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Extreme Points of Certain Subsets of Hermitian Elements in Banach Algebras

Gerd Herzog and Christoph Schmoeger

Abstract

We consider the real Banach spaces $\mathcal{H}(\mathcal{A})$ of all hermitian elements of a complex Banach algebra \mathcal{A} . We prove that if an even power of $a \in \mathcal{H}(\mathcal{A})$ is hermitian, then a is an extreme point of the unit ball of $\mathcal{H}(\mathcal{A})$ if and only if $a^2 = \mathbf{1}$. Moreover, if an odd power of $a \in \mathcal{H}(\mathcal{A})$ is hermitian and a is an extreme point of the unit ball of $\mathcal{H}(\mathcal{A})$, then $a^3 = a$.

Key words and phrases: Extreme points, hermitian elements.

1. Terminology and Introduction

Let K be a subset of a (real or complex) vector space X . A point $x \in K$ is called an *extreme point* of K if $y, z \in K$, $0 < \mu < 1$ and

$$x = \mu y + (1 - \mu)z$$

imply that $x = y = z$. We denote the set of extreme points of K by $ext(K)$.

Throughout this paper \mathcal{A} denotes a complex unital Banach algebra with unit $\mathbf{1}$. If \mathcal{B} is a subset of \mathcal{A} we write $(\mathcal{B})_1$ for the set

$$(\mathcal{B})_1 = \{a \in \mathcal{B} : \|a\| \leq 1\}.$$

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For $a \in \mathcal{A}$ we denote by $\sigma(a)$ the spectrum of a and by $r(a)$ the spectral radius of a . Let \mathcal{A}' denote the topological dual space of \mathcal{A} . Given $a \in \mathcal{A}$ the set

$$V(a) = \{\varphi(a) : \varphi \in \mathcal{A}', \varphi(\mathbf{1}) = 1 = \|\varphi\|\}$$

is called the *numerical range* of a .

An element $a \in \mathcal{A}$ is called *hermitian* if $V(a) \subseteq \mathbb{R}$. The set of all hermitian elements in \mathcal{A} is denoted by $\mathcal{H}(\mathcal{A})$. Observe that if \mathcal{A} is a B^* -algebra, then $a \in \mathcal{H}(\mathcal{A})$ if and only if $a^* = a$ (see [1, page 47]).

Proposition 1.1

- (1) If $a \in \mathcal{A}$, then $\sigma(a) \subseteq V(a)$;
- (2) If $a \in \mathcal{H}(\mathcal{A})$, then $\sigma(a) \subseteq \mathbb{R}$;
- (3) $\mathcal{H}(\mathcal{A})$ is a real Banach space;
- (4) If $a \in \mathcal{H}(\mathcal{A})$, then $r(a) = \|a\|$; and
- (5) $a \in \mathcal{H}(\mathcal{A}) \iff \|\exp(ita)\| = 1$ for all $t \in \mathbb{R}$.

Proof. (1) [1, Theorem 2.6]. (2) follows from (1). (3) [1, Lemma 5.4]. (4) [5]. (5) [1, Lemma 5.2]. □

The following result is well-known. For the convenience of the reader we include a proof.

Proposition 1.2 Let $a \in \mathcal{A}$ and $\mathcal{M} \subseteq \mathcal{A}$.

- (1) Suppose that $a \in (\mathcal{A})_1$, is invertible and that $a^{-1} \in (\mathcal{A})_1$. Then $a \in \text{ext}((\mathcal{A})_1)$.
- (2) If $a \in (\mathcal{M})_1$ is invertible and $a^{-1} \in (\mathcal{A})_1$, then $a \in \text{ext}((\mathcal{M})_1)$.

Proof. (1) Let $a = \mu b + (1 - \mu)c$ with $b, c \in (\mathcal{A})_1$ and $0 < \mu < 1$. Then

$$\mathbf{1} = \mu b a^{-1} + (1 - \mu) c a^{-1}.$$

For $\varphi \in \mathcal{A}'$ with $\varphi(\mathbf{1}) = 1 = \|\varphi\|$ it follows that

$$1 = \mu \varphi(b a^{-1}) + (1 - \mu) \varphi(c a^{-1}).$$

Since $b, c, a^{-1} \in (\mathcal{A})_1$ we have $|\varphi(ba^{-1})| \leq 1$ and $|\varphi(ca^{-1})| \leq 1$. Since 1 is an extreme point of the closed unit disc in \mathbb{C} , we get $\varphi(ba^{-1}) = \varphi(ca^{-1}) = 1$. This shows that

