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Trace Classes and Fixed Points for the Extended Modular Group $\overline{\Gamma}$

Özden Koruoğlu, Recep Şahin and Sebahattin İkkikardeş

Abstract

The extended modular group $\overline{\Gamma} = PGL(2, \mathbb{Z})$ is the group obtained by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group $\Gamma = PSL(2, \mathbb{Z})$. In this paper, we find the trace classes of the extended modular group $\overline{\Gamma}$. Using this, we classify the elements of $\overline{\Gamma}$.

Key Words: Extended modular group, trace class, fixed points

1. Introduction

$PSL(2, \mathbb{R})$ is the group of all conformal automorphisms of the upper half plane \mathcal{U} , i.e.,

$$PSL(2, \mathbb{R}) = \left\{ z \mapsto \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

By adding all anti-conformal automorphisms of \mathcal{U} to the $PSL(2, \mathbb{R})$, we obtain the group $G = PSL(2, \mathbb{R}) \cup G'$ where

$$G' = \left\{ z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d} : a, b, c, d \in \mathbb{R}, ad - bc = -1 \right\}.$$

The modular group $\Gamma = PSL(2, \mathbb{Z})$ is generated by two linear fractional transforma-

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tions

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + 1.$$

Let $S = TU$, i.e.

$$S(z) = -\frac{1}{z+1}.$$

Then the modular group Γ is isomorphic to the free product of two finite cyclic groups of orders 2 and 3 and it has a presentation

$$\Gamma = \langle T, S \mid T^2 = S^3 = I \rangle \cong C_2 * C_3.$$

The extended modular group $\bar{\Gamma} = PGL(2, \mathbb{Z})$ is defined by adding the reflection $R(z) = 1/\bar{z}$ to the generators of the modular group Γ (see [2, 4 and 10]). Thus the extended modular group has the presentation

$$\bar{\Gamma} = \langle T, S, R \mid T^2 = S^3 = R^2 = (TR)^2 = (SR)^2 = I \rangle \cong D_2 *_{\mathbb{Z}_2} D_3. \quad (1.1)$$

It is well-known that the extended modular group $PGL(2, \mathbb{Z})$ is equal to $GL(2, \mathbb{Z})/\{\pm I\}$ and the modular group $PSL(2, \mathbb{Z})$ is equal to $SL(2, \mathbb{Z})/\{\pm I\}$. (Throughout this paper, we identify each matrix A in $GL(2, \mathbb{Z})$ with $-A$, so that they each represent the same element of $PGL(2, \mathbb{Z})$). Thus we can represent the generators of the extended modular group $\bar{\Gamma}$ as

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, the extended modular group $\bar{\Gamma} = PGL(2, \mathbb{Z})$ is $PSL(2, \mathbb{Z}) \cup M'$, where

$$M' = \{z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d} : a, b, c, d \in \mathbb{Z}, ad - bc = -1\}.$$

The modular group $PSL(2, \mathbb{Z})$, and its normal subgroups, have especially been of great interest in many fields of Mathematics, for example number theory, automorphic function theory and group theory (see for example [5, 6 and 7]). The extended modular

group $\bar{\Gamma}$ was intensively studied. For the examples of these studies, see [2, 8 and 10]. In [8], we have investigated the power and free subgroups of the extended modular group $\bar{\Gamma}$.

We mention here types of the elements in the extended modular group $\bar{\Gamma}$. In standard terminology, a point $z \in \mathbb{C} \cup \{\infty\}$ is called a *fixed point* of $V(z) \in \bar{\Gamma} = \Gamma \cup M'$, if $V(z) = z$, and the trace of $V(z)$ is defined by $tr(V) = a + d$. If we take $V(z) \in \bar{\Gamma}$ then $V(z)$ has the matrix presentation $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$. There is a relation between the fixed points and the trace of a transformation of $\bar{\Gamma}$. Thus we can determine fixed points location in $\mathbb{C} \cup \{\infty\}$ with trace.

If $V(z) \in \Gamma$, then the number of fixed points of $V(z)$ is at most two. Also, if

- $|tr(V)| > 2$ then there are two fixed points in $\mathbb{R} \cup \{\infty\}$ and $V(z)$ is called a hyperbolic element.
- $|tr(V)| = 2$ then there is one fixed point in $\mathbb{R} \cup \{\infty\}$ and $V(z)$ is called a parabolic element.
- $|tr(V)| < 2$ then there are two conjugate fixed points in $\mathbb{C} \cup \{\infty\}$ and $V(z)$ is called an elliptic element.

If $V(z) \in M'$, then it has two fixed points or the set of fixed points is a circle. Also, if

- $tr(V) \neq 0$, then there are two distinct fixed points on the $\mathbb{R} \cup \{\infty\}$ and $V(z)$ is called a glide reflection.
- $tr(V) = 0$, then the set of fixed points is the circle of radius $\frac{1}{|c|}$ centered at $(\frac{a}{c}, 0)$ and $V(z)$ is called a reflection.

In [1], Fine studied trace classes in the modular group Γ . He gave an effective algorithm to determine for each integer d a complete set of representatives for the trace classes in trace d . This algorithm has been extended by Schmidt and Sheingorn to the general Hecke groups in [9].

In this paper, we find the trace classes of the extended modular group $\bar{\Gamma}$. To do this, we will use the notations and the method used for modular group Γ in [1]. Additionally, using this we classify the elements of $\bar{\Gamma}$. Finally, we give the types of the elements of $\bar{\Gamma}''$ as an application of this classification.

2. Trace Classes in the Extended Modular Group

From (1.1), we know that the extended modular group $\bar{\Gamma}$ is a free product with amalgamation as $\bar{\Gamma} = D_2 *_{\mathbb{Z}_2} D_3$. Each element of a free product with amalgamation has a normal form. Thus if $g \in \bar{\Gamma}$, then g has one of two representations as a reduced word $W(T, S, R)$ in T, S and R . That is either $g = T^{a_1} S^{b_1} \dots T^{a_n} S^{b_n}$ or $g = T^{a_1} S^{b_1} \dots T^{a_n} S^{b_n} R$, where $a_1 = 0$ or 1 , $a_i = 1$, for $i = 2, \dots, n$ and $b_i = 0, 1$ or 2 for $i = 1, 2, \dots, n$. Note that these results can be obtained by the presentation of $\bar{\Gamma}$.

To find trace classes we need the following transformations:

$$TS : z \mapsto z + 1, \quad TS^2 : z \mapsto \frac{z}{z+1}, \quad R : z \mapsto \frac{1}{z}.$$

Conjugate matrices have the same trace. The converse is not true. For example, S and $(TS)R$ have the same trace, but these elements are not the conjugate. Thus the conjugacy classes in $\bar{\Gamma}$ are partitioned by trace.

Now let us try to determine specific representatives for each trace class.

A reduced word $W(T, S, R) \in \bar{\Gamma}$ is called a *cyclically reduced word* