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Abstract

Let R be a commutative ring with a unit and M an R-module. In this paper we give a comparison between the $\mathcal{F}$-closure in M of an R-submodule having a minimal extension and the closure of this minimal extension for the same Gabriel topology defined on the ring R. If $J(R) \in \mathcal{F}$ we prove that both closures are the same. Moreover, if R is Artinian or semi-simple then the converse also holds.

Key Words: Jacobson radical and closure of minimal extensions.

1. Introduction and Preliminaries

Throughout this paper $\mathcal{F}$ denotes a non-trivial Gabriel topology on a commutative ring R with unit, and $J(R)$ its Jacobson radical (For more details on Gabriel topology, see [1], [2], [3], [4]).

If M is an R-module, then $N \leq M$ means that N is an R-submodule of M, and its closure with respect to the Gabriel topology $\mathcal{F}$ in M will be denoted by $Cl^M_{\mathcal{F}}(N) = \{x \in M : \exists I \in \mathcal{F} \mid Ix \subseteq N\}$, and if $N = Cl^M_{\mathcal{F}}(N)$, the submodule N is called $\mathcal{F}$-closed. An R-module M is $\mathcal{F}$-multiplication module if for each $\mathcal{F}$-closed submodule $N = Cl^M_{\mathcal{F}}(N)$ there exists an ideal $I \leq R$ such that $N = Cl^M_{\mathcal{F}}(IN)$ (see [1],[2]). We say that L is a minimal extension of N if N is a R-submodule of L and if there exists no R-submodule T of L such that $N \subsetneq T \subsetneq L$.

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2. The Minimal Extensions and Jacobson Radical

The following result which will be used later concerns the closure of an arbitrary extension.

Proposition 2.1 *Let $N \leq L \leq M$ be three R-modules. Then,*

$$Cl^L_{\mathcal{F}}(N) = L \quad \text{if and only if } Cl^M_{\mathcal{F}}(N) = Cl^M_{\mathcal{F}}(L).$$

Proof. If $Cl^L_{\mathcal{F}}(N) = L$ then we have $Cl^L_{\mathcal{F}}(N) = Cl^M_{\mathcal{F}}(N) \cap L = L$. Therefore $L \subseteq Cl^M_{\mathcal{F}}(N)$. Thus $Cl^M_{\mathcal{F}}(N) \subseteq Cl^M_{\mathcal{F}}(L)$,which implies that

$$Cl^M_{\mathcal{F}}(N) = Cl^M_{\mathcal{F}}(L).$$

Conversely, if $Cl^M_{\mathcal{F}}(N) = Cl^M_{\mathcal{F}}(L)$, then

$$Cl^M_{\mathcal{F}}(N) \cap L = Cl^L_{\mathcal{F}}(N) = Cl^M_{\mathcal{F}}(L) \cap L = L.$$

□

The main result in this paper is the following:

Theorem 2.2 *Let $N \leq L \leq M$ be three R-modules where L is a minimal extension of N. If $J(R) \in \mathcal{F}$, then $Cl^M_{\mathcal{F}}(N) = Cl^M_{\mathcal{F}}(L)$.*

Proof. To prove this theorem, we first show the following result: Let $N \leq L \leq M$ be three R-modules with L a minimal extension of N, then $Cl^L_{\mathcal{F}}(N) = N$ if and only if for all x in $L \backslash N$ and for any ideal I in $\mathcal{F}$ we have $L = N + Ix$.

Let us suppose by way of contradiction, that there exists an x in $L \setminus N$ and I in $\mathcal{F}$ such that $N + Ix \subsetneq L$. But $N \subseteq N + Ix$ and since L is a minimal extension of N then $N = N + Ix$, which implies that $Ix \subseteq N$ and therefore $x \in Cl^L_{\mathcal{F}}(N) = N$, this is a contradiction. Conversely, suppose that for any x in $L \backslash N$ and any ideal I in $\mathcal{F}$, we have $L = N + Ix$. Then if $x_0 \in Cl^L_{\mathcal{F}}(N)$ and $x_0 \notin N$ then there exists J in $\mathcal{F}$ such that $Jx_0 \subseteq N$, but $x_0 \in L \backslash N$, then $L = N + Jx_0$ and hence $L \subseteq N + Jx_0 \subseteq N$, wish is impossible.

