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H. G. NAGARAJA

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On Cartan Spaces with (α, β) -metric

H. G. Nagaraja

Abstract

É. Cartan [2] has originally introduced a Cartan space, which is considered as dual of Finsler space. H. Rund [10], F. Brickell [1] and others studied the relation between these two spaces. The theory of Hamilton spaces was introduced and studied by R. Miron ([8] , [9]). He proved that Cartan space is a particular case of Hamilton space. T. Igrashi ([5], [6]) introduced the notion of the (α, β) -metric in Cartan spaces and obtained the metric tensor and the invariants ρ and r which characterize the special classes of Cartan spaces with (α, β) -metric. This paper presents a study of Cartan spaces with (α, β) -metric admitting h-metrical d-connection. We prove the conditions for these spaces to be locally Minkowski and conformally flat.

Key Words: Cartan space, (α, β) -metric, h-metrical d-connection, Conformally flat.

1. Introduction

Let M be a real smooth manifold and (T^*M, π, M) its cotangent bundle. Let $C^n = (M, K(x, p))$, where $K : T^*M \rightarrow R$ is a scalar function which is differentiable on $T^*\widetilde{M} = T^*M - \{0\}$, and is homogeneous on the fibres of T^*M . The hessian of K^2 , i.e. $g^{ij}(x, p) = \frac{1}{2}\dot{\partial}^i\dot{\partial}^j K^2$, where $\dot{\partial}^i = \frac{\partial}{\partial p_i}$, is positively defined on $T^*\widetilde{M}$. Here C^n is called the Cartan space and the functions $K(x, p)$ and $g^{ij}(x, p)$ are called, respectively, the fundamental function and the metric tensor of the Cartan space C^n .

The reciprocal $g_{ij}(x, p)$ of $g^{ij}(x, p)$ is given by $g_{ij}(x, p)g^{ij}(x, p) = \delta_j^k$, where $g_{ij}(x, p)$

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and $g^{ij}(x, p)$ are both symmetric and homogeneous of order 0 in p_j .

A Cartan space $C^n = (M, K)$ is said to be with (α, β) -metric if $K(x, p)$ is a function of the variables $\alpha(x, p) = (a^{ij} p_i p_j)^{\frac{1}{2}}$, $\beta(x, p) = p_i b^i(x)$, where $a^{ij}(x)$ is a Riemannian metric and $b^i(x)$ is a vector field depending only on x . Clearly K must satisfy the conditions imposed to the fundamental function of a Cartan space.

In this paper, we consider the Cartan spaces with (α, β) -metric admitting h-metrical d-connection in section 2 and their conformal change in section 3. The fundamental tensor $g^{ij}(x, p)$ and its reciprocal $g_{ij}(x, p)$ of the Cartan space $C^n = (M, K(\alpha, \beta))$ are given by [6] the relation

$$g^{ij} = \rho a^{ij} + \rho b^i b^j + \rho_{-1} (b^i p^j + b^j p^i) + \rho_{-2} p^i p^j, \quad (1.1)$$

where $\rho, \rho_0, \rho_{-1}, \rho_{-2}$ are the invariants given by

$$\begin{aligned} \rho &= \frac{1}{2} \alpha^{-1} K_{\alpha}, \quad \rho_{-1} = \frac{1}{2} \alpha^{-1} K_{\alpha\beta}, \quad \rho_{-2} = \frac{1}{2} \alpha^{-2} (K_{\alpha\alpha} - \alpha^{-1} K_{\alpha}) \\ \rho_0 &= \frac{1}{2} K_{\beta\beta}, \end{aligned} \quad (1.2)$$

and

$$g_{ij} = \sigma a_{ij} - \sigma_0 b_i b_j + \sigma_{-1} (b_i p_j + b_j p_i) + \sigma_{-2} p_i p_j \quad (1.3)$$

where

$$\sigma = \frac{1}{\rho}, \quad \sigma_0 = \frac{\rho_0}{\rho\tau}, \quad \tau = \sigma + \sigma_0 B^2 + \rho_{-1} \beta, \quad \sigma_{-1} = \frac{\rho_{-1}}{\rho\tau}, \quad \sigma_{-2} = \frac{\rho_{-2}}{\rho\tau}, \quad (1.4)$$

and where $B^2 = b^i b_i$.

