M is a Kaehler product manifold $M = M_1 \times M_2$, ([6]).

Let M be an invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then, $M = M_1 \times M_2$, where M_1 is a invariant submanifold of \bar{M}^m and M_2 is an invariant submanifold of \bar{M}^n . Then we have

$$T\bar{M} = T\bar{M}^m \oplus T\bar{M}^n \quad (3.1)$$

$$= \{TM_1 \oplus TM_2\} \oplus \{TM_1^\perp \oplus TM_2^\perp\}. \quad (3.2)$$

For any $X \in \Gamma(TM_1)$ and $N_1 \in \Gamma(TM_1^\perp)$, we put

$$JX = J_m X = F_1 X + \omega_1 X, JN_1 = J_n N_1 = B_1 N_1 + C_1 N_1, \quad (3.3)$$

where $F_1 X, B_1 N_1 \in \Gamma(TM_1)$ and $C_1 N_1, \omega_1 X \in \Gamma(TM_1^\perp)$ Similarly, for any $Y \in \Gamma(TM_2)$ and $N_2 \in \Gamma(TM_2^\perp)$, we put

$$JY = J_n Y = F_2 Y + \omega_2 Y, JN_2 = J_n N_2 = B_2 N_2 + C_2 N_2, \quad (3.4)$$

where $F_2Y, B_2N_2 \in \Gamma(TM_2)$ and $C_2N_2, \omega_2Y \in \Gamma(TM_2^\perp)$.

Theorem 3.1 *Let M be an F -invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. If M is a slant submanifold of \bar{M} , then M is a slant product manifold $M_1 \times M_2$, where M_1 is a slant submanifold of \bar{M}^m and M_2 is a slant submanifold of \bar{M}^n .*

Proof. Let us assume that M is a slant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then, the angle $\theta(X)$ between JX and the tangent space T_xM at $x \in M$ is constant for $X \in T_xM$, i.e, it is independent of the choice $x \in M$ and $X \in T_xM$. Then we have

$$\cos\theta(X) = \frac{\bar{g}(JX, \phi X)}{|X| |\phi X|}$$

where ϕX is the tangential part of JX .

Thus for $X_1 \in \Gamma(TM_1)$, we have

$$\begin{aligned} \cos\theta(X_1) &= \frac{\bar{g}(JX_1, \phi X_1)}{|X| |\phi X|} \\ &= \frac{\bar{g}(J_m \bar{P}X_1 + J_n \bar{Q}X_1, \phi X_1)}{|X| |F_1X_1|} \\ &= \frac{\bar{g}(J_m X_1 + \bar{Q}JX, \phi X_1)}{|X| |F_1X_1|}. \end{aligned}$$

Hence, we have $\cos\theta(X_1) = \frac{\bar{g}(J_m X_1, F_1X_1)}{|X| |F_1X_1|}$. This means that the angle $\theta(X_1)$ between $J_m X$ and the tangent space T_xM_1 is constant. Since M_1 is a submanifold in \bar{M}^m , we conclude that M_1 is slant submanifold of \bar{M}^m . Similarly M_2 is slant submanifold of \bar{M}^n . \square

Now, we denote the Weingarten operators of M, M_1 and M_2 in \bar{M}, \bar{M}^m and \bar{M}^n by A, A^1 and A^2 , respectively. Also denote the second fundamental forms of M, M_1 and M_2 in \bar{M}, \bar{M}^m and \bar{M}^n by h, h_1 and h_2 , respectively. Then we have ([4])

$$h(X, Y) = h_1(X_1, Y_1) + h_2(X_2, Y_2), \quad (3.5)$$

where $X = X_1 + X_2, Y = Y_1 + Y_2 \in \Gamma(TM)$. We also note that, since M is a Riemannian product manifold, we have

$$TM_1 = \{X \in \Gamma(TM) \mid fX = X\}, TM_2 = \{X \in \Gamma(TM) \mid fX = -X\}. \quad (3.6)$$

