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Abstract

In this paper, we explore the relations among reduced cases of algebraic models for homotopy 3-types for groups such as braided crossed and quadratic modules and reduced simplicial groups with Moore complex of length 2.

Key Words: Braided Crossed modules, Cat-groups, Simplicial groups, Quadratic Modules.

1. Introduction

Whitehead [19] obtained an algebraic description of homotopy type of any 3-dimensional complex, and he gave the notion of crossed modules which model homotopy 2-type. Mac Lane used them to describe the third cohomology of a group, moreover, Mac Lane and Whitehead, [14], gave a description of 3-type in terms of a crossed module.

Conduché [8] introduced the notion of 2-crossed module of groups model homotopy 3-type. Simplicial groups were studied by Kan [12]. Conduché also gave an equivalence between 2-crossed modules and simplicial groups with Moore complex of length 2. This equivalence establishes the role of 2-crossed modules as algebraic models of homotopy 3-types since the homotopy properties of a simplicial group are given by its Moore complex. It is known that since crossed modules model homotopy 2-type, the category of crossed modules is equivalent to the category of simplicial groups with Moore complex of length 1.

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Brown and Gilbert [6] defined the braided, regular crossed modules which model homotopy 3-types. They proved that this structure is equivalent to the simplicial groups with Moore complex of length 2. This equivalence ensured that the braided, regular crossed modules model homotopy 3-types. Furthermore, they showed that the category of braided, regular crossed modules is equivalent to that of 2-crossed modules. The reduced case of braided, regular crossed module of groupoids is called a braided crossed module of groups (cf. [6]).

Another algebraic model of homotopy 3-type is quadratic module of groups. This structure was introduced by Baues [3]. Baues defined a functor from simplicial groups to quadratic modules. In fact, a quadratic module is a 2-crossed module with additional *nilpotent* conditions. The reduced case of quadratic module is called a reduced quadratic module (cf. [3]).

This article intends to work on relations among reduced cases of algebraic models of homotopy 3-types such as braided crossed modules, reduced quadratic modules, reduced simplicial groups, and braided categorical groups.

2. Braided Crossed and Reduced Quadratic Modules

Crossed modules were given by Whitehead in [19]. A crossed module (C_2, C_1, ∂) is a group homomorphism $\partial : C_2 \to C_1$, together with an action of C_1 on C_2 written x^y for $y \in C_1$ and $x, x' \in C_2$, satisfying $\partial(x^y) = y^{-1}(\partial x)y$ and $x^{\partial x'} = x'^{-1}xx'$. The second condition is called a Peiffer identity. If ∂ satisfies only the first condition, then it is called a pre-crossed module. Clearly, a crossed module is a pre-crossed module. We denote such a crossed module by (C_2, C_1, ∂). A morphism of crossed modules from (C_2, C_1, ∂) to (C_2', C_1', ∂') is pair of group morphisms, $\varphi : C_2 \to C_2'$ and $\psi : C_1 \to C_1'$ such that $\varphi(x^y) = \psi(x)^{\varphi(y)}$ and $\partial'\varphi(x) = \psi\partial(x)$ for $x \in C_2$ and $y \in C_1$. Before giving the definition of reduced quadratic module, we should recall some basic structures from [3].

We denote the commutator in a group G by

$$[x, y] = x^{-1}y^{-1}xy$$

for $x, y \in G$ and we denote the Peiffer commutator in a pre-crossed module $\partial : C_2 \to C_1$ by

$$\langle x, y \rangle = x^{-1}y^{-1}xy^{\partial x}$$

for $x, y \in C_2$. Thus, a pre-crossed module $\partial : C_2 \to C_1$ is a crossed module if $\langle x, y \rangle = 1$ for all $x, y \in C_2$. Furthermore, in a group G, there exists a lower central series

$$\cdots \Gamma_{n+1} \subset \Gamma_n \subset \cdots \subset \Gamma_2 \subset \Gamma_1 = G$$

where $\Gamma_n = \Gamma_n(G)$ is the subgroup of G generated by all iterated commutators $[x_1, ..., x_n]$ of length n. Where $\Gamma_2(G)$ is the commutator subgroup of G. Similarly, there exists a lower Peiffer central series

$$\cdots P_{n+1} \subset P_n \subset \cdots \subset P_2 \subset C_2$$

in a pre-crossed module $\partial : C_2 \to C_1$. Where $P_n = P_n(\partial)$ is the subgroup of C_2 generated by all iterated Peiffer commutators $\langle x_1, ..., x_n \rangle$ of length n in C_2.

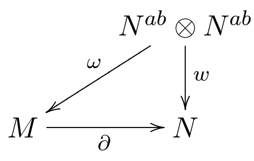
A group G is nilpotent of class 2 if $\Gamma_3(G) = 1$ and $\Gamma_2(G) \neq 1$, in this case we call G a *nil*(2)-group. A *nil(2)-module* is a pre-crossed module $\partial : C_2 \to C_1$ with additional "nilpotency" condition. This condition is $P_3(\partial) = 1$ where $P_3(\partial)$ is generated by Peiffer elements $\langle x_1, x_2, x_3 \rangle$ of length 3. Thus a *nil(2)-module* can be considered as generalizations of nil(2)-groups.