To prove Theorem 2.2, we suppose that $Cl^M_{\mathcal{F}}(N) \subsetneq Cl^M_{\mathcal{F}}(L)$. By Proposition 2.1, we have $Cl^L_{\mathcal{F}}(N) = N$ since $N \subseteq Cl^L_{\mathcal{F}}(N) \subsetneq L$ and L is a minimal extension of N. Let $x \in Cl^M_{\mathcal{F}}(L) \backslash Cl^M_{\mathcal{F}}(N)$, then there exists I in $\mathcal{F}$ such that $Ix \subseteq L$ and $Ix \nsubseteq N$. So, we can find an i in I such that $ix \in L \backslash N$. By the above result, we have $L = N + J(R)ix$ and since $ix \in L$ then there exist n in N and λ in $J(R)$ such that $ix = n + \lambda ix$, then $(1-\lambda)ix = n$ thus $(1-\lambda)ix \in N$, and since $(1-\lambda)$ is invertible in R thus $ix \in N$, which is impossible. □

Corollary 2.3 *Let R be a commutative ring with a unit such that $J(R) \in \mathcal{F}$, and let M be an R-module. Then an $\mathcal{F}$-closed R-submodule of M does not have a minimal extension in M.*

Corollary 2.4 *If R is a commutative ring with unit, $J(R) \in \mathcal{F}$ and M is an Artinian R-module, then the unique $\mathcal{F}$-closed R-submodule of M is M.*

Proof. Let M be an Artinian R-module and N an R-submodule of M, $\mathcal{F}$-closed and $N \subsetneq M$, then N has a minimal extension L in M, and since $J(R) \in \mathcal{F}$, $Cl_{\mathcal{F}}^{M}(L) = Cl_{\mathcal{F}}^{M}(N) = N \subsetneq L \subseteq Cl_{\mathcal{F}}^{M}(L)$, which is absurd. □

Conversely, if R is an Artinian or semi-simple ring, we have the following theorem.

Theorem 2.5 *Let R be an Artinian or semi-simple ring. Then $J(R) \in \mathcal{F}$ if and only if for any R-modules $N \leq L \leq M$, where L is a minimal extension of N, we have $Cl_{\mathcal{F}}^{M}(N) = Cl_{\mathcal{F}}^{M}(L)$.*

Proof. By Theorem 2.2, if $J(R) \in \mathcal{F}$ then $Cl_{\mathcal{F}}^{M}(N) = Cl_{\mathcal{F}}^{M}(L)$.

Conversely: if R is an Artinian ring, suppose that $J(R) \notin \mathcal{F}$, then $Cl_{\mathcal{F}}^{R}(J(R)) \neq R$, and since R is Artinian, $Cl_{\mathcal{F}}^{R}(J(R))$ has a minimal extension I in R. By hypothesis we have $Cl_{\mathcal{F}}^{R}(Cl_{\mathcal{F}}^{R}(J(R))) = Cl_{\mathcal{F}}^{R}(I)$, but $Cl_{\mathcal{F}}^{R}(I) = Cl_{\mathcal{F}}^{R}(Cl_{\mathcal{F}}^{R}(J(R))) = Cl_{\mathcal{F}}^{R}(J(R)) \subsetneq I \subseteq Cl_{\mathcal{F}}^{R}(I)$. This is impossible.

