A Decomposition Method for Solving Unsteady Convection-Diffusion Problems

SHAHER MOMANI

A Decomposition Method for Solving Unsteady Convection-Diffusion Problems

Shaher Momani

Abstract

In this study, a decomposition method for approximating the solutions of unsteady convection-diffusion problems is implemented. The approximate solution is calculated in the form of a convergent series with easily computable components. The calculations are accelerated by using the noise terms phenomenon for nonhomogeneous problems. Numerical examples are investigated to illustrate the pertinent features of the proposed algorithm.

Key Words: Convection-diffusion equation; Decomposition method; Noise terms.

1. Introduction

Consider the following convection-diffusion equation:

$$
\begin{cases}
\frac{\partial u}{\partial t} + b_1(x,y)\frac{\partial u}{\partial x} + b_2(y)\frac{\partial u}{\partial y} - \left(a_1\frac{\partial^2 u}{\partial x^2} + a_2\frac{\partial^2 u}{\partial y^2} \right) = f(t,x,y), & \text{in } \Omega \times J, \\[2ex]
u(x,y,t) = g_1(x,t), & \text{on } \partial\Omega \times J, \quad (1.1) \\[2ex]
u(x,y,0) = g_2(x,y), & \text{in } \Omega,
\end{cases}
$$

where $\Omega = (0,1) \times (0,1), J = (0,T), b_1(x,y), b_2(y)$ are smooth functions and a_1, a_2 are positive constants. This equation may be seen in computational hydraulics and fluid dynamics to model convection-diffusion of quantities such as mass, heat, energy, vorticity,

etc. [1]. Several numerical methods have been proposed to solve convection-diffusion problems approximately. Among them are restrictive Taylor's approximation [2], the alternating direction implicit (ADI) method [3], the upwind method [4], and the explicit predictor method [5].

Recently, the Adomian decomposition method (in short, ADM) [6, 7] has emerged as an alternative method for solving a wide ranges of problems whose mathematical models involve algebraic, differential, integro-differential, and partial differential equations. The decomposition method yields rapidly convergent series solutions for both linear and nonlinear deterministic and stochastic equations. The technique has many advantages over the classical techniques, mainly, it avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculations and avoidance of physically unrealistic assumptions.

The convergence of the decomposition series has been investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered in the literature [8–12]. They obtained some results about the speed of convergence of this method. In recent work of Abbaoui et al [13] have proposed a new approach of convergence of the decomposition series. The authors have given a new condition for obtaining convergence of the decomposition series to the classical presentation of the ADM in [13].

In this paper, various convection-diffusion equations can be handled easily, quickly, and elegantly by implementing the ADM rather than the traditional methods for finding analytical as well as numerical solutions.

2.   Analysis of the Method

In this section, we outline the steps to obtain analytic solution of the convection-diffusion equation (1.1) using the ADM. To begin, it is convenient to rewrite the equation in the standard operator from

$$L_t u + b_1(x,y)u_x + b_2(y)u_y - \Big(a_1 L_x u + a_2 L_y u\Big) = f(x,y,t), \qquad (2.1)$$

where $L_t = \frac{\partial}{\partial t}$, $L_x = \frac{\partial^2}{\partial x^2}$, and $L_y = \frac{\partial^2}{\partial y^2}$. The inverse of the operator L_t exists and it can conveniently be taken as the one-fold integration operator L_t^{-1}. Thus, applying the inverse operator L_t^{-1} to (2.1) yields

$$L_t^{-1} L_t u = L_t^{-1}\Big(-b_1(x,y)u_x - b_2(y)u_y + a_1 L_x u + a_2 L_y u + f(x,y,t)\Big). \qquad (2.2)$$

Therefore, it follows that

$$u(x,y,t) = u(x,y,0) + L_t^{-1}\Big(-b_1(x,y)u_x - b_2(y)u_y + a_1 L_x u + a_2 L_y u + f(x,y,t)\Big). \qquad (2.3)$$