For any group G, the group $G^{ab} = G/\Gamma_2(G)$ is the abelianization of the group G. The crossed module

$$\partial^{cr} : C_2^{cr} = C_2/P_2(\partial) \to C_1$$

is called the *crossed module associated to pre-crossed module* $\partial : C_2 \to C_1$ (cf. [3]). Where $P_2(\partial) = \langle C_2, C_2 \rangle$ is the Peiffer subgroup of C_2. Baues gives the notion of ∂^{cr} to define the quadratic module structure in [3]. However, in definition of the reduced quadratic module, the notion of nil(2)-module corresponds to the nil(2)-group. Because, in a quadratic module, if its last component is trivial, the reduced quadratic module can be obtained.

Definition 2.1 ([3]) *A reduced quadratic module* (ω, ∂) *of groups is a diagram*



of homomorphism between groups such that the following axioms are satisfied:
1. *The group N is a $nil(2)$-group and the quotient map $N \to N^{ab}$ to the abelianization N^{ab} of N is denoted by $x \longmapsto \overline{x}$.*
2. *The composition $\partial\omega = w$ is the commutator map, or equivalently for $x, y \in N$*

$$\partial\omega(\overline{x} \otimes \overline{y}) = w(\overline{x} \otimes \overline{y}) = [x, y].$$

3. *For $a \in M$ and $x \in N$;*

$$1 = \omega\left((\overline{\partial a} \otimes \overline{x})(\overline{x} \otimes \overline{\partial a})\right).$$

4. *For $a, b \in M$,*

$$\omega(\overline{\partial a} \otimes \overline{\partial b}) = [a, b].$$

A map $(l, m) : (\omega, \partial) \to (\omega', \partial')$ between reduced quadratic modules is a pair of homomorphisms $l : M \to M'$, $m : N \to N'$ with $m\partial = \partial' l$ and $l\omega = \omega'$.

We denote the category of reduced quadratic modules of groups and of maps as above by **RQM**.

Braided regular crossed module and its reduced case called braided crossed module were given by Brown and Gilbert [6] as models for homotopy 3-types.

Definition 2.2 ([6]) *A braided crossed module of groups*

$$C_2 \xrightarrow{\partial} C_1$$

is a crossed module of groups together with a map $\{-,-\} : C_1 \times C_1 \to C_2$ called braiding map satisfying the following axioms:

$BC1$- $\{x, yy'\} = \{x, y\}^{y'}\{x, y'\}$
$BC2$- $\{xx', y\} = \{x', y\}\{x, y\}^{x'}$
$BC3$- $\partial\{x, y\} = [y, x]$
$BC4$- $\{x, \partial a\} = a^{-1}a^x$
$BC5$- $\{\partial b, y\} = (b^{-1})^y b$

for all $x, x'y, y' \in C_1$ and $a, b \in C_2$.

From $BC4$ and $BC5$, for $a, b \in C_2$, obviously

$$\begin{aligned} \{\partial b, \partial a\} &= a^{-1}a^{\partial b} \\ &= a^{-1}b^{-1}ab \quad (\because \partial \text{ is a cross. mod.}) \\ &= [a, b]. \end{aligned}$$

Thus, we can add an axiom to the axioms of braided crossed module for later use, as

$$BC6\text{-} \{\partial b, \partial a\} = [a, b]$$

for $a, b \in C_2$. A morphism of braided crossed modules is a morphism of crossed modules which is compatible with the braiding map. We denote the category of braided crossed modules by **BCM**. Now, we give the relation between braided crossed modules and reduced quadratic modules of groups:

Proposition 2.3 *There is a functor from the category of braided crossed modules to that of reduced quadratic modules of groups.*

Proof. Let

$$\partial : C_2 \to C_1$$

be a braided crossed module. We construct a reduced quadratic module from this structure. Let

$$N = C_1/\Gamma_3(C_1)$$

be a quotient group. Then N becomes a $nil(2)$-group since the triple commutators are trivial on itself. Let

$$q_1 : C_1 \to N$$

be a quotient map. Let $C = N^{ab}$ and let

$$\begin{array}{ccc} N & \twoheadrightarrow & C \\ q_1 x & \longmapsto & \overline{q_1 x} \end{array}$$

be a quotient map. Consider the subgroup P of C_2 generated by the elements of the form

$$\{[x, y], z\} \text{ and } \{x, [y, z]\}$$

for $x, y, z \in C_1$. Here, $\{-,-\}$ is the braiding map. Since the elements $[x, y]$ and $[y, z]$ are in $\Gamma_2(C_1)$ and $\{-,-\}$ is the braiding map, it can be shown that P is a normal subgroup of C_2. Now, consider the quotient group $M = C_2/P$ and quotient map $q_2 : C_2 \to M$. For all $x \in C_1$ and $[y, z] \in \Gamma_2(C_1)$ and $\{x, [y, z]\} \in P$, from $BC3$ we can write,