If R is a semi-simple ring, then $J(R) = \bigcap_{i=1}^{n} \mathcal{M}_i$ where $(\mathcal{M}_i)_{1 \leq i \leq n}$ is the family of all maximal ideals of R, then R is a minimal extension of every $\mathcal{M}_i$, by hypothesis $Cl_{\mathcal{F}}^{R}(\mathcal{M}_i) = R$ $(i = 1, 2, \ldots, n)$, and since $Cl_{\mathcal{F}}^{R}(J(R)) = Cl_{\mathcal{F}}^{R}(\bigcap_{i=1}^{n} \mathcal{M}_i) = \bigcap_{i=1}^{n} Cl_{\mathcal{F}}^{R}(\mathcal{M}_i) = R$, so $J(R) \in \mathcal{F}$. □

Corollary 2.6 *If R is a commutative ring with unit, and $J(R) \in \mathcal{F}$ then R has a proper ideal without a minimal extension.*

Proof. By absurdity, let us suppose that all proper ideals of R have a minimal extension in R. $\mathcal{F}$ is not trivial; thus there is an ideal I which does not belong to $\mathcal{F}$, then $Cl^R_{\mathcal{F}}(I) \neq R$ and hence the ideal $Cl^R_{\mathcal{F}}(I)$ has an minimal extension J in R. But $J(R) \in \mathcal{F}$; then $Cl^R_{\mathcal{F}}(Cl^R_{\mathcal{F}}(I)) = Cl^R_{\mathcal{F}}(J) = Cl^R_{\mathcal{F}}(I) \subsetneq J$, which is absurd. □

If $N \leq L \leq M$ are three R-modules, where L is a minimal extension of N. The following proposition states two properties on R-modules that are between $Cl^M_{\mathcal{F}}(N)$ and $Cl^M_{\mathcal{F}}(L)$.

Proposition 2.7 *Let $N \leq L \leq M$ be three R-modules where L is a minimal extension of N and N_0 a submodule of M such that $Cl^M_{\mathcal{F}}(N) \leq N_0 \leq Cl^M_{\mathcal{F}}(L)$. We have:*

i- If $Cl^M_{\mathcal{F}}(N) \neq N_0$ then $L \subseteq N_0$.

ii- If $Cl^M_{\mathcal{F}}(N) \neq N_0$ and N_0 is $\mathcal{F}$-closed in M then $N_0 = Cl^M_{\mathcal{F}}(L)$.

Proof. i- Let us suppose that $Cl^M_{\mathcal{F}}(N) \subsetneq N_0$, then there exists $x \in N_0 \setminus Cl^M_{\mathcal{F}}(N)$, then $x \in Cl^M_{\mathcal{F}}(L)$ then there exists I in $\mathcal{F}$ such that $Ix \subseteq L$ and $Ix \nsubseteq N$. Let λ in I such that $\lambda x \in L \setminus N$, and since $Cl^M_{\mathcal{F}}(N) \subsetneq N_0 \subseteq Cl^M_{\mathcal{F}}(L)$ then $Cl^M_{\mathcal{F}}(N) \neq Cl^M_{\mathcal{F}}(L)$. By Proposition 2.1, we have $Cl^L_{\mathcal{F}}(N) = N$, and also by the result proved in Theorem 2.2, then for any J in $\mathcal{F}$: $L = N + J\lambda x \subseteq N_0$. ii- If $Cl^M_{\mathcal{F}}(N) \neq N_0$. By i- $L \subseteq N_0$ then $Cl^M_{\mathcal{F}}(L) \subseteq Cl^M_{\mathcal{F}}(N_0)$ however $Cl^M_{\mathcal{F}}(N_0) = N_0$ of or $Cl^M_{\mathcal{F}}(L) = N_0$. □

Remark 2.8 For a Gabriel topology $\mathcal{F}$ defined on R such that $J(R) \in \mathcal{F}$, the closure of an R-module and the closure of a minimal extension of this R-module are the same. But this result is not true in general as shown in the following example.