Now, we decompose the unknown function $u(x,y,t)$ a sum of components defined by the series

$$u(x,y,t) = \sum_{n=0}^{\infty} u_n(x,y,t). \qquad (2.4)$$

The zeroth component is usually taken to be all terms arise from the initial conditions and the integration of the source term $f(x,y,t)$, i.e.,

$$u_0 = u(x,y,0) + L_t^{-1} f(x,y,t). \qquad (2.5)$$

The remaining components $u_n(x,y,t)$, $n \geq 1$, can be completely determined such that each term is computed by using the previous term. Since u_0 is known,

$$u_n = L_t^{-1}\Big(-b_1(x,y)(u_{n-1})_x - b_2(y)(u_{n-1})_y + a_1 L_x u_{n-1} + a_2 L_y u_{n-1}\Big), \;\; n \geq 1. \quad (2.6)$$

A slight modification to the ADM was proposed by Wazwaz [14] that gives some flexibility in the choice of the zeroth component u_0 to be any simple term and modify the term u_1 accordingly. Since the computation in (2.6) depends heavily on u_0 the whole computations to find the solution will be simplified considerably. For example an alternative to (2.6) might be

$$\begin{aligned}
u_0 &= u(x,y,0) \\
u_1 &= L_t^{-1} f(x,y,t) + L_t^{-1}\Big(-b_1(x,y)(u_0)_x - b_2(y)(u_0)_y + a_1 L_x u_0 + a_2 L_y u_0\Big), \\
u_n &= L_t^{-1}\Big(-b_1(x,y)(u_{n-1})_x - b_2(y)(u_{n-1})_y + a_1 L_x u_{n-1} + a_2 L_y u_{n-1}\Big), \;\; n \geq 2.
\end{aligned} \qquad (2.7)$$

Finally an N-term approximate solution is given by

$$\Phi_N(x, y, t) = \sum_{n=0}^{N-1} u_n(x, y, t), \quad N \geq 1, \tag{2.8}$$

and the exact solution is $u(x, y, t) = \lim_{N \to \infty} \Phi_N$.

To show the effectiveness of the proposed decomposition method and to give a clear overview of the methodology, some examples of the convection-diffusion problem (1.1) will be discussed in the following section.

3. Numerical Examples

We shall illustrate the numerical scheme by three examples. These examples are somewhat artificial in the sense that the exact answer is known in advance and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the numerical scheme. All the results are calculated by using the symbolic calculus software Mathematica.

Example 1 *We consider the following homogeneous Convection-diffusion problem*

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y, t) \in \Omega \times J, \tag{3.1}$$

subject to the initial condition

$$u(x, y, 0) = \exp[-(x - 0, 05)^2 - (y - 0.5)^2]. \tag{3.2}$$

The exact solution is given by [3]

$$u(x, y, t) = \frac{1}{4t + 1} \exp\left[-\frac{(x - t - 0.05)^2}{4t + 1} - \frac{(y - t - 0.05)^2}{4t + 1}\right]. \tag{3.3}$$

To find the approximate solution of the initial value problem (3.1) and (3.2), we apply the scheme (2.6). This gives

$$u_0 = \exp[-(x - 0,05)^2 - (y - 0.5)^2],$$

$$u_1 = \exp[-(x - 0,05)^2 - (y - 0.5)^2](4x^2 - 2x + 4(y - 1.28078)(y + 0.780778))t,$$

$$u_2 = e^{-(x-0,05)^2-(y-0.5)^2}(8x^2 - 8x^3 + 16x^2(y - 1.64194)(y + 1.14194)$$
$$-8x(y - 1.68614)(y + 1.18614) + 8(y - 2.13746)(y - 1)(y + 0.5)(y + 1.63746))t^2,$$

$$u_2 = \exp[-(x - 0,05)^2 - (y - 0.5)^2](10.6667x^6 - 16x^5 + 32x^4(y - 1.92705)(y + 1.42705)$$
$$-32x^3(y - 1.98805)(y + 1.48805) + 32x^2(y - 2.42821)(y - 1.312)(y + 0.813201)$$
$$(y + 1.92821) - 16x(y - 2.45578)(y - 1.37237)(y + 0.872374)(y + 1.95574)$$
$$+10.6667(y - 2.7968)(y - 1.77639)(y - 0.859479)(y + 0.359479)(y + 1.27639)$$
$$(y + 2.2968))t^3,$$
$$\vdots$$

$$(3.4)$$

in this manner the components of the decomposition series (2.4) are obtained as far as we like.