$\partial\{x,[y,z]\} = [x,[y,z]] \in \Gamma_3(C_1)$. Similarly, $[x,y] \in \Gamma_2(C_1)$ and $z \in C_1$ and $\{[x,y],z\} \in P$, we can write $\partial\{[x,y],z\} = [[x,y],z] \in \Gamma_3(C_1)$. Thus we obtain $\partial(P) \subseteq \Gamma_3(C_1)$. Then, we have a well defined homomorphism $\overline{\partial} : M \to N$ given by $\overline{\partial}(aP) = (\partial a)\Gamma_3(C_1)$ for $aP \in M$. Indeed, if $aP = bP$, we have $ab^{-1} \in P$ and then $\partial(ab^{-1}) \in \partial(P)$. Since $\partial(P) \subseteq \Gamma_3(C_1)$, we obtain $\partial(ab^{-1}) \in \Gamma_3(C_1)$ and since ∂ is a homomorphism we obtain $\partial a \partial b^{-1} \in \Gamma_3(C_1)$ and

$$(\partial a)\Gamma_3(C_1) = (\partial b)\Gamma_3(C_1).$$

Thus, we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\overline{\partial}} & N \\ {\scriptstyle q_2}\uparrow & & \uparrow{\scriptstyle q_1} \\ C_2 & \xrightarrow[\partial]{} & C_1 \end{array}$$

Let

$$\begin{array}{rccc} w: & C \otimes C & \longrightarrow & N \\ & \overline{q_1 x} \otimes \overline{q_1 y} & \longmapsto & [x,y] \end{array}$$

be commutator map. We can define the quadratic map using the braiding map

$$\omega: \quad C \otimes C \quad \longrightarrow \quad M$$

by $\omega\left(\overline{q_1 x} \otimes \overline{q_1 y}\right) = q_2\{y,x\}$. Here, $\{-,-\}$ is the braiding map. Therefore

$$\begin{array}{ccc} & & C \otimes C \\ & {\scriptstyle \omega}\swarrow & \downarrow{\scriptstyle w} \\ M & \xrightarrow[\overline{\partial}]{} & N \end{array}$$

becomes a reduced quadratic module. Now, we show that all axioms of reduced quadratic module are satisfied.

1. For elements $x\Gamma_3(C_1), y\Gamma_3(C_1), z\Gamma_3(C_1) \in C_1/\Gamma_3(C_1) = N$, since

$$\begin{aligned} [[x\Gamma_3(C_1), y\Gamma_3(C_1)], z\Gamma_3(C_1)] &= [[x,y],z]\Gamma_3(C_1) \\ &= \Gamma_3(C_1) \quad (\because [[x,y],z] \in \Gamma_3(C_1)) \end{aligned}$$

and

$$\begin{aligned}[x\Gamma_3(C_1),[y\Gamma_3(C_1),z\Gamma_3(C_1)]] &= [x,[y,z]]\Gamma_3(C_1) \\ &= \Gamma_3(C_1), \quad (\because [x,[y,z]] \in \Gamma_3(C_1)),\end{aligned}$$

where the group N is a $nil(2)$-group.

2. For $\overline{q_1 x}, \overline{q_1 y} \in C$, we obtain

$$\begin{aligned}\overline{\partial}\omega\left(\overline{q_1 x} \otimes \overline{q_1 y}\right) &= \overline{\partial} q_2\{y,x\} \\ &= q_1\partial\{y,x\} \\ &= q_1([x,y]) \quad (\text{by } BC3) \\ &= [q_1 x, q_1 y].\end{aligned}$$

3. For $q_2 a \in M$ and $q_1 x \in N$, we obtain

$$\begin{aligned}\omega\left([\overline{\partial} q_2 a] \otimes [q_1 x][q_1 x] \otimes [\overline{\partial} q_2 a]\right) &= q_2\left(\{x,\partial a\}\{\partial a, x\}\right) \\ &= q_2(1). \quad (\text{by } BC4 \text{ and } BC5).\end{aligned}$$

4. For $q_2 a, q_2 b \in M$, we obtain

$$\begin{aligned}\omega\left(\overline{\overline{\partial} q_2 a} \otimes \overline{\overline{\partial} q_2 b}\right) &= \omega(\overline{q_1 \partial a} \otimes \overline{q_1 \partial b}) \\ &= q_2\{\partial b, \partial a\} \\ &= q_2[a,b] \quad (\text{by } BC6) \\ &= [q_2 a, q_2 b].\end{aligned}$$

Thus all the axioms of reduced quadratic module are satisfied. We can define a functor from the category of braided crossed modules to that of reduced quadratic modules;

$$\Delta : \mathbf{BCM} \to \mathbf{RQM}.$$

□

3. Braided Cat-Groups, Crossed and Reduced 2-Crossed Modules

Cat-groups were given by Loday in [13]. In the following, **Cat**(**Gp**) will denote the category of internal categories in the category of groups. An object of **Cat**(**Gp**), called a cat-group, will be represented by a diagram of groups and group morphisms

$$A \underset{I}{\overset{s,t}{\rightrightarrows}} O$$

such that $sI = tI = id_O$, and the composition of two morphisms $x, y \in A$ with $t(x) = s(y)$ will be denoted $x \circ y$. The following definition can be found in the literature [4], [10], [11].

