

1-1-2008

Some Characterizations of Rectifying Curves in the Euclidean Space E^4

KAZIM İLARSLAN

EMILIJA NESOVIC

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>



Part of the [Mathematics Commons](#)

Recommended Citation

İLARSLAN, KAZIM and NESOVIC, EMILIJA (2008) "Some Characterizations of Rectifying Curves in the Euclidean Space E^4 ," *Turkish Journal of Mathematics*: Vol. 32: No. 1, Article 3. Available at: <https://journals.tubitak.gov.tr/math/vol32/iss1/3>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact academic.publications@tubitak.gov.tr.

Some Characterizations of Rectifying Curves in the Euclidean Space \mathbb{E}^4

Kazım İlarıslan, Emilija Neřović

Abstract

In this paper, we define a rectifying curve in the Euclidean 4-space as a curve whose position vector always lies in orthogonal complement N^\perp of its principal normal vector field N . In particular, we study the rectifying curves in \mathbb{E}^4 and characterize such curves in terms of their curvature functions.

Key Words: Rectifying curve, Frenet equations, curvature.

1. Introduction

In the Euclidean 3-space, rectifying curves are introduced by B. Y. Chen in [1] as space curves whose position vector always lies in its rectifying plane, spanned by the tangent and the binormal vector fields T and B of the curve. Accordingly, the position vector with respect to some chosen origin, of a rectifying curve α in \mathbb{E}^3 , satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),$$

where $\lambda(s)$ and $\mu(s)$ are arbitrary differentiable functions in arclength parameter $s \in I \subset \mathbb{R}$.

The Euclidean rectifying curves are studied in [1, 2]. In particular, it is shown in [2] that there exist a simple relationship between the rectifying curves and the centrodes, which play some important roles in mechanics, kinematics as well as in differential

1991 *AMS Mathematics Subject Classification:* 53C50, 53C40.

geometry in defining the curves of constant precession. The rectifying curves are also studied in [2] as the extremal curves. In the Minkowski 3-space \mathbb{E}_1^3 , the rectifying curves are investigated in [4].

In this paper, in analogy with the Euclidean 3-dimensional case, we define the rectifying curve in the Euclidean space \mathbb{E}^4 as a curve whose position vector always lies in the orthogonal complement N^\perp of its principal normal vector field N . Consequently, N^\perp is given by

$$N^\perp = \{W \in \mathbb{E}^4 \mid \langle W, N \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{E}^4 . Hence N^\perp is a 3-dimensional subspace of \mathbb{E}^4 , spanned by the tangent, the first binormal and the second binormal vector fields T, B_1 and B_2 respectively. Therefore, the position vector with respect to some chosen origin, of a rectifying curve α in \mathbb{E}^4 , satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B_1(s) + \nu(s)B_2(s), \quad (1)$$

for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ in arclength function s . Next, we characterize rectifying curves in terms of their curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ and give the necessary and the sufficient conditions for arbitrary curve in \mathbb{E}^4 to be a rectifying. Moreover, we obtain an explicit equation of a rectifying curve in \mathbb{E}^4 .

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be arbitrary curve in the Euclidean space \mathbb{E}^4 . Recall that the curve α is said to be of unit speed (or parameterized by arclength function s) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product of \mathbb{E}^4 given by

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

for each $X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4) \in \mathbb{E}^4$. In particular, the norm of a vector $X \in \mathbb{E}^4$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along the unit speed curve α , where T, N, B_1 and B_2 denote respectively the tangent, the principal normal, the first binormal

and the second binormal vector fields. Then the Frenet formulas are given by (see [3])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \quad (2)$$

The functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ are called, respectively, the first, the second and the third curvature of the curve α . If $k_3(s) \neq 0$ for each $s \in I \subset \mathbb{R}$, the curve α lies fully in \mathbb{E}^4 . Recall that the unit sphere $\mathbb{S}^3(1)$ in \mathbb{E}^4 , centered at the origin, is the hypersurface defined by

$$\mathbb{S}^3(1) = \{X \in \mathbb{E}^4 \mid \langle X, X \rangle = 1\}.$$

3. Some Characterizations of Rectifying Curves in \mathbb{E}^4

In this section, we firstly characterize the rectifying curves in \mathbb{E}^4 in terms of their curvatures. Let $\alpha = \alpha(s)$ be a unit speed rectifying curve in \mathbb{E}^4 , with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. By definition, the position vector of the curve α satisfies the equation (1), for some differentiable functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$. Differentiating the equation (1) with respect to s and using the Frenet equations (2), we obtain

$$T = \lambda'T + (\lambda k_1 - \mu k_2)N + (\mu' - \nu k_3)B_1 + (\mu k_3 + \nu')B_2.$$

