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Cover for Modules and Injective Modules

N. Amiri

Abstract

Let R be a commutative ring with identity and M be an R -module with $\text{Spec}(M) \neq \emptyset$. A cover of the R -submodule K of M is a subset C of $\text{Spec}(M)$ satisfying that for any $x \in K, x \neq 0$, there is $N \in C$ such that $\text{ann}(x) \subset (N : M)$. If we denote by $J = \bigcap_{N \in C} (N : M)$ and assume that M is finitely generated, then $JM = M$ implies that $M = 0$, M is called C -injective provided each R -homomorphism $\phi : (N : M) \rightarrow M$ with $N \in C$ can be lifted to an R -homomorphism $\lambda : R \rightarrow M$. If R is a commutative Noetherian ring and $C' = \text{Spec}(R)$, where $C' = \{(N : M) | N \in C\}$, then every C -injective R -module is injective.

Key Words: Commutative ring, D -prime module cover, prime submodule, injective module, quasi-injective and injective hull.

Definition. Let M be an R -module. A proper submodule P of M is a prime submodule, if $rm \in P$, for $r \in R$ and $m \in M$ implies that either $m \in P$ or $rM \subset P$. The set of all prime submodules of M is called the spectrum of M and denoted by $\text{Spec}(M)$.

Definition. Let M be an R -module. A subset C of $\text{Spec}(M)$ is a cover of M , if for every $0 \neq x \in M$ there exists $P \in C$ such that $\text{ann}(x) \subset (P : M)$. If C is a finite set, then C is called a finite cover.

Definition. An R -module M is called D -prime provided that $M \neq 0$ and $\text{ann}(N) = \text{ann}(M)$, for all non-zero submodule N of M .

1. Cover for Modules and Localization

Lemma 1. Let M be a non-zero R -module and C a cover of M and $J = \bigcap_{P \in C} (P : M)$ if $JM = M$, then $M = 0$.

Proof. Suppose that $M \neq 0$ and $JM = M$, then there exists $r \in R$ such that $r - 1 \in J$ and $rM = 0$, so $rm = 0$ for all $m \in M$ and $r \in \text{ann}(m)$. Hence $r \in J$, that is a contradiction. \square

Lemma 2. Let R be a Noetherian ring, M is a finitely generated R -module, C a cover of M , $I \subset \bigcap_{P \in C} (P : M)$. Then $\bigcap_{n=1}^{\infty} I^n M = 0$.

Proof. Let $\bigcap_{n=1}^{\infty} I^n M = K$. Then by Krull's Theorem $IK = K$ and by Lemma 1, $K = 0$. \square

Lemma 3. Let C be a finite subset of $\text{Spec}(M)$ such that $(P : M)$ is maximal for every $P \in C$, and $J = \bigcap_{P \in C} (P : M)$. If $\bigcap_{n=1}^{\infty} J^n M = 0$, then C is a finite cover of M .

Proof. If C is not a cover of M , then there is an element $0 \neq x \in M$ such that $\text{ann}(x) \not\subset (P : M)$ for all $P \in C$. Hence $\text{ann}(x) + (P : M) = R$. Let $1 = r + s$ with $s \in (P : M)$ and $r \in \text{ann}(x)$. Then for every $n \in \mathbb{N}$, $1^n = (r + s)^n = r' + s'$, $r' \in \text{ann}(x)$, $s' \in (P : M)^n$, so $x = r'x + s'x = s'x$. Hence $Rx = (P : M)^n x$, for every $P \in C$, and so $J^n x = Rx$. Hence $\bigcap_{n=1}^{\infty} J^n M \neq 0$, which is a contradiction. \square

Theorem 4. Let R be a Noetherian ring and M a faithful finitely generated R -module. Then M has a finite cover C and $\bigcap_{n=1}^{\infty} J^n M = 0$, where $J = \bigcap_{P \in C} (P : M)$. In particular, if $M = R$, then $\bigcap_{n=1}^{\infty} J^n = 0$.

Proof. See [1. Theorem 6]. \square

Theorem 5. Let M be a finitely generated R -module and C is a subset of $\text{Spec}(M)$. If for every prime ideal P of R and $N \in C$, $N_P \neq M_P$, then C is a cover for M over

R if and only if C_P is a cover for M_P over R_P , for every prime ideal P of R , where $C_P = \{N_P | N \in C\}$.