Definition 3.1 *A braiding for a cat-group*

$$G : A \underset{I}{\overset{s,t}{\rightleftarrows}} O$$

is a map

$$\begin{array}{ccc} O \times O & \stackrel{\tau}{\longrightarrow} & A \\ (a,b) & \longmapsto & \tau_{a,b} \end{array}$$

which satisfies the following conditions:

a) $s\tau_{a,b} = ba$ *and* $t\tau_{a,b} = ab$.

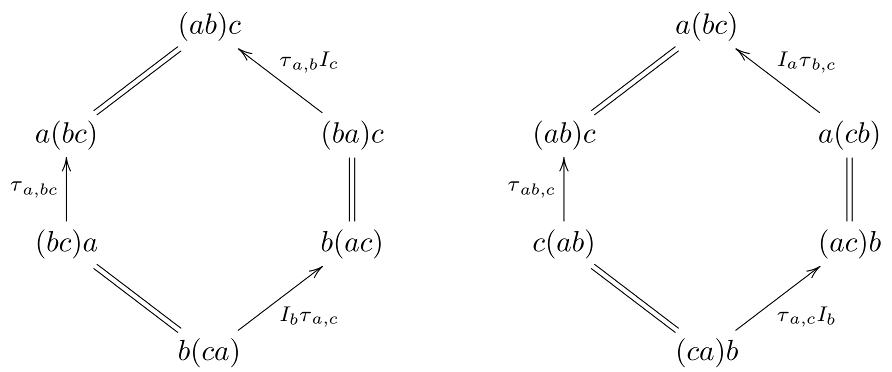
b) Naturality:

Given $x, y \in A$; $x : a \to a'$, $y : b \to b'$, *the following diagram is commutative.*

$$\begin{array}{ccc} ba & \xrightarrow{yx} & b'a' \\ {\scriptstyle \tau_{a,b}}\downarrow & & \downarrow{\scriptstyle \tau_{a',b'}} \\ ab & \xrightarrow[xy]{} & a'b' \end{array}$$

c) Hexagon axiom:

For $a, b, c \in O$ *the following diagrams are commutative.*



d) $\tau_{1,a} = \tau_{a,1} = I_a$.

A cat-group together with a braiding map is usually called a braided cat-group. Given braided cat-groups (G,τ), (G',τ'), a morphism between them is a morphism of cat-groups which is compatible with τ in the sense that the following square is commutative.

$$\begin{array}{ccc} O\times O & \xrightarrow{\tau} & A \\ {\scriptstyle f_0\times f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ O'\times O' & \xrightarrow[\tau']{} & A' \end{array}$$

BCat(**Gp**) will denote the category of braided cat-groups.

Now, we give the relation between braided crossed modules and braided cat-groups. It is well-known that crossed modules are equivalent to internal categories in the category of groups (cf. [10] and [13]). By using this equivalence, we give the following proposition to see the role of the notion of braiding map between these structures from Joyal and Street [11].

Proposition 3.2 *The category of braided crossed modules is equivalent to that of braided cat-groups.*

Proof. Let $\partial : C_2 \to C_1$ be a braided crossed module. Then, we know from [10] and [13] that

$$G : C_1 \rtimes C_2 \underset{I}{\overset{s,t}{\rightrightarrows}} C_1$$

together with $t(x,y)=x$, $s(x,y)=x(\partial y)$ and $I(x)=(x,0)$, is a cat-group. It is easy to see that the composition of two morphisms is

$$(x,y)\circ(x',y')=(x,yy')$$

if $x'=x(\partial y)$ for (x,y), $(x',y')\in C_1\rtimes C_2$. Let $C_1=O$ and $C_1\rtimes C_2=A$. The braiding map on this cat-group is given by

$$\begin{array}{rccc} \tau : & O\times O & \longrightarrow & A \\ & (a,b) & \longmapsto & (ba,\{b,a\}) \end{array}$$

for $a, b \in O$, where $\{-,-\}$ is the braiding map on the crossed module ∂. Then, (G, τ) becomes a braided cat-group. Indeed,

$$\begin{aligned} s\tau_{a,b} &= s(ba, \{b, a\}) \\ &= ba\delta\{b, a\} \\ &= baa^{-1}b^{-1}ab \quad (\text{by } BC3) \\ &= ab \end{aligned}$$