Now, since $h(X_2, X_1) = 0$, using (2.10) we have $A_{N_1}X_2 \in \Gamma(TM_2)$. On the other hand, from (2.12) we have $g(A_{N_1}X_2, Y_2) = g(h_2(X_2, Y_2), N_1)$, thus from (3.1) we get $g(A_{N_1}X_2, Y_2) = 0$, hence $A_{N_1}X_2 \in \Gamma(TM_1)$. Thus we obtain

$$A_{N_1}X_2 = 0. \quad (3.7)$$

In similar way, we have

$$A_{N_2}X_1 = 0. \quad (3.8)$$

Thus from (3.7) and (3.8) we obtain

$$A_N X = A_{N_1}X_1 + A_{N_2}X_2, \quad (3.9)$$

where $X = X_1 + X_2 \in \Gamma(TM)$ and $N = N_1 + N_2 \in \Gamma(TM^\perp)$. Moreover, using (2.10) we have

$$A_N X = A_{N_1}^1 X_1 + A_{N_2}^2 X_2. \quad (3.10)$$

Theorem 3.2 *Let M be an F -invariant submanifold of a Kehlerian product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. If M is a Kaehler slant submanifold in \bar{M} , then M_1 is a Kaehler slant submanifold of \bar{M}^m and M_2 is a Kaehler slant submanifold of \bar{M}^n .*

Proof. Let M be a Kaehler slant submanifold of \bar{M} . Then using (2.9), (2.8), (3.3), (3.4) and taking into account (3.2), we obtain

$$(\nabla_{\bar{P}X} F_1) \bar{P}Y = A_{\omega_1 \bar{P}Y}^1 \bar{P}X - B_1 h_1(\bar{P}X, \bar{P}Y) \quad (3.11)$$

$$(\nabla_{\bar{Q}X} F_2) \bar{Q}Y = A_{\omega_2 \bar{Q}Y}^2 \bar{Q}X - B_2 h_2(\bar{Q}X, \bar{Q}Y) \quad (3.12)$$

for $X, Y \in \Gamma(TM)$. In similar way,

$$(\nabla_X \phi)Y = A_{\omega Y} X - B h(X, Y) \quad (3.13)$$

for $X, Y \in \Gamma(TM)$. Let $X_1, Y_1 \in \Gamma(TM_1)$ in (3.13), then if $(\nabla_X \phi) = 0$, i.e., M is Kaehler slant submanifold in \bar{M} , from (3.5) and (3.10) we have

$$A_{\omega_1 Y_1}^1 X_1 - B_1 h_1(X_1, Y_1) = 0,$$

hence $(\nabla_{\bar{P}X} F_1) = 0$. Similarly, $(\nabla_{\bar{Q}X} F_2) \bar{Q}Y = 0$. Then our assertion follows from Theorem 3.1 \square

4. Curvature Inequalities

In this section, we give some inequalities for invariant, anti-invariant and slant submanifolds of a Kaehler product manifolds in terms of Ricci tensor and scalar curvature. First we need the following.

Lemma 4.1. *Let M be a $(k + l)$ -dimensional F -invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. If M is an invariant submanifold of $\bar{M} = \bar{M}^m \times \bar{M}^n$, then M is an invariant product manifold $M = M_1 \times M_2$, where M_1 is minimal in \bar{M}^m and M_2 is minimal in \bar{M}^n .*

Proof, Let M be an F -invariant submanifold of $\bar{M} = \bar{M}^m \times \bar{M}^n$, then we have ([4])

$$H = \frac{k}{k+l}H_1 + \frac{l}{k+l}H_2, \quad (4.1)$$

where H , H_1 and H_2 are mean curvature vector fields of M , M_1 and M_2 . On the other hand, it is known that invariant submanifolds of Kaehler manifolds are minimal ([6]); thus, $H = 0$, $H_1 = 0$ and $H_2 = 0$.

Let M be an F -invariant submanifold of Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then from (2.10) and (2.11) we obtain

$$\begin{aligned} g(R(X, Y)Z, W) &= \frac{1}{16}(c_1 + c_2)[g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\ &+ g(JY, Z)g(JX, W) - g(JX, Z)g(JY, W) \\ &+ 2g(X, JY)g(JZ, W) + 2g(FY, Z)g(FX, W) \\ &- g(FX, Z)g(FY, W) + g(JY, FZ)g(JX, FW) \\ &- g(JX, FZ)g(JY, FW) + 2g(FX, JY)g(JZ, FW)] \\ &+ \frac{1}{16}(c_1 - c_2)[g(FY, Z)g(X, W) - g(FX, Z)g(Y, W) \\ &+ g(Y, Z)g(FX, W) - g(X, Z)g(FY, W) \\ &+ g(JY, FZ)g(JX, W) - g(JX, FZ)g(JY, W) \\ &+ g(JY, Z)g(JX, FW) - g(JX, Z)g(JY, FW) \\ &+ 2g(FX, JY)g(JZ, W) - 2g(X, JY)g(FZ, JW) \\ &+ g(h(X, W), h(Y, Z)) - g(h(Y, Z), h(X, W))]. \end{aligned} \quad (4.2)$$