Proof. Let $\frac{m}{s} \in M_P$. Since $m \in M$ and C is a cover for M , there exists $N \in C$ such that $\text{ann}(m) \subset (N : M)$. Let $r/s \in \text{ann}(\frac{m}{s})$. Since $\text{ann}(m_P) \subset (N_P : M_P)$, $r/s \in (N_P : M_P)$ and $\text{ann}(\frac{m}{s}) \subset (N_P : M_P)$ so C_P is a cover for M_P over R_P . Let $m \in M$, then $\frac{m}{1} \in M_P$ so there exists $N_P \in C_P$ such that $\text{ann}(\frac{m}{1}) \subset (N_P : M_P)$. Now let $r \in \text{ann}(m)$. Then $\frac{r}{1} \cdot \frac{y}{1} \in N_P$, where $\frac{y}{1} \in M_P$, so $\frac{ry}{1} = \frac{n}{s}$ for some $n \in N$; and so there exists $s' \in R - P$ such that $rss'y = s'n \in N$. Hence $ss'(ry) \in N$, and since $ss' \notin (N : M)$, $ry \in N$, so $rM \subset N$, and $\text{ann}(x) \subset (N : M)$. \square

Theorem 6. Let R be a reduced ring and C is a subset of $\text{Spec}(R)$. Then C is a cover for R as an R -module if and only if $C[[x]]$ is a cover for $R[[x]]$, where $C[[x]] = \{P[[x]] | P \in C\}$.

Proof. Let C be a cover for R and $g(x) \in \text{ann}(f(x))$ for $f(x), g(x) \in R[[x]]$. If $g(x) = \sum_{n=0}^{\infty} b_n x^n$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then for every i , $b_i f(x) = 0$, so for every i , $b_i \in \text{ann}(a_0) \subset P$, for some $P \in C$ and hence $g(x) \in P[[x]]$. Conversely if $C[[x]]$ is a cover for $R[[x]]$ and let $a \in R, r \in R$ such that $r \in \text{ann}(a) \subset P[[x]]$, for some $P[[x]] \in C[[x]]$. So $ra = 0$ and hence $r \in P[[x]] \cap R$, so $r \in P$. Then $\text{ann}(a) \subset P$. Hence C is a cover for R . \square

Proposition 7. Let R be a ring and C is a subset of $\text{Spec}(R)$. Then C is a cover for R as an R -module if and only if $C[x] = \{P[x] | P \in C\}$ is a cover for $R[x]$ as an $R[x]$ -module.

Proof. Let C be a cover for R and $f(x) \in R[x], g(x) \in \text{ann}(f(x))$, then $f(x)g(x) = 0$. If $g(x) = \sum_{i=0}^k b_i x^i$, and $f(x) = \sum_{i=1}^m a_i x^i$, then there is an element a such that $ag(x) = 0$ so $b_i \in \text{ann}(a) \subset P$ for some $P \in C$. So $g(x) \in P[x]$ and hence $C[x]$ is a cover for $R[x]$.

Conversely, let $C[x]$ be a cover for $R[x]$, and let $a \in R, r \in R$, and $r \in \text{ann}(a)$. As $\text{ann}(a) \subset P[x]$ so $r \in P[x] \cap R = P$. Thus C is a cover for R . \square

2. C -injective Modules

Definition. Let R be a ring, M, X are R -modules, C is a cover of M . We say that X is C -injective provided every R -homomorphism $\phi : (N : M) \rightarrow X$, where $N \in C$ can be lifted to an R -homomorphism $\lambda : R \rightarrow X$. In the next results we shall be interested in ring R with the following properties:

(P1) for every proper ideal I there exists a finite set of prime ideals P_1, P_2, \dots, P_n such that $P_1 P_2 \cdots P_n \leq I \leq P_1 \cap P_2 \cap \cdots \cap P_n$.

(P2) The ascending chain condition on prime ideals.

Proposition 8. Let R be a Noetherian ring. Then R satisfies (P1) and (P2).

Proof. Since R is Noetherian then R satisfies (P2). Suppose R does not satisfy (P1).

Let $S = \{J \mid \text{(P1) fails for } J\}$. Suppose I be a maximal element of S . Then I is not prime ideal, so there exists ideal I_1 and I_2 properly containing I such that $I_1 I_2 \leq I$. By the choice of I , (P1) holds for each I_1 and I_2 , and hence for I , which is a contradiction. \square

Proposition 9. Let R be a ring which satisfies (P1) and (P2). Then every non-zero R -module contains a D -prime submodule.

Proof. Let M be a non-zero R -module. Let $I = \text{ann}(M)$. Then there exists prime ideal P_1, P_2, \dots, P_n such that $P_1 P_2 \cdots P_n \leq I \leq P_1 \cap P_2 \cap \cdots \cap P_n$. Thus $P_1 P_2 \cdots P_n M = 0$ and it follows that there exists P_k such that $P_k m = 0$, for some $m \in M$. Suppose $B = \{P : P \text{ is a prime ideal and } Px = 0 \text{ for some } x \in M\}$. Let Q be a maximal element of B and let $y \in M$ such that $Qy = 0$. We show that $N = Ry$ is a D -prime submodule of M . Let K be a non-zero submodule of N . Then $Q \leq \text{ann}(K)$, we show that $Q = \text{ann}(K)$. Let $Q \neq \text{ann}(K)$.