and

$$\begin{aligned} t\tau_{a,b} &= t(ba, \{b, a\}) \\ &= ba, \end{aligned}$$

and this is axiom (a) of braided cat-group. Other axioms can be shown similarly. This enables us to define a functor

$$\Theta : \mathbf{BCM} \longrightarrow \mathbf{BCat(Gp)}.$$

Conversely, let

$$G : A \underset{I}{\overset{s,t}{\rightrightarrows}} O$$

be a braided cat-group. Then $t : \ker s \to O$ is a crossed module associated to the cat-group G together with the action given by $l^x = (Ix)^{-1} l (Ix)$. The braiding map on this crossed module is given by

$$\begin{aligned} \{-,-\} : \quad O \times O &\longrightarrow \ker s \\ (a, b) &\longmapsto (Ib)^{-1}(Ia)^{-1}\tau_{a,b.} \end{aligned}$$

For example, the equalities
for $a, b \in O$

$$\begin{aligned} t\{a, b\} &= t((Ib)^{-1}(Ia)^{-1}\tau_{a,b}) \\ &= b^{-1}a^{-1}ba \\ &= [b, a], \end{aligned}$$

for $a \in O$, $y \in \ker s$

$$\begin{aligned}\{a, t(y)\} &= (Ity)^{-1}(Ia)^{-1}\tau_{a,ty} \\ &= y^{-1}I(a)^{-1}yI(a) \\ &= y^{-1}(y)^a,\end{aligned}$$

and for $x \in \ker s$ and $b \in O$,

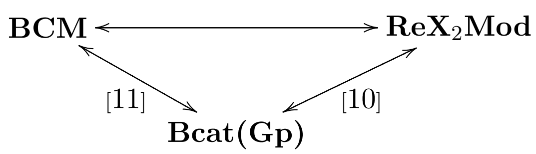
$$\begin{aligned}\{t(x), b\} &= (Ib)^{-1}(I(tx))^{-1}\tau_{tx,b} \\ &= (Ib)^{-1}x^{-1}I(b)x \\ &= (x^{-1})^b x\end{aligned}$$

are axioms $BC3$, $BC4$, and $BC5$, respectively. The other axioms can be shown similarly. Then, this crossed module becomes a braided crossed module. Thus we can define a functor

$$\Delta : \mathbf{BCat(Gp)} \longrightarrow \mathbf{BCM}.$$

□

Garzon and Miranda showed in [10] that the category of braided cat-groups is equivalent to $\mathbf{ReX_2Mod}$, the category of reduced 2-crossed modules given by Conduché in [8]. Also, we can easily say that the category of braided crossed modules is equivalent to that of reduced 2-crossed modules. Therefore, we can give the following diagram of equivalences of categories:



4. Simplicial Groups and Moore Complex

We refer the reader to May's book [15] and Mutlu and Porter's article [16] for the basic properties of simplicial groups.

Denoting the usual category of finite ordinals by Δ, we obtain for each $k \geq 0$, a subcategory $\Delta_{\leq k}$ determined by the objects $[j]$ of Δ with $j \leq k$. A simplicial group $\mathbf{G}$

consists of a family of groups G_n together with face and degeneracy maps $d_i^n : G_n \to G_{n-1}$, $0 \leq i \leq n$ $(n \neq 0)$ and $s_i^n : G_n \to G_{n+1}$, $0 \leq i \leq n$ satisfying the usual simplicial identities:

$$\begin{array}{llcll}
1. & d_i^{n-1}d_j^n & = & d_{j-1}^{n-1}d_i^n, & (0 \leq i < j \leq n), \\
2. & s_i^{n+1}s_j^n & = & s_{j+1}^{n+1}s_i^n, & (0 \leq i \leq j \leq n), \\
3. & d_i^{n+1}s_j^n & = & s_{j-1}^{n-1}d_i^n, & (0 \leq i < j \leq n), \\
4. & d_i^{n+1}s_j^n & = & id, & (i = j \text{ or } i = j+1), \\
5. & d_i^{n+1}s_j^n & = & s_j^{n-1}d_{i-1}^n & (0 \leq j < i-1 \leq n)
\end{array}$$

given by May [15]. In fact it can be completely described as a functor $\mathbf{G} : \Delta^{op} \to \mathbf{Grp}$, where Δ is the category of finite ordinals. A *reduced* simplicial group is a simplicial group whose last component is trivial. A k-truncated simplicial group is a functor from $\Delta^{op}_{\leq k}$ to $\mathbf{Grp}$. We will denote the category of simplicial groups by **SimpGrp** and the category of k-truncated simplicial groups by $\mathbf{Tr}_k\mathbf{SimpGrp}$. By a *k-truncation of a simplicial group,* we mean a k-truncated simplicial group $\mathbf{tr}_k\mathbf{G}$ obtained by forgetting dimensions of order $> k$ in a simplicial group **G.** This gives a truncation functor $\mathbf{tr}_k : \mathbf{SimpGrp} \to \mathbf{Tr}_k\mathbf{SimpGrp}$ which admits a right adjoint $\mathbf{cosk}_k : \mathbf{Tr}_k\mathbf{SimpGrp} \to \mathbf{SimpGrp}$ called the *k-coskeleton functor,* and a left adjoint $\mathbf{sk}_k : \mathbf{Tr}_k\mathbf{SimpGrp} \to \mathbf{SimpGrp}$, called the *k-skeleton functor.* For the explicit constructions of these see [9].