It follows that

$$\begin{aligned} \lambda' &= 1, \\ \lambda k_1 - \mu k_2 &= 0, \\ \mu' - \nu k_3 &= 0, \\ \mu k_3 + \nu' &= 0, \end{aligned} \quad (3)$$

and therefore

$$\begin{aligned} \lambda(s) &= s + c, \\ \mu(s) &= \frac{k_1(s)(s + c)}{k_2(s)}, \\ \nu(s) &= \frac{k_1(s)k_2(s) + (s + c)(k_1'(s)k_2(s) - k_1(s)k_2'(s))}{k_2^2(s)k_3(s)}, \end{aligned} \quad (4)$$

where $c \in \mathbb{R}$. In this way the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ are expressed in terms of the curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ of the curve α . Moreover, by using the last equation in (3) and relation (4), we easily find that the curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$ satisfy the equation

$$\frac{k_1(s)k_3(s)(s+c)}{k_2(s)} + \left(\frac{k_1(s)k_2(s) + (s+c)(k_1'(s)k_2(s) - k_1(s)k_2'(s))}{k_2^2(s)k_3(s)} \right)' = 0, \quad c \in \mathbb{R}. \quad (5)$$

Conversely, assume that the curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$, of an arbitrary unit speed curve α in \mathbb{E}^4 , satisfy the equation (5). Let us consider the vector $X \in \mathbb{E}^4$ given by

$$X(s) = \alpha(s) - (s+c)T(s) - \frac{k_1(s)(s+c)}{k_2(s)}B_1(s) - \frac{k_1(s)(k_2(s) - (s+c)k_2'(s)) + k_1'(s)k_2(s)(s+c)}{k_2^2(s)k_3(s)}B_2(s).$$

By using the relations (2) and (5), we easily find $X'(s) = 0$, which means that X is a constant vector. This implies that α is congruent to a rectifying curve. In this way, the following theorem is proved.

Theorem 3.1 *Let $\alpha(s)$ be unit speed curve in \mathbb{E}^4 , with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. Then α is congruent to a rectifying curve if and only if*

$$\frac{k_1(s)k_3(s)(s+c)}{k_2(s)} + \left(\frac{k_1(s)k_2(s) + (s+c)(k_1'(s)k_2(s) - k_1(s)k_2'(s))}{k_2^2(s)k_3(s)} \right)' = 0, \quad c \in \mathbb{R}.$$

In particular, assume that all the curvature functions $k_1(s)$, $k_2(s)$ and $k_3(s)$ of rectifying curve α in \mathbb{E}^4 , are constant and different from zero. Then equation (5) easily implies a contradiction. Hence we obtain the following theorem.

Theorem 3.2 *There are no rectifying curves lying fully in \mathbb{E}^4 , with non-zero constant curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$.*

Moreover, if two of the curvature functions are constant, we may consider the following cases.

Suppose that $k_1(s) = \text{constant} > 0$, $k_2(s) = \text{constant} \neq 0$ and $k_3(s)$ is non-constant function. By using the equation (5), we find differential equation

$$k_3'(s) - k_3^3(s)(s+c) = 0, \quad c \in \mathbb{R}.$$

The solution of the previous differential equation is given by

$$k_3(s) = \frac{1}{\sqrt{|-s^2 - 2cs - 2c_1|}}, \quad c, c_1 \in \mathbb{R}.$$

Similarly, assume that $k_2(s) = \text{constant} \neq 0$, $k_3(s) = k_3 = \text{constant} \neq 0$ and $k_1(s)$ is non-constant function. Then equation (5) implies differential equation

$$k_3^2 k_1(s)(s+c) + (k_1(s)(s+c))' = 0, \quad c \in \mathbb{R}, \quad k_3 \in \mathbb{R}_0,$$

whose solution has the form

$$k_1(s) = \frac{c_1}{e^{k_3^2 s}(s+c)}, \quad c_1 \in \mathbb{R}^+.$$

Finally, if $k_1(s) = \text{constant} > 0$, $k_3(s) = k_3 = \text{constant} \neq 0$ and $k_2(s)$ is non-constant function, by using equation (5) we get differential equation

$$k_3^2(s+c)/k_2(s) + ((s+c)/k_2(s))' = 0, \quad c \in \mathbb{R}, \quad k_3 \in \mathbb{R}_0.$$

The previous differential equation has the solution

$$k_2(s) = c_1 e^{k_3^2 s}(s+c), \quad c_1 \in \mathbb{R}^+.$$

In this way, we obtain the following theorem.