In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the numerical solutions using the N-term approximation (2.8). Table 1 shows the difference of analytical solution and numerical solutions of the absolute errors. It is to be noted that six terms only were used in evaluating the approximate solutions. We achieved a very good approximation with the actual solution of the equation by using only 6-terms of the decomposition series solution derived above. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

Example 2 *We next consider the following Convection-diffusion problem*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y, t) \in \Omega \times J, \tag{3.5}$$

subject to the initial condition

$$u(x, y, 0) = \sin(\pi x)\sin(\pi y) \tag{3.6}$$

Using (2.5) and (2.6)to determine the individual terms of the decomposition series (2.4), we find

Table 1. The absolute difference between the present solution Φ_6 and the exact solution of the equation (3.1) with initial values (3.2) when $t = 0.1$.

y/x	0.2	0.4	0.6	0.8	1.0
0.2	1.07142E-03	1.00905E-03	3.73240E-04	3.08426E-04	5.00628E-04
0.4	1.00905E-03	9.38058E-04	2.91148E-04	3.83625E-04	5.40541E-04
0.6	3.73240E-04	2.91148E-04	1.65995E-04	5.59178E-04	4.98117E-04
0.8	3.08426E-04	3.83625E-04	5.59178E-04	5.81031E-04	2.92605E-04
1.0	5.00628E-04	5.40541E-04	4.98117E-04	2.92605E-04	4.67080E-05

$$u_0 = \sin(\pi x)\sin(\pi y),$$

$$u_1 = -2\pi^2 t \sin(\pi x)\sin(\pi y),$$

$$u_2 = \frac{4\pi^4 t^2}{2!}\sin(\pi x)\sin(\pi y),$$

$$\tag{3.7}$$

$$u_3 = \frac{-8\pi^6 t^3}{3!}\sin(\pi x)\sin(\pi y),$$

$$\vdots$$

and so on; in this manner the rest of components of the decomposition series (2.4) can be obtained. The solution for the convection-diffusion equation (3.5) in a series form is given by

$$u(x, y, t) = \sin(\pi x)\sin(\pi y)\left[1 - 2\pi^2 t + \frac{4\pi^4 t^2}{2!} - \frac{8\pi^6 t^3}{3!} + \dots\right]. \tag{3.8}$$

It can be easily observed that (3.8) is equivalent to the exact solution

$$u(x, y, t) = e^{-2\pi^2 t} \sin(\pi x) \sin(\pi y). \tag{3.9}$$

This can be verified through substitution.

It is worth noting that exact solution (3.8) is obtained by using the initial condition only. Moreover, the obtained solution can be used to justify the given boundary conditions. It is also worth to point out that the Adomian decomposition method does not require discretization of the variables, i.e. time and space, it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The approach is implemented directly in a straightforward manner without using restrictive assumptions or linearization.

Example 3 *We next consider the following nonhomogeneous Convection-diffusion problem*

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 3x^2 - 6x + 2t + 1, \quad (x, y, t) \in \Omega \times J, \tag{3.10}$$

subject to the initial conditions

$$u(x, y, 0) = x^3 + y. \tag{3.11}$$

Since the computations depends heavily on u_0, we will use the modified ADM (2.7) for defining the components of the decomposition series. This will ease the computation considerably. Thus

$$
\begin{aligned}
u_0 &= x^3 + y \\
u_1 &= 3tx^2 - 6tx + t^2 + t + L_t^{-1}\Big(- b_1(x, y)(u_0)_x - b_2(y)(u_0)_y + a_1 L_x u_0 + a_2 L_y u_0 \Big), \\
u_n &= L_t^{-1}\Big(- b_1(x, y)(u_{n-1})_x - b_2(y)(u_{n-1})_y + a_1 L_x u_{n-1} + a_2 L_y u_{n-1} \Big), \quad n \geq 2.
\end{aligned}
$$