Recall that given a simplicial group **G,** *the Moore complex* $(\mathbf{NG},\partial)$ of **G** is the normal chain complex defined by

$$NG_n = \bigcap_{i=0}^{n-1} \ker d_i^n$$

with $\partial_n : NG_n \to NG_{n-1}$ induced from the face map d_n^n by restriction. The nth *homotopy group* $\pi_n(\mathbf{G})$ of **G** is the nth homology of the Moore complex of **G**, i.e.

$$\begin{aligned}
\pi_n(\mathbf{G}) &\cong H_n(\mathbf{NG},\partial) \\
&= \bigcap_{i=0}^{n} \ker d_i^n / d_{n+1}^{n+1}\Big(\bigcap_{i=0}^{n} \ker d_i^{n+1}\Big).
\end{aligned}$$

We say that the Moore complex **NG** of a simplicial group is of *length k* if $NG_n = 1$ for all $n \geq k+1$, so that a Moore complex of length k is also of length l for $l > k$.

Corollary 4.1 ([8]) *Let* $\mathbf{G}'$ *be* $(n\text{-}1)$*-truncated simplicial group. Then there is a simplicial group* $\mathbf{G}$ *with* $\mathbf{tr}_k\mathbf{G} \cong \mathbf{G}'$ *if and only if* $\mathbf{G}'$ *satisfies the following property:*

For all nonempty sets of indices $(I \neq J), I, J \subset [n-1]$ *with* $I \cup J = [n-1]$,

$$[\ \bigcap_{i\in I} kerd_i, \bigcap_{j\in J} kerd_j\ ] = 1.$$

This normal subgroup N_n^G depends functorially on G, but we will usually abbreviate N_n^G to N_n, when no change of group is involved.

4.1. Braided Cat-groups and Reduced Simplicial Groups

In this section, we give an equivalence between the category of braided cat-groups and the category of reduced simplicial groups with Moore complex of length 2. This result is a combination of the equivalence between braided 2-groups and braided crossed modules (cf. [11]) and the equivalence between reduced 2-crossed modules and reduced simplicial groups with Moore complex of length 2 (cf. [8]).

Firstly, we give a functor from the category of reduced simplicial groups to that of braided cat-groups.

Let $\mathbf{G}$ be reduced simplicial group with Moore complex $\mathbf{NG}$. We construct a braided cat-group

$$C : A \underset{I}{\overset{s,t}{\rightleftarrows}} O\ .$$

Let $O = NG_1$. By using the action of NG_1 on NG_2 via s_1, define the semi-direct product group $A = NG_1 \rtimes NG_2/\partial_3(NG_3)$. The source and target maps are given by $s(x, \overline{a}) = x$ and $t(x, \overline{a}) = x\partial_2 a$ respectively. The composition can be defined by

$$(x, \overline{a}) \circ (y, \overline{b}) = (x, \overline{ab})$$

for $y = x\partial_2 a$. The identity map $I : O \to A$ is given by $I(x) = (x, \overline{1})$. Where $\overline{a}$ represents a coset of element a of NG_2 in $NG_2/\partial_3(NG_3)$. The group operation in $NG_1 \rtimes NG_2/\partial_3(NG_3)$ is given by

$$(x, \overline{a})(y, \overline{b}) = (xy, (\overline{s_1 y a s_1 y^{-1}})\overline{b})$$

for $x, y \in NG_1$ and $a, b \in NG_2$. Then, the interchange law holds. That is, we have a cat-group

$$C : A \underset{I}{\overset{s,t}{\rightrightarrows}} O \ .$$

Define the braiding map on this cat-group by

$$\begin{array}{rccl} \tau : & O \times O & \longrightarrow & A \\ & (x, y) & \longmapsto & \tau_{x,y} = (yx, \overline{s_0y^{-1}s_1x^{-1}s_0ys_1y^{-1}s_1xs_1y}) \end{array}$$

for $x, y \in O$. Now, we show that some axioms of braided cat-groups are satisfied.