Theorem 3.3 *Let $\alpha = \alpha(s)$ be unit speed curve in \mathbb{E}^4 , with curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. Then α is congruent to a rectifying curve if*

(a) $k_1(s) = \text{constant} > 0$, $k_2(s) = \text{constant} \neq 0$ and $k_3(s) = 1/\sqrt{|-s^2 - 2cs - 2c_1|}$, $c, c_1 \in \mathbb{R}$;

(b) $k_2(s) = \text{constant} \neq 0$, $k_3(s) = k_3 = \text{constant} \neq 0$ and $k_1(s) = c_1/(e^{k_3^2 s}(s+c))$, $c \in \mathbb{R}$, $c_1 \in \mathbb{R}^+$;

(c) $k_1(s) = \text{constant} > 0$, $k_3(s) = k_3 = \text{constant} \neq 0$ and $k_2(s) = c_1 e^{k_3^2 s}(s+c)$, $c \in \mathbb{R}$, $c_1 \in \mathbb{R}^+$.

In the next theorem, we give the necessary and the sufficient conditions for the curve α in \mathbb{E}^4 to be a rectifying curve.

Theorem 3.4 *Let $\alpha(s)$ be unit speed rectifying curve in \mathbb{E}^4 , with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. Then the following statements hold:*

(i) The distance function $\rho(s) = \|\alpha(s)\|$ satisfies $\rho^2(s) = s^2 + c_1s + c_2$, $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}_0$.

(ii) The tangential component of the position vector of the curve is given by $\langle \alpha(s), T(s) \rangle = s + c$, $c \in \mathbb{R}$.

(iii) The normal component $\alpha^N(s)$ of the position vector of the curve has constant length and the distance function $\rho(s)$ is non-constant.

(iv) The first binormal component and the second binormal component of the position vector of the curve are respectively given by

$$\begin{aligned} \langle \alpha(s), B_1(s) \rangle &= \frac{k_1(s)(s+c)}{k_2(s)}, \\ \langle \alpha(s), B_2(s) \rangle &= \frac{k_1(s)k_2(s) + (s+c)(k_1'(s)k_2(s) - k_1(s)k_2'(s))}{k_2^2(s)k_3(s)}, \quad c \in \mathbb{R}. \end{aligned} \quad (6)$$

Conversely, if $\alpha(s)$ is a unit speed curve in \mathbb{E}^4 with non-zero curvatures $k_1(s)$, $k_2(s)$, $k_3(s)$ and one of the statements (i), (ii), (iii) or (iv) holds, then α is a rectifying curve.

Proof. Let us first suppose that $\alpha(s)$ is a unit speed rectifying curve in \mathbb{E}^4 with non-zero curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$. The position vector of the curve α satisfies the equation (1), where the functions $\lambda(s)$, $\mu(s)$ and $\nu(s)$ satisfy relation (3). Multiplying the third equation in (3) with $-\nu'(s)$ and the last equation in (3) with $\mu'(s)$ and adding, we get $k_3(s)(\mu(s)\mu'(s) + \nu(s)\nu'(s)) = 0$. It follows that $\mu(s)\mu'(s) + \nu(s)\nu'(s) = 0$ and consequently

$$\mu^2(s) + \nu^2(s) = a^2, \quad (7)$$

for some constant $a \in \mathbb{R}_0^+$. From relation (1) we have $\langle \alpha(s), \alpha(s) \rangle = \lambda^2(s) + \mu^2(s) + \nu^2(s)$, which together with (4) and (7) gives $\langle \alpha(s), \alpha(s) \rangle = (s+c)^2 + a^2$. Therefore, $\rho^2(s) = s^2 + c_1s + c_2$, $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}_0$, which proves statement (i).

But using the relations (1) and (4) we easily get $\langle \alpha(s), T(s) \rangle = s + c$, $c \in \mathbb{R}$, so the statement (ii) is proved.