Let M be an F -invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Since M is also a product manifold $M = M_1 \times M_2$, we can choose an orthonormal basis $\{e_1, \dots, e_k, \tilde{e}_1, \dots, \tilde{e}_l\}$ for $T_p M$ such that $\{e_1, \dots, e_k\}$ is tangent to M_1 and $\{\tilde{e}_1, \dots, \tilde{e}_l\}$ is tangent to M_2 . Thus from (4.1) and (4.2), using (3.6) we obtain

$$\begin{aligned} 2\tau &= \frac{1}{16}c_1[5k^2 - 4k + l^2 + 12 \| P_1 \|^2] + \frac{1}{16}c_2[5l^2 - 4l + k^2 + 12 \| P_2 \|^2] \\ &+ k^2 \| H_1 \|^2 + l^2 \| H_2 \|^2 - \| h \|^2, \end{aligned} \quad (4.3)$$

where $\| h \|^2 = \sum_{a,b=1}^{(k+l)} g(h(e_a, e_b), h(e_a, e_b))$, $\| P_1 \|^2 = \sum_{i,j=1}^k g(e_i, F_1 e_j)^2$ and $\| P_2 \|^2 = \sum_{\alpha,\beta=1}^l g(\tilde{e}_\alpha, F_2 \tilde{e}_\beta)^2$ and τ is the scalar curvature of M . \square

Thus (4.3) enables us to state the following theorem:

Theorem 4.1. *Let M be an F -invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then the following statements are true:*

1. *We have*

$$\begin{aligned} \tau &\leq \frac{1}{32}c_1[5k^2 - 4k + l^2 + 12 \| P_1 \|^2] + \frac{1}{32}c_2[5l^2 - 4l + k^2 + 12 \| P_1 \|^2] \\ &+ \frac{k^2}{2} \| H_1 \|^2 + \frac{l^2}{2} \| H_2 \|^2. \end{aligned} \quad (4.4)$$

2. *If M is θ slant submanifold, then*

$$\begin{aligned} \tau &\leq \frac{k^2}{2} \| H_1 \|^2 + \frac{k^2}{2} \| H_2 \|^2 + \frac{1}{32}c_1[5k^2 + l^2 - 4m(1 - 3\cos^2\theta)] \\ &+ \frac{1}{32}c_2[5l^2 + k^2 - 4l(1 - 3\cos^2\theta)]. \end{aligned} \quad (4.5)$$

3. *If M is an invariant submanifold, then*

$$\tau \leq \frac{1}{32}c_1[5k^2 + 8k + l^2] + \frac{1}{32}c_2[5l^2 + 8l + k^2]. \quad (4.6)$$

4. If M is an anti-invariant submanifold, then

$$\begin{aligned} \tau \leq & \frac{k^2}{2} \|H_1\|^2 + \frac{k^2}{2} \|H_2\|^2 + \frac{1}{32}c_1[5k^2 - 4k + l^2] \\ & + \frac{1}{32}c_2[5l^2 - 4l + k^2]. \end{aligned} \quad (4.7)$$

5. Equality cases 1.,2.,3, and 4 hold if and only if M is totally geodesic.

Proof. Let M be an F -invariant submanifold of Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then M is also a product manifold $M = M_1 \times M_2$. Thus (1) follows from (4.2) and (4.3). If M is slant submanifold, then M_1 and M_2 are slant submanifolds of \bar{M}^m and \bar{M}^n . It is known that a slant submanifold of a Kaehler manifold satisfies the following

$$g(PX, PY) = \cos^2\theta g(X, Y).$$

Hence we have

$$\|P_1\|^2 = k\cos^2\theta, \quad \|P_2\|^2 = l\cos^2\theta. \quad (4.8)$$

Thus, using (4.8) in (4.4) we obtain (2). Putting $\theta = \frac{\pi}{2}$ in (4.5) we have (3). Since invariant M implies $H = 0$, from Lemma 4.1 we get (4). The last statement is clear from (4.3).□

Theorem 4.2. Let M be a $(k + l)$ -dimensional F -invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m(c_1) \times \bar{M}^n(c_2)$. Then the following statements are true:

(1) We have

$$\begin{aligned} S(X, X) \leq & \frac{1}{16}c_1[(5k - 4) \|X_1\|^2 + l \|X_2\|^2 + 12 \|PX_1\|^2] \\ & + \frac{1}{16}c_2[(5l - 4) \|X_1\|^2 + k \|X_1\|^2 + 12 \|PX_2\|^2] \\ & + kg(H_1, h_1(X_1, X_1)) + lg(H_2, h_2(X_2, X_2)), \end{aligned} \quad (4.9)$$

where S is the Ricci tensor, $X = X_1 + X_2$, $\|X_1\|^2 = \sum_{i=1}^k g(e_i, JX_1)^2$ and $\|X_1\|^2 = \sum_{\alpha=1}^l g(\tilde{e}_\alpha, JX_2)^2$.