a)

$$\begin{array}{rcl} s\tau_{x,y} & = & s(yx, \overline{s_0y^{-1}s_1x^{-1}s_0ys_1y^{-1}s_1xs_1y}) \\ & = & yx \end{array}$$

and

$$\begin{array}{rcl} t\tau_{x,y} & = & t(yx, \overline{s_0y^{-1}s_1x^{-1}s_0ys_1y^{-1}s_1xs_1y}) \\ & = & yxd_2(s_0y^{-1}s_1x^{-1}s_0ys_1y^{-1}s_1xs_1y) \\ & = & yx(s_0d_1y^{-1}x^{-1}s_0d_1yy^{-1}xy) \\ & = & yx(x^{-1})^{d_1y}y^{-1}xy \quad \text{(by action)} \\ & = & yxx^{-1}y^{-1}xy \quad \text{(by reduced condition)} \\ & = & xy. \end{array}$$

b) for $x = (a, \overline{k})$ and $y = (b, \overline{l})$, $s(x) = a$, $t(x) = ad_2k = a'$ and $s(y) = b$, $t(y) = bd_2l = b'$, we must show that

$$\tau_{a,b} \circ xy = yx \circ \tau_{a',b'}.$$

$xy = (a, \overline{k})(b, \overline{l}) = (ab, \overline{(s_1b)^{-1}ks_1bl})$ and $\tau_{a,b} = (ba, \overline{s_0b^{-1}s_1a^{-1}s_0bs_1b^{-1}s_1as_1b})$ and then since $t\tau_{a,b} = ab = s(xy)$, we have

$$\begin{array}{rcl} xy \circ \tau_{a,b} & = & (ba, \overline{s_0b^{-1}s_1a^{-1}s_0bs_1b^{-1}s_1as_1b}) \circ (ab, \overline{(s_1b)^{-1}ks_1bl}) \\ & = & (ba, \overline{s_0b^{-1}s_1a^{-1}s_0bs_1b^{-1}s_1aks_1bl}) \end{array}$$

and we have $s(xy \circ \tau_{a,b}) = ba$ and

$$\begin{aligned} t(xy \circ \tau_{a,b}) &= bas_0d_1b^{-1}a^{-1}s_0d_1bb^{-1}ad_2kbd_2l \\ &= ba(a^{-1})^{d_1b}b^{-1}ad_2kbd_2l \quad \text{(by action)} \\ &= baa^{-1}b^{-1}ad_2kbd_2l \quad \text{(by reduced condition)} \\ &= ad_2kbd_2l \\ &= a'b'. \end{aligned}$$

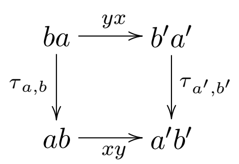
Furthermore, $yx = (b, \overline{l})(a, \overline{k}) = (ba, \overline{(s_1a)^{-1}ls_1ak})$
and $\tau_{a',b'} = (b'a', \overline{(s_0b')^{-1}(s_1a')^{-1}s_0b'(s_1b')^{-1}s_1a's_1b'})$ and

$$\begin{aligned} yx \circ \tau_{a',b'} &= (ba, \overline{(s_1a)^{-1}ls_1ak}) \circ (b'a', \overline{(s_0b')^{-1}(s_1a')^{-1}s_0b'(s_1b')^{-1}s_1a's_1b'}) \\ &= (ba, \overline{(s_1a)^{-1}ls_1ak(s_0b')^{-1}(s_1a')^{-1}s_0b'(s_1b')^{-1}s_1a's_1b'}) \\ &= (ba, s_1a^{-1}ls_1aks_0(b')^{-1}s_1(ad_2k)^{-1}s_0(b')s_1(bd_2l)^{-1}s_1(ad_2k)s_1(bd_2l)) \\ &= (ba, s_1a^{-1}ls_1ak(s_1(ad_2k)^{-1})^{d_1(b')}s_1(bd_2l)^{-1}s_1(ad_2k)s_1(bd_2l)) \quad \text{(by action)} \\ &= (ba, s_1a^{-1}ls_1ak(s_1(ad_2k)^{-1})s_1(bd_2l)^{-1}s_1(ad_2k)s_1(bd_2l)) \\ &\quad \text{(by reduced condition)} \end{aligned}$$

and we have $s(yx \circ \tau_{a',b'}) = ba$ and

$$\begin{aligned} t(yx \circ \tau_{a',b'}) &= baa^{-1}d_2lad_2k(ad_2k)^{-1}(bd_2l)^{-1}(ad_2k)(bd_2l) \\ &= (bd_2l)(ad_2k)(ad_2k)^{-1}(bd_2l)^{-1}(ad_2k)(bd_2l) \\ &= (ad_2k)(bd_2l) \\ &= a'b'. \end{aligned}$$

Thus, the diagram



is commutative.