Note that the position vector of an arbitrary curve α in \mathbb{E}^4 can be decomposed as $\alpha(s) = m(s)T(s) + \alpha^N(s)$, where $m(s)$ is arbitrary differentiable function and $\alpha^N(s)$ is the normal component of the position vector. If α is a rectifying curve, relation (1) implies $\alpha^N(s) = \mu(s)B_1(s) + \nu(s)B_2(s)$ and therefore $\langle \alpha^N(s), \alpha^N(s) \rangle = \mu^2(s) + \nu^2(s)$. Moreover, by using (7), we find $\|\alpha^N(s)\| = a$, $a \in \mathbb{R}_0^+$. By statement (i), $\rho(s)$ is non-constant function, which proves statement (iii).

Finally, using (1) and (4) we easily obtain (6), which proves statement (iv).

Conversely, assume that statement (i) holds. Then $\langle \alpha(s), \alpha(s) \rangle = s^2 + c_1s + c_2$, $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}_0$. Differentiating the previous equation two times with respect to s and using (2), we obtain $\langle \alpha(s), N(s) \rangle = 0$, which implies that α is a rectifying curve.

If statement (ii) holds, in a similar way it follows that α is a rectifying curve.

If statement (iii) holds, let us put $\alpha(s) = m(s)T(s) + \alpha^N(s)$, where $m(s)$ is arbitrary differentiable function. Then

$$\langle \alpha^N(s), \alpha^N(s) \rangle = \langle \alpha(s), \alpha(s) \rangle - 2 \langle \alpha(s), T(s) \rangle m(s) + m^2(s).$$

Since $\langle \alpha(s), T(s) \rangle = m(s)$, it follows that

$$\langle \alpha^N(s), \alpha^N(s) \rangle = \langle \alpha(s), \alpha(s) \rangle - \langle \alpha(s), T(s) \rangle^2,$$

where $\langle \alpha(s), \alpha(s) \rangle = \rho^2(s) \neq \text{constant}$. Differentiating the previous equation with respect to s and using (2), we find

$$k_1(s) \langle \alpha(s), T(s) \rangle \langle \alpha(s), N(s) \rangle = 0.$$

It follows that $\langle \alpha(s), N(s) \rangle = 0$ and hence the curve α is a rectifying.

If statement (iv) holds, by taking the derivative of the equation

$$\langle \alpha(s), B_1(s) \rangle = \frac{k_1(s)(s+c)}{k_2(s)}$$

with respect to s and using (2), we obtain

$$-k_2(s) \langle \alpha(s), N(s) \rangle + k_3(s) \langle \alpha(s), B_2(s) \rangle = \left(\frac{k_1(s)(s+c)}{k_2(s)} \right)'$$

By using (6), the last equation becomes $\langle \alpha(s), N(s) \rangle = 0$, which means that α is a rectifying curve. This proves the theorem. \square

In the next theorem, we find the parametric equation of a rectifying curve.

Theorem 3.5 *Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be a curve in \mathbb{E}^4 given by $\alpha(t) = \rho(t)y(t)$, where $\rho(t)$ is arbitrary positive function and $y(t)$ is a unit speed curve in the unit sphere $\mathbb{S}^3(1)$. Then α is a rectifying curve if and only if*

$$\rho(t) = \frac{a}{\cos(t+t_0)}, \quad a \in \mathbb{R}_0, \quad t_0 \in \mathbb{R}. \quad (8)$$

Proof. Let α be a curve in \mathbb{E}^4 given by

$$\alpha(t) = \rho(t)y(t),$$

where $\rho(t)$ is arbitrary positive function and $y(t)$ is a unit speed curve in $\mathbb{S}^3(1)$. By taking the derivative of the previous equation with respect to t , we get

$$\alpha'(t) = \rho'(t)y(t) + \rho(t)y'(t).$$

Hence the unit tangent vector of α is given by

$$T(t) = \frac{\rho'(t)}{v(t)}y(t) + \frac{\rho(t)}{v(t)}y'(t), \quad (9)$$

where $v(t) = \|\alpha'(t)\|$ is the speed of α . Differentiating the equation (9) with respect to t , we find

$$T' = \left(\frac{\rho'}{v}\right)'y + \left(\frac{2\rho'}{v} - \frac{\rho\rho'(\rho + \rho'')}{v^3}\right)y' + \left(\frac{\rho}{v}\right)y''. \quad (10)$$