(2) If M is θ - slant submanifold, then

$$\begin{aligned} S(X, X) &\leq \frac{1}{16}c_1[(5k - 4) \| X_1 \|^2 + l \| X_2 \|^2 + 12\cos^2\theta \| X_1 \|^2] \\ &+ \frac{1}{16}c_2[(5l - 4) \| X_1 \|^2 + k \| X_1 \|^2 + 12\cos^2\theta \| X_2 \|^2] \\ &+ kg(H_1, h_1(X_1, X_1)) + lg(H_2, h_2(X_2, X_2)). \end{aligned} \quad (4.10)$$

(3) If M is invariant, then

$$\begin{aligned} S(X, X) &\leq \frac{1}{16}c_1[(5k + 8) \| X_1 \|^2 + l \| X_2 \|^2] \\ &+ \frac{1}{16}c_2[(5l + 8) \| X_1 \|^2 + k \| X_1 \|^2]. \end{aligned} \quad (4.11)$$

(4) If M is anti-invariant

$$\begin{aligned} S(X, X) &\leq \frac{1}{16}c_1[(5k - 4) \| X_1 \|^2 + l \| X_2 \|^2] \\ &+ \frac{1}{16}c_2[(5l - 4) \| X_1 \|^2 + k \| X_1 \|^2] \\ &+ kg(H_1, h_1(X_1, X_1)) + lg(H_2, h_2(X_2, X_2)). \end{aligned} \quad (4.12)$$

Proof. Let M be an F -invariant submanifold of a Kaehler product manifold $\bar{M} = \bar{M}^m \times \bar{M}^n$. Then M is a product manifold $M = M_1 \times M_2$. We choose an orthonormal basis $\{e_1, \dots, e_k, \tilde{e}_1, \dots, \tilde{e}_l\}$ for T_pM such that $\{e_1, \dots, e_k\}$ is tangent to M_1 and $\{\tilde{e}_1, \dots, \tilde{e}_l\}$ is tangent to M_2 . Then, from (4.3) and (3.6), we obtain,

$$\begin{aligned} \sum_{i=1}^k g(R(e_i, X_1)Y_1, e_i) &= \frac{1}{16}c_1(5k - 4)g(X_1, Y_1) + \frac{1}{16}c_2kg(X_1, Y_1) \\ &+ \sum_{i=1}^k \{12g(e_i, JX_1)g(e_i, JY_1) \\ &- g(h_1(e_i, X_1), h_1(Y_1, e_i))\} \\ &+ kg(H_1, h_1(X_1, Y_1)) \end{aligned} \quad (4.13)$$

and

$$\begin{aligned}
 \sum_{\alpha=1}^l g(R(\tilde{e}_\alpha, X_2)Y_2, \tilde{e}_\alpha) &= \frac{1}{16}c_2(5l-4)g(X_2, Y_2) + \frac{1}{16}c_2lg(X_2, Y_2) \\
 &+ \sum_{\alpha=1}^l \{12g(\tilde{e}_\alpha, JX_2)g(\tilde{e}_\alpha, JY_2) \\
 &- g(h_2(\tilde{e}_\alpha, X_2), h_2(Y_2, \tilde{e}_\alpha))\} \\
 &+ lg(H_2, h_2(X_2, Y_2)).
 \end{aligned} \tag{4.14}$$

Thus, from (4.13) and (4.14), we have

$$\begin{aligned}
 S(X, X) &= \frac{1}{16}c_1[(5k-4) \| X_1 \|^2 + l \| X_2 \|^2] \\
 &+ \frac{1}{16}c_2[(5l-4) \| X_2 \|^2 + k \| X_1 \|^2] \\
 &+ \sum_{i=1}^k 12g(e_i, JX_1)^2 - g(h_1(e_i, X_1), h_1(Y_1, e_i)) \\
 &+ \sum_{\alpha=1}^l 12g(\tilde{e}_\alpha, JX_2)^2 - g(h_2(\tilde{e}_\alpha, X_2), h_2(Y_2, \tilde{e}_\alpha)) \\
 &+ kg(H_1, h_1(X_1, Y_1)) + lg(H_2, h_2(X_2, Y_2)).
 \end{aligned} \tag{4.15}$$

Then (1) follows from (4.15). The proof of the other assertions are similar to the assertions of Theorem 4.1. \square

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