The axiom $c)$ takes more work and it can be completely showed similarly to axiom $b)$. Then, we leave the detailed calculations to the reader.

d) We must show that $\tau_{1,a} = \tau_{a,1} = I_a$. Where $I_a = (a,1)$. We have

$$\begin{aligned} \tau_{1,a} &= (a, \overline{s_0 a^{-1} s_1 1^{-1} s_0 a s_1 a^{-1} s_1 1 s_1 a}) \\ &= (a, s_0 a^{-1} s_0 a s_1 a^{-1} s_1 a) \\ &= (a,1) \\ &= I_a \end{aligned}$$

and

$$\begin{aligned} \tau_{a,1} &= (a, \overline{s_0 1^{-1} s_1 a^{-1} s_0 1 s_1 1^{-1} s_1 a s_1 1}) \\ &= (a, s_1 a^{-1} s_1 a) \\ &= (a,1) \\ &= I_a. \end{aligned}$$

Thus we can define a functor from reduced simplicial groups to braided cat-groups;

$$\Gamma : \mathbf{ReSimpGrp} \longrightarrow \mathbf{BCat(Gp)}.$$

Theorem 4.2 *The category of reduced simplicial groups with Moore complex of length 2 is equivalent to that of braided cat-groups.*

Proof. In the above statements, we have already defined a functor from the category of reduced simplicial groups to that of braided cat-groups. Therefore we can define a functor from the category of reduced simplicial groups with Moore complex of length 2 to that of braided cat-groups;

$$\Gamma : \mathbf{ReSimpGrp}_{\leq 2} \longrightarrow \mathbf{BCat(Gp)}.$$

Conversely, let

$$C : A \underset{I}{\overset{s,t}{\rightrightarrows\!\!\leftarrow}} O$$

be a braided cat-group. We construct a reduced simplicial group. Let $e \in O$ be identity element. Suppose that $G_0 = \{e\}$ and $G_1 = O$. Then we have a 1-truncated simplicial group with trivial homomorphisms $\{G_1, G_0\}$. The group O acts on $\ker s$ by I. That is, for $x \in O$ and $a \in \ker s$, $a^x = I(x)^{-1} a I(x) \in \ker s$. Indeed, $s(a^x) = s(I(x)^{-1} a I(x)) = x^{-1} 1 x = 1$. By using this action, we can create the semi-direct product group

$$O \rtimes \ker s$$

with the group operation

$$(x,a)(x',a') = (xx', I(x')^{-1}aI(x)a').$$

On the other hand, the group O acts on $O \rtimes \ker s$ by

$$(x,a)^{x'} = (xx', I(x')^{-1}aI(x))$$

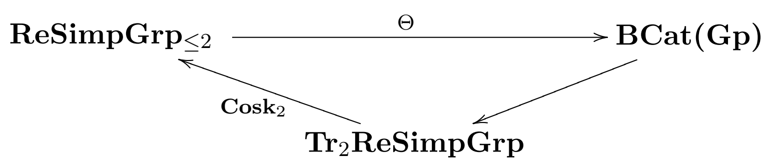
for $(x,a) \in O \rtimes \ker s$ and $x' \in O$. By using this action, we can create the semi-direct product group $O \rtimes (O \rtimes \ker s)$. Let $G_2 = O \rtimes (O \rtimes \ker s)$. We have

$$\begin{array}{ll} d_0^2(c_1,c_2,a) = c_1 & d_1^2(c_1,c_2,a) = c_1c_2 \\ d_2^2(c_1,c_2,a) = c_2 & s_0^1(c_1) = (c_1,1,1), \quad s_1^1(c_2) = (1,c_2,1). \end{array}$$

These maps satisfy the simplicial identities. We thus have a reduced 2-truncated simplicial group

$$\{G_2, G_1, G_0\}.$$

There is a $\mathbf{Cosk}_2$ functor from the category of 2- truncated simplicial groups to that of simplicial groups. We can write the following diagram;

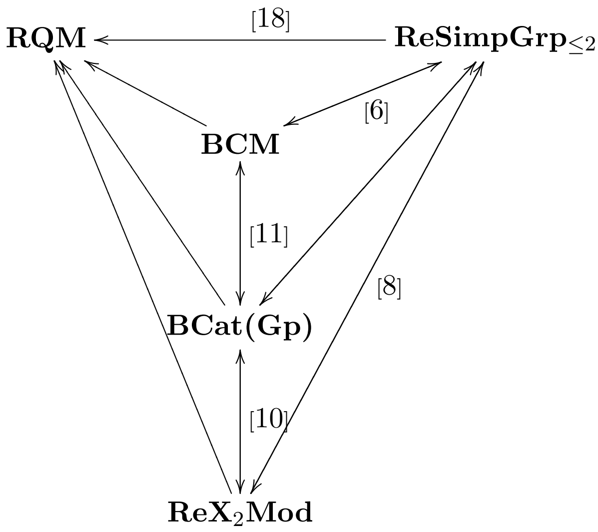


and this enables us to define a functor

$$\Delta : \mathbf{BCat(Gp)} \longrightarrow \mathbf{ReSimpGrp}_{\leq 2}.$$

□

Thus, we can picture the following diagram of equivalences of categories;



The numbers in this diagram correspond the references.

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