Let Y be the unit vector field in \mathbb{E}^4 satisfying the equations $\langle Y, y \rangle = \langle Y, y' \rangle = \langle Y, y \times y' \rangle = 0$. Then $\{y, y', y \times y', Y\}$ is the orthonormal frame of \mathbb{E}^4 . Therefore, decomposition of y'' with respect to the frame $\{y, y', y \times y', Y\}$ reads

$$y'' = \langle y'', y \rangle y + \langle y'', y' \rangle y' + \langle y'', y \times y' \rangle y \times y' + \langle y'', Y \rangle Y. \quad (11)$$

Since $\langle y, y \rangle = \langle y', y' \rangle = 1$, it follows that $\langle y'', y \rangle = -1$ and $\langle y'', y' \rangle = 0$, so the equation (11) becomes

$$y'' = -y + \langle y'', y \times y' \rangle y \times y' + \langle y'', Y \rangle Y. \quad (12)$$

Substituting (12) into (10) and applying Frenet formulas for arbitrary speed curves in \mathbb{E}^4 , we find

$$\begin{aligned} \kappa_1 v N = & \left(\left(\frac{\rho'}{v}\right)' - \frac{\rho}{v} \right) y + \left(\frac{2\rho'}{v} - \frac{\rho\rho'(\rho + \rho'')}{v^3} \right) y' + \frac{\langle y'', y \times y' \rangle}{v} \alpha \times y' \\ & + \left(\frac{\rho}{v} \right) \langle y'', Y \rangle Y. \end{aligned} \quad (13)$$

Since $\langle y, y \rangle = 1$, we have $\langle y, y' \rangle = 0$ and thus $\langle \alpha, y' \rangle = 0$. We also have $\langle \alpha, Y \rangle = 0$. By definition, α is a rectifying curve in \mathbb{E}^4 if and only if $\langle \alpha, N \rangle = 0$.

Therefore, after taking the scalar product of (13) with α , we have $\langle \alpha, N \rangle = 0$ if and only if

$$\left(\frac{\rho'}{v}\right)' - \frac{\rho}{v} = 0.$$

The previous differential equation is equivalent to the equation

$$\rho\rho'' - 2\rho'^2 - \rho^2 = 0. \tag{14}$$

whose nontrivial solutions are given by (8). This proves the theorem. \square

Example: Let us consider the curve $\alpha(s) = (a/(\sqrt{2}\cos(s+s_0)))(\sin(s), \cos(s), \sin(s), \cos(s))$, $a \in \mathbb{R}_0$, $s_0 \in \mathbb{R}$ in \mathbb{E}^4 . This curve has a form $\alpha(s) = \rho(s)y(s)$, where $\rho(s) = a/\cos(s+s_0)$ and $y(s) = (1/\sqrt{2})(\sin(s), \cos(s), \sin(s), \cos(s))$. Since $\|y(s)\| = 1$ and $\|y'(s)\| = 1$, $y(s)$ is a unit speed curve in the unit sphere $\mathbb{S}^3(1)$. According to the theorem 3.5, $\alpha(s)$ is a rectifying curve lying fully in \mathbb{E}^4 .

Acknowledgement

The authors are very grateful to the referee for his/her useful comments and suggestions which improved the first version of the paper.

References

- [1] Chen, B. Y.: When does the position vector of a space curve always lie in its rectifying plane?, Amer. Math. Monthly **110**, 147-152 (2003).
- [2] Chen, B. Y., Dillen, F.: Rectifying curves as centrodes and extremal curves, Bull. Inst. Math. Academia Sinica **33**, No. 2, 77-90 (2005).
- [3] Gluck, H.: Higher curvatures of curves in Euclidean space, Amer. Math. Monthly **73**, 699-704, (1966).
- [4] İlarıslan, K., Neřović, E., Petrović-Torgařev, M.: Some characterizations of rectifying curves in the Minkowski 3-space, Novi Sad J. Math. Vol. **33**, No. 2, 23-32, (2003).
- [5] Millman, R. S., Parker, G. D.: Elements of differential geometry, Prentice-Hall, New Jersey, 1977.

İLARSLAN, NEŠOVIĆ

- [6] Struik, D. J.: Differential geometry, second ed., Addison-Wesley, Reading, Massachusetts, 1961.

Kazım İLARSLAN
Kırıkkale University
Faculty of Sciences and Arts
Department of Mathematics
Kırıkkale-TURKEY
e-mail: kilarslan@yahoo.com

Received 21.09.2006

Emilija NEŠOVIĆ
Faculty of Science
Department of mathematics and informatics
Radoja Domanovića 12
34 000 Kragujevac SERBIA
e-mail: emines@ptt.yu