$$V(ba^{-1}) = V(ca^{-1}) = \{1\}.$$

Hence $ba^{-1}, ca^{-1} \in \mathcal{H}(\mathcal{A})$. Proposition 1.1 (1) gives $\sigma(ba^{-1}) = \sigma(ca^{-1}) = \{1\}$. By Proposition 1.1 (3), $ba^{-1} - \mathbf{1}, ca^{-1} - \mathbf{1} \in \mathcal{H}(\mathcal{A})$. From $\sigma(ba^{-1} - \mathbf{1}) = \sigma(ca^{-1} - \mathbf{1}) = \{0\}$ we see (Proposition 1.1 (4)) that

$$\|ba^{-1} - \mathbf{1}\| = \|ca^{-1} - \mathbf{1}\| = 0,$$

thus $a = b = c$.

(2) If $a = \mu b + (1 - \mu)c$ with $b, c \in (\mathcal{M})_1$ and $0 < \mu < 1$, then $b, c \in (\mathcal{A})_1$. It follows from (1) that $a = b = c$, thus $a \in \text{ext}((\mathcal{M})_1)$. \square

Proposition 1.3 *Let $a \in (\mathcal{A})_1$ and suppose that $\{-1, 1\} \subseteq \sigma(a)$. Let*

$$\langle \mathbf{1}, a \rangle_{\mathbb{R}} = \{\alpha \mathbf{1} + \beta a : \alpha, \beta \in \mathbb{R}\} \quad \text{and} \quad \langle \mathbf{1}, a \rangle_{\mathbb{C}} = \{\alpha \mathbf{1} + \beta a : \alpha, \beta \in \mathbb{C}\}.$$

Then $a \in \text{ext}((\langle \mathbf{1}, a \rangle_{\mathbb{R}})_1)$ and $a \in \text{ext}((\langle \mathbf{1}, a \rangle_{\mathbb{C}})_1)$.

Proof. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. First we show that $\mathbf{1}$ and a are linearly independent. To this end assume that $\alpha, \beta \in \mathbb{K}$ and $0 = \alpha \mathbf{1} + \beta a$. Since $1, -1 \in \sigma(a)$, the spectral mapping theorem gives $\alpha + \beta = 0 = \alpha - \beta$. Thus $\alpha = \beta = 0$.

Now let $a = \mu b + (1 - \mu)c$ with $b, c \in (\langle \mathbf{1}, a \rangle_{\mathbb{K}})_1$ and $0 < \mu < 1$. There are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{K}$ such that $b = \alpha_1 \mathbf{1} + \alpha_2 a$ and $c = \beta_1 \mathbf{1} + \beta_2 a$, hence

$$a = (\mu \alpha_1 + (1 - \mu) \beta_1) \mathbf{1} + (\mu \alpha_2 + (1 - \mu) \beta_2) a.$$

Since $\mathbf{1}$ and a are linearly independent,

$$\mu \alpha_1 + (1 - \mu) \beta_1 = 0$$

and

$$\mu \alpha_2 + (1 - \mu) \beta_2 = 1.$$

It follows that

$$\mu(\alpha_2 + \alpha_1) + (1 - \mu)(\beta_2 + \beta_1) = 1 \tag{1.1}$$

and

$$\mu(\alpha_2 - \alpha_1) + (1 - \mu)(\beta_2 - \beta_1) = 1. \quad (1.2)$$

From $-1, 1 \in \sigma(a)$ we get $\alpha_1 + \alpha_2, \alpha_1 - \alpha_2 \in \sigma(b)$. Since $\|b\| \leq 1$, $|\alpha_1 + \alpha_2|, |\alpha_1 - \alpha_2| \leq 1$. Similarly, $|\beta_1 + \beta_2|, |\beta_1 - \beta_2| \leq 1$. From (1.1) and (1.2) we now see that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1 = \alpha_2 - \alpha_1 = \beta_2 - \beta_1$, hence $\alpha_2 = \beta_2 = 1$ and $\alpha_1 = \beta_1 = 0$, thus $a = b = c$. \square

2. Extreme points of subsets of $\mathcal{H}(\mathcal{A})$

We first give an example, due to M. J. Crabb, which shows the existence of a hermitian element of which the square is not hermitian.

Example 2.1 ([1, page 57]).

Let $\mathcal{A} = \mathbb{C}^3$ with pointwise multiplications, and let $p : \mathbb{C}^3 \rightarrow [0, \infty)$ be defined by

$$p(\alpha, \beta, \gamma) = \sup\{|\lambda^{-1}\alpha + \beta + \lambda\gamma| : \lambda \in \mathbb{C}, |\lambda| = 1\}.$$

Define the norm $\|\cdot\|$ on \mathcal{A} by

$$\|a\| = \sup\{p(xa) : x \in \mathcal{A}, p(x) = 1\}.$$

Then \mathcal{A} is a complex unital (commutative) Banach algebra with respect to $\|\cdot\|$.