Example 2.9 *Let R be a commutative ring with unit and R' an Artinian commutative domain. Consider the ring $B = R \times R'$, thus $P = R \times (0)$ is a prime ideal of B. Let $\mathcal{A}$ be an ideal of R' minimal in the set $\{I \text{ ideal of } R' : (0) \neq I\}$, thus the ideal $Q = R \times \mathcal{A}$ is a minimal extension of P. If we consider the set $\mathcal{F} = \{I \text{ ideal in } B : I \nsubseteq P\}$ which defines a Gabriel topology on B, then $P \notin \mathcal{F}$ and $Cl^B_{\mathcal{F}}(P) = P$, and $Q \in \mathcal{F}$ and $Cl^B_{\mathcal{F}}(Q) = B$.*

3. The Minimal Extensions and $\mathcal{F}$-Multiplication Modules

Proposition 3.1 *Let M be an $\mathcal{F}$-multiplication R-module. If $J(R) \in \mathcal{F}$ then every maximal R-submodule of M is $\mathcal{F}$-multiplication.*

Proof. If N is a maximal R-submodule of M then M is a minimal extension of N. Moreover, $J(R) \in \mathcal{F}$ then $Cl^M_{\mathcal{F}}(N) = M$, and by Theorem 3.7 [1] N is $\mathcal{F}$-multiplication. □

An R-module M is called a multiplication module if for every submodule $N \leq M$ there exists an ideal $I \leq R$ such that $N = IM$. Recall that an R-module M is called $\mathcal{F}$-cyclic if $M = Cl^M_{\mathcal{F}}(Rm)$ for some $m \in M$.

Proposition 3.2 *Let M be an R-module, if $J(R) \in \mathcal{F}$ and M does not have any proper $\mathcal{F}$-multiplication R-submodules, then M is not a multiplication module.*

Proof. By absurdity, let us suppose that M a multiplication R-module. Therefore it is $\mathcal{F}$-multiplication, and Theorem 2.5 [5] gives us the existence of a maximal R-submodule of M, that one notes by N, if $J(R) \in \mathcal{F}$ then $Cl^M_{\mathcal{F}}(N) = M$ and by Theorem 3.7[1] N is $\mathcal{F}$-multiplication, which is absurd. □

Definition 3.3 *An R-module M is called of finite length if there exists a sequence of R-submodules $(M_i)_{1 \leq i \leq n}$ of M verifying: $(0) = M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_n = M$, with M_{i+1} minimal extension of M_i for $1 \leq i \leq n-1$.*

Theorem 3.4 *If M is an R-module of finite length and $J(R) \in \mathcal{F}$, then M is $\mathcal{F}$-multiplication.*

Proof. Assume M is an R-module of finite length n. There exists a sequence of R-submodules $(M_i)_{1 \leq i \leq n}$ verifying: $(0) = M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_n = M$, with M_{i+1} minimal extension of M_i for $1 \leq i \leq n-1$, in addition $J(R) \in \mathcal{F}$ thus $Cl^{M_{i+1}}_{\mathcal{F}}(M_i) = M_{i+1} = Cl^M_{\mathcal{F}}(M_i) \bigcap M_{i+1}$, then $M_{i+1} \subseteq Cl^M_{\mathcal{F}}(M_i)$, and consequently $Cl^M_{\mathcal{F}}(M_{i+1}) \subseteq Cl^M_{\mathcal{F}}(Cl^M_{\mathcal{F}}(M_i)) = Cl^M_{\mathcal{F}}(M_i)$ and hence $M = Cl^M_{\mathcal{F}}(M_n) \subseteq Cl^M_{\mathcal{F}}(M_{n-1}) \subseteq \ldots \subseteq Cl^M_{\mathcal{F}}((0))$, then $Cl^M_{\mathcal{F}}((0)) = M$. Therefore M is $\mathcal{F}$-cyclic and by the Corollary 3.9 [1] M is $\mathcal{F}$-multiplication. □

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