The Cartan tensor C^{ijk} is given by

$$\begin{aligned} C^{ijk} &= -\frac{1}{2} [r_{-1} b^i b^j b^k + \{\rho_{-1} a^{ij} b^k + \rho_{-2} a^{ij} p^k + r_{-2} b^i b^j p^k + \\ &\quad r_{-3} b^i p^j p^k + i/j/k\} + r_{-4} p^i p^j p^k], \end{aligned} \quad (1.5)$$

where

$$\begin{aligned} r_{-1} &= \frac{1}{2} K_{\beta\beta\beta}, \quad r_{-2} = \frac{1}{2} \alpha^{-1} K_{\alpha\beta\beta}, \quad r_{-3} = \frac{1}{2} \alpha^{-2} (K_{\alpha\alpha\beta} - \alpha^{-1} K_{\alpha\beta}) \\ r_{-4} &= \frac{1}{2} \alpha^{-3} \{K_{\alpha\alpha\alpha} - 3\alpha^{-1} K_{\alpha\alpha} + 3\alpha^{-2} K_{\alpha}\}. \end{aligned} \quad (1.6)$$

Let $'\cdot'$ denote the covariant differentiation with respect to Christoffel symbols γ_{jk}^i constructed from a_{ij} . Since $a^{ij}{}_{;k} = 0$ and $p_{i;k} = 0$, if $b^i{}_{;k} = 0$, then $g^{ij}{}_{;k} = 0$. Using the Christoffel symbols $\Gamma_{jk}^i(p) = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk})$ constructed from $g_{ij}(x, p)$, we can define canonical N -connection

$$N_{ij} = \Gamma_{ij}^k p_k - \frac{1}{2} \Gamma_{hr}^k p_k p^r \dot{\partial}^h g_{ij}. \tag{1.7}$$

We consider the canonical d-connection

$$D\Gamma = (N_{jk}, H_{jk}^i, C_i^{jk}) \tag{1.8}$$

where

$$H_{jk}^i = \frac{1}{2} g^{ir} (\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}). \tag{1.9}$$

The d -tensor field of type (2,1) C_i^{jk} is given by

$$C_i^{jk} = -\frac{1}{2} g_{ir} \dot{\partial}^r g^{jk} = g_{ir} C^{rjk}, \tag{1.10}$$

Let $'|k'$ denote the h -covariant differentiation with respect to $D\Gamma$.

Definition 1.1 A d -connection $D\Gamma$ of a Cartan space C^n with (α, β) -metric is called the h -metrical d -connection if it satisfies the conditions

- h -deflection tensor $D_{ij}(= p_{i|j}) = 0$;
- $\alpha^{ij}{}_{|k} = 0$;
- $g^{ij}{}_{|k} = 0$.

2. Cartan Spaces with (α, β) -metric admitting h -metrical d-connection

If the connection $D\Gamma$ is h -metrical, then $\alpha|_h = 0$, from which we get that

$$0 = K|_h = \alpha|_h K_\alpha + \beta|_h K_\beta = \beta|_h K_\beta$$

and

$$\beta|_h = b^i{}_{|h} p_i = 0 \tag{2.1}$$

From (1.1), we have

$$g^{ij}{}_{|k} = b^i{}_{|k}(\rho_0 b^j + \rho_{-1} p^j) + b^j{}_{|k}(\rho_0 b^i + \rho_{-1} p^i) = 0$$

Transvecting the above with p_i , and by virtue of (2.1) we get

$$b^j{}_{|k}(\rho_0 \beta + \rho_{-1} \alpha) = 0,$$

which gives $b^j{}_{|k} = 0$.

Now from $a^{ij}{}_{|k} = 0$, we can get $H^i{}_{jk} = \gamma^i{}_{jk}$. Hence we have

$$b^i{}_{:k} = 0, \tag{2.2}$$

and also the curvature tensor $D^i{}_{hjk}$ of $D\Gamma$ coincides with the curvature tensor $R^i{}_{hjk}$ of Riemannian connection $R\Gamma = (\gamma^i{}_{jk}, \gamma^i{}_{jk} y_i, 0)$.

If $R^i{}_{hjk} = 0$ then $D^i{}_{hjk} = 0$. Thus we have the following proposition.

Proposition 2.1 *A Cartan space C^n with (α, β) metric admitting a h -metrical d -connection is locally flat if and only if the associated Riemannian space is locally flat.*

If the connection $D\Gamma$ is h -metrical, then $g^{ij}{}_{|h} = 0$, $\alpha_{|h} = 0$, $a^{ij}{}_{|h} = 0$, $b^k{}_{|h} = 0$, $p^k{}_{|h} = 0$, from which we get $r_{-1|h} = 0$, $r_{-2|h} = 0$, $r_{-3|h} = 0$ and $r_{-4|h} = 0$.