Let $a = (-1, 0, 1)$. The following properties are shown in [1]:

- (1) $a \in \mathcal{H}(\mathcal{A})$, $a^2 \notin \mathcal{H}(\mathcal{A})$, $\sigma(a) = \{-1, 0, 1\}$;
- (2) $\mathcal{A} = \{\alpha\mathbf{1} + \beta a + \gamma a^2 : \alpha, \beta, \gamma \in \mathbb{C}\}$;
- (3) $\mathcal{H}(\mathcal{A}) = \{\alpha\mathbf{1} + \beta a : \alpha, \beta \in \mathbb{R}\}$.

We have $a^3 = a$ and, by Proposition 1.3 and (3), $a \in \text{ext}((\mathcal{H}(\mathcal{A}))_1)$. This is not an accident, as we see in what follows.

Before we state the main results of this paper, we need the following lemma.

Lemma 2.2 *Let the functions $f_1, f_2, f_3, f_4 : [-1, 1] \rightarrow \mathbb{R}$ be defined by*

$$f_1(t) = t + \frac{1}{2k}(1 - t^{2k}), \quad f_2(t) = t - \frac{1}{2k}(1 - t^{2k}),$$

$$f_3(t) = t + \frac{1}{2k+1}(t - t^{2k+1}) \quad \text{and} \quad f_4(t) = t - \frac{1}{2k+1}(t - t^{2k+1}),$$

where $k \in \mathbb{N}$. Then

$$f_j([-1, 1]) = [-1, 1] \quad (j = 1, \dots, 4).$$

Proof. Routine. □

Theorem 2.3 *Suppose that $a \in (\mathcal{H}(\mathcal{A}))_1$, and that $a^{2k} \in \mathcal{H}(\mathcal{A})$ for some $k \in \mathbb{N}$. Then the following assertions are equivalent:*

(1) $a \in \text{ext}((\mathcal{H}(\mathcal{A}))_1)$.

(2) $a^2 = \mathbf{1}$.

Proof. (1) \Rightarrow (2): Let $h = \frac{1}{2k}(\mathbf{1} - a^{2k})$. Then $h, a + h, a - h \in \mathcal{H}(\mathcal{A})$. Let $\lambda \in \sigma(a + h)$. By the spectral mapping theorem, $\lambda = \alpha + \frac{1}{2k}(1 - \alpha^{2k})$ for some $\alpha \in \sigma(a)$. Since $\sigma(a) \subseteq [-1, 1]$, $\lambda = f_1(\alpha)$ with $\alpha \in [-1, 1]$, where f_1 is as in Lemma 2.2. Hence $\lambda \in [-1, 1]$. This shows that $\sigma(a + h) \subseteq [-1, 1]$. Hence, by Proposition 1.1 (4), $\|a + h\| = r(a + h) \leq 1$. Thus $a + h \in (\mathcal{H}(\mathcal{A}))_1$. A similar argument (use the function f_2 in Lemma 2.2) shows that $a - h \in (\mathcal{H}(\mathcal{A}))_1$. We have

$$a = \frac{1}{2}(a + h) + \frac{1}{2}(a - h).$$

Since a is an extreme point of $(\mathcal{H}(\mathcal{A}))_1$, $a = a + h = a - h$, hence $h = 0$. Therefore $a^{2k} = \mathbf{1}$. If the entire function g is defined by $g(z) = z^{2k} - 1$, then g has only simple zeros and $g(a) = 0$. It follows from [2, Proposition 8.11] that

$$\sigma(a) = \{\lambda_1, \dots, \lambda_r\}, \quad g(\lambda_j) = 0 \quad (j = 1, \dots, r)$$

and

$$(a - \lambda_1 \mathbf{1})(a - \lambda_2 \mathbf{1}) \cdots (a - \lambda_r \mathbf{1}) = 0.$$

Since $|\lambda_j| = 1$ ($j = 1, \dots, r$) and $\sigma(a) \subseteq [-1, 1]$, it follows that $\sigma(a) \subseteq \{-1, 1\}$, hence $a^2 = \mathbf{1}$.