$$\tag{3.12}$$

Solving these equations recursively we obtain

$$u_0 = x^3 + y,$$

$$u_1 = t^2 - t + 6xt - 3tx^2 + t - 6xt + 3tx^2,$$

$$u_2 = 0,$$

$$u_3 = 0,$$
(3.13)

$$\vdots$$

We observe the appearance of noise terms between the components of u_1. By canceling the noise terms from u_1 and verifying that the remaining terms of u_0 and u_1 justify the equation we obtain the exact solution in the form

$$u(x, y, t) = x^3 + y + t^2.$$
(3.14)

One important note to be made here is, we obtained the exact solution by using two components only. This is due to the fact that nonhomogeneeous equations may give rise to noise terms that accelerate the convergence of the solution, as presented in Example 3.

4. Conclusions

In conclusion, Adomian decomposition method was used for finding exact and approximate solutions of the convection-diffusion problems (1.1). The numerical results obtained justify the advantage of this methodology. The nonhomogeneous case was effectively handled by employing the effect of noise terms phenomenon, where the exact solution was obtained by using two components only. It may be concluded that Adomian methodology is a very powerful and efficient technique in finding exact and approximate solutions for wide classed of problems.

There are two important points to make here. First, as the decomposition method does not require discretization of the variable, i.e., time and space, it is not effected

by computation round off error and necessity of large computer and time. Second, the technique avoids the cumbersome of the computational methods.

The method was analyzed and tested on two homogeneous and one nonhomogeneous problems from the literature.

References

[1] Roach, P. J.: Computational Fluid Dynamics. Hermosa, Albuquerque, NM, 1976.

[2] Ismail, H. N. A., Elbarbary, E. M. E. and Salem, G. S. E.: Restrictive Taylors's approximation for solving convection-diffusion equation. Appl. Math. Comp. 147, 355-363 (2204).

[3] Karaa, S. and Zhang, J.: Higher order ADI method for solving unsteady convection-diffusion problems. J. Comp. Physics 198, 1-9 (2004).

[4] Rui, H.: An alternative direction iterative method with second-order upwind scheme for convection-diffusion equations. Intern. J. Computer Math. 80(4), 527-533 (2003).

[5] Rivera, W., Zhu, J. and Huddleston, D.: An efficient parallel algorithm with application to computational fluid dynamics. Comp. Math. Appl. 45, 165-188 (2003).

[6] Adomian, G.: A review of the decomposition method in applied mathematics. J. Math. Anal. Appl. 135, 501-544 (1988).

[7] Adomian, G.: Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic Publishers, Boston (1994).

[8] Seng, V., Abbaoui, K. and Cherruault, Y.: Adomian's polynomials for nonlinear operators. Math. Comput. Modelling 24(1), 59-65 (1996).

[9] Cherruault, Y.: Convergence of Adomian's method. Kybernetes 18, 31-38 (1989).

[10] Rèpaci, A.: Nonlinear dynamical systems: On the accuracy of Adomian's decomposition method. Appl. Mth. Lett. 3(3), 35-39 (1990).

[11] Cherruault, Y. and Adomian, G.: Decomposition methods: A new proof of convergence. Math. Comput. Modelling 18, 103-106 (1993).

[12] Abbaoui, K. and Cherruault, Y.: Convergence of Adomian's method applied to differential equations. Computers Math. Applic. 28(5), 103-109 (1994).

[13] Abbaoui, K. and Cherruault, Y.: New ideas for proving convergence of decomposition methods. Computers Math. Applic. 29(7), 103-108 (1995).

[14] Wazwaz, A. M.: A reliable modification of Adomian's decomposition method. Appl. Math. Comp. 92, 1-7 (1998).

Shaher MOMANI

Department of Mathematics,

Mutah University-JORDAN

e-mail: shahermm@yahoo.com

60