Hence from (1.5), (1.6) and (1.10), we have

$$C^{ij}{}_{k|h} = 0. \tag{2.3}$$

Definition 2.1 *A Cartan space C^n is a Berwald space if and only if $C^{ij}{}_{k|h} = 0$.*

Hence from (2.3) we have the following proposition.

Proposition 2.2 *A Cartan space with (α, β) metric admitting a h -metrical d -connection is a Berwald space.*

As it is well known [11] a locally Minkowski space is a Berwald space in which the curvature tensor vanishes.

Hence from Proposition (2.1) and Proposition (2.2), we have the following proposition.

Theorem 2.1 *A Cartan space with (α, β) metric admitting a h -metrical d -connection is locally Minkowski if and only if the associated Riemannian space is locally flat.*

3. Conformal Change of a Cartan Space

Let $C^n = (M, K)$ be an n -dimensional Cartan space with (α, β) -metric $K = K(\alpha, \beta)$. By a conformal change $\sigma : K \rightarrow \bar{K} : \bar{K}(\bar{\alpha}, \bar{\beta}) = e^\sigma K(\alpha, \beta)$, we have another Cartan space $\bar{C}^n = (M, \bar{K}(\bar{\alpha}, \bar{\beta}))$, where $\bar{\alpha} = e^\sigma \alpha$ and $\bar{\beta} = e^\sigma \beta$.

Putting $\alpha = (a^{ij}(x)p_i p_j)^{\frac{1}{2}}$ and $\beta = b^i(x)p_i$, we get $\bar{\alpha}^{ij} = e^{2\sigma} a^{ij}$ and $\bar{b}^i = e^\sigma b^i$. Then the Christoffel symbols $\bar{\gamma}_{jk}^i$ constructed from $\bar{\alpha}^{ij}$ are written as

$$\bar{\gamma}_{jk}^i = \gamma_{jk}^i + B_{jk}^i \tag{3.1}$$

where $B_{jk}^i = \delta_j^i \sigma_k + \delta_k^i \sigma_j - \sigma^i a_{jk}$, $\sigma^i = \sigma_j a^{ij}$.

Taking covariant derivative of \bar{b}^i with respect to $\bar{\gamma}_{jk}^i$, we get

$$\bar{b}^i{}_{:k} = e^\sigma (b^i{}_{:k} + 2\sigma_k b^i + b^r \sigma_r \delta_k^i - \sigma^i b^r a_{rk}).$$

Transvecting the above by \bar{b}^k , and putting

$$M^i = \frac{1}{B^2} \left\{ b^k b_{:k}^i - \frac{b^r{}_{:r} b^i}{n+4} \right\}, \tag{3.2}$$

we have $\sigma^i = \bar{M}^i - M^i$, from which we get $\sigma_i = \bar{M}_i - M_i$. Substituting this in (3.1) and putting

$D_{hj}^i = \gamma_{hj}^i + \delta_h^i M_j + \delta_j^i M_h - M^i a_{hj}$, we have

$$\bar{D}_{hj}^i = D_{hj}^i. \tag{3.3}$$

D_{hj}^i is a symmetric conformally invariant linear connection on M .

Thus we have the following proposition.

Proposition 3.1 *In a Cartan space with (α, β) -metric, there exists a conformally invariant symmetric linear connection D_{jk}^i .*

If we denote by D_{hjk}^i the curvature tensor of D_{jk}^i , we have from (3.3)

$$\bar{D}_{hjk}^i = D_{hjk}^i. \tag{3.4}$$

Since $b^i{}_{:k} = 0$, we have $M^i = 0$. Hence $D_{jk}^i = \gamma_{jk}^i$ and $D_{hjk}^i = R_{hjk}^i$. Thus we have the next proposition.

Proposition 3.2 *In a Cartan space C^n with (α, β) -metric admitting h -metrical d -connection, $M^i = 0$ and there exists a conformally invariant symmetric linear connection D_{jk}^i such that $D_{jk}^i = \gamma_{jk}^i$ and its curvature tensor $D_{hjk}^i = R_{hjk}^i$.*

If the associated Riemannian space (M, α) is locally flat ($R_{hjk}^i = 0$), then from (3.4) and Proposition (3.2), we have $\overline{D}_{hjk}^i = 0$, i.e. the space C^n is conformally flat.

Thus we conclude that.

Theorem 3.1 *A Cartan space $C^n = (M, K(\alpha, \beta))$ with (α, β) metric admitting h -metrical d -connection is conformally flat if and only if associated Riemannian space is locally flat.*

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H. G. NAGARAJA
Department Of Mathematics, Central College,
Bangalore University, Bangalore-560 001,
Karnataka-INDIA
e-mail: hgnraj@yahoo.com

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