(2) \Rightarrow (1): Since $a \in \mathcal{H}(\mathcal{A})$ and $a = a^{-1}$, $1 = r(a^2) = r(a)^2 = \|a\|^2$, hence $a \in (\mathcal{H}(\mathcal{A}))_1$. With $\mathcal{M} = \mathcal{H}(\mathcal{A})$ we see that $a \in \text{ext}((\mathcal{M})_1)$, by Proposition 1.2 (2). \square

Theorem 2.4 *If $a \in (\mathcal{H}(\mathcal{A}))_1$ is an extreme point of $(\mathcal{H}(\mathcal{A}))_1$ and if $a^{2k+1} \in \mathcal{H}(\mathcal{A})$ for some $k \in \mathbb{N}$, then $a^3 = a$.*

Proof. Let $h = \frac{1}{2^{k+1}}(a - a^{2k+1})$. Then $h, a + h, a - h \in \mathcal{H}(\mathcal{A})$. As in the proof of Theorem 2.3 we see that $\sigma(a \pm h) \subseteq [-1, 1]$ (use the functions f_3 and f_4 of Lemma 2.2). Thus $\|a \pm h\| = r(a \pm h) \leq 1$, therefore $a \pm h \in (\mathcal{H}(\mathcal{A}))_1$. Since a is an extreme point of $(\mathcal{H}(\mathcal{A}))_1$ and since

$$a = \frac{1}{2}(a + h) + \frac{1}{2}(a - h),$$

it follows that $a = a + h = a - h$, and so $h = 0$. If the entire function g is defined by $g(z) = z - z^{2k+1}$, then g has only simple zeros and $g(a) = 0$. As in the proof of Theorem 2.3 we derive $\sigma(a) \subseteq \{0, 1, -1\}$ and $a^3 = a$. \square

Corollary 2.5 *If $a \in (\mathcal{H}(\mathcal{A}))_1$ is an extreme point of $(\mathcal{H}(\mathcal{A}))_1$ and if $a^n \in \mathcal{H}(\mathcal{A})$ for some $n \in \mathbb{N}$, $n \geq 2$, then $a^3 = a$.*

We say that $a \in \mathcal{H}(\mathcal{A})$ is *positive* if $\sigma(a) \subseteq [0, \infty)$. We denote by $\text{pos}(\mathcal{A})$ the set of all positive elements of \mathcal{A} .

Corollary 2.6 *If $p \in (\text{pos}(\mathcal{A}))_1$ and $p^2 \in \mathcal{H}(\mathcal{A})$, then the following assertions are equivalent:*

(1) $p \in \text{ext}((\text{pos}(\mathcal{A}))_1)$.

(2) $p^2 = p$.

Proof. Since

$$p \in (\text{pos}(\mathcal{A}))_1 \iff 2 \left(p - \frac{1}{2} \mathbf{1} \right) \in (\mathcal{H}(\mathcal{A}))_1$$

and

$$p^2 \in \mathcal{H}(\mathcal{A}) \iff \left(2\left(p - \frac{1}{2}\mathbf{1}\right)\right)^2 \in \mathcal{H}(\mathcal{A}),$$

it follows from Theorem 2.3 that p is an extreme point of $(\text{pos}(\mathcal{A}))_1$ if and only if $(2(p - \frac{1}{2}\mathbf{1}))^2 = \mathbf{1}$. \square

Now let H denote a complex Hilbert space and consider the B^* -algebra $\mathcal{A} = \mathcal{B}(H)$, the Banach algebra of all bounded linear operators on H . Then

$$\mathcal{H}(\mathcal{A}) = \{A \in \mathcal{B}(H) : A \text{ is selfadjoint}\}$$

and

$$\text{pos}(\mathcal{A}) = \{A \in \mathcal{H}(\mathcal{A}) : A \geq 0\},$$

thus $(\text{pos}(\mathcal{A}))_1 = \{A \in \mathcal{H}(\mathcal{A}) : 0 \leq A \leq I\}$, where I denotes the identity operator on H . Observe that if $A \in \mathcal{H}(\mathcal{A})$, then $A^n \in \mathcal{H}(\mathcal{A})$ for all $n \in \mathbb{N}$.

As an immediate consequence of Theorem 2.3 and Corollary 2.6, we get the following well-know results (see [4, 2.5.6]):

Corollary 2.7 *Let H and \mathcal{A} be as above.*

- (1) $A \in (\mathcal{H}(\mathcal{A}))_1$ is an extreme point of $(\mathcal{H}(\mathcal{A}))_1$, if and only if $A^2 = I$.
- (2) $P \in (\text{pos}(\mathcal{A}))_1$ is an extreme point of $(\text{pos}(\mathcal{A}))_1$ if and only if $P^2 = P$.

For a characterisation of the extreme points of the unit ball of a general B^* -algebra see [3, Theorem 9.5.16].

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