Then there exists prime ideal q_1, q_2, \dots, q_m such that $q_1 q_2 \cdots q_m \leq \text{ann}(K) \leq q_1 \cap q_2 \cap \cdots \cap q_m$. It follows that $q_1 q_2 \cdots q_m K = 0$, and there exists $x \in K$ such that $q_i x = 0$ for some i . But $Q < \text{ann}(K) \leq q_i$ and this contradicts the choice of Q . Hence $Q = \text{ann}(K)$, and so N is D -prime submodule of M .

Theorem 10. Let M be an R -module and R satisfies (P1) and (P2), C a cover of M . Then M is C -injective if and only if M is an injective R -module.

Proof. Let M be a C -injective and I be an ideal of R and $\phi : I \rightarrow M$ an R -

homomorphism. By zorn lemma there exists an ideal J containing I maximal with respect to the property that ϕ can be lifted to a homomorphism $\lambda : J \rightarrow M$. We show that $J = R$. Suppose $J \neq R$. Thus $\frac{R}{J}$ is a non-zero R -module and so $\frac{R}{J}$ has a D -prime submodule. \square

Let K be an ideal containing J such that $\frac{K}{J}$ is a D -prime submodule of $\frac{R}{J}$. Let $k \in K, k \notin J$. Then $\frac{(Rk+J)}{J}$ is a D -prime module. Let $P = \{r \in R | rk \in J\}$. Then $\frac{R}{P} \simeq \frac{Rk+J}{J}$, and hence P is a prime ideal of R . As $P = (N : M)$, where $N \in C$, define $\gamma : P \rightarrow M$ $\gamma(x) = \lambda(kx)$. Then γ is a homomorphism, and because $P = (N : M)$ for $N \in C$, there exists $m \in M$ such that $\gamma(x) = mx$. Now define $\theta : kR + J \rightarrow M$ by $\theta(rk + j) = rm + \lambda(j)$, so θ is well-defined, θ is a homomorphism and θ extends λ and hence ϕ . This contradiction shows that $J = R$. it follows that M is injective.

3. Quasi-Injective Modules

Definition. An R -module M is said to be quasi-injective if every R -homomorphism $\phi : N \rightarrow M, N$ a submodule of M , is induced by an R -endomorphism of M .

Notation. Let C be a cover for R -module M , denote $C(M) = \{x \in M | (N : M) \subset ann(x), \text{ for some } N \in C\}$.

Lemma 11. $C(M)$ is a submodule of M .

Proof. It is obvious. \square

Theorem 12. An R -module M is quasi-injective if and only if $M = E[C(M)]$, where $E[C(M)]$ is injective hull of $C(M)$.

Proof. If M is quasi-injective. Then $M \leq E[C(M)]$, we show that $E[C(M)] \leq M$. Let $y \in E[C(M)]$, then there exists $N \in C$ such that $(N : M) \subset ann(y)$; and since C is a cover for M there exists $x \in M$ such that $ann(x) \subset (N : M)$. We define $\alpha : Rx \rightarrow Ry$ by $\alpha(x) = y$. Let $E = E[M]$, so we have the mapping

$$\begin{array}{ccccccc} 0 & \rightarrow & Rx & \rightarrow & M & \rightarrow & E \\ & & \alpha \downarrow & & \swarrow \lambda & & \\ & & & & E & & \end{array}$$

Now $\phi = \lambda/M$ maps x onto y ; and since M is quasi-injective, it is fully invariant in E , then $y \in M$ so that $E[C(M)] \leq M$, and equality holds. Conversely, suppose that

$M = E[C(M)]$, since $E[C(M)]$ is a injective R -module so is M , and since every injective R -module is quasi-injective. Hence M is quasi-injective R -modules. \square

Corollary 13. Let C be a cover for an R -module M . Then the following are equivalent.

- (1) M is quasi-injective R -module.
- (2) M is a injective R -module.
- (3) $M = E[C(M)]$.

References

- [1] Amiri, N., Ershad, M. and Sharif, H.: Cover for Modules, To appear in International Journal Of Commutative Ring.
- [2] Anderson, F. W. and Fuller, K. P.: Ring and Categories of Modules Springer-Verlag (1974).
- [3] Gooderl, K. R.: Von Neumann Regular Ring, Pitman, 1979.
- [4] Matsumura, H.: Commutative ring theory, Cambridge university Press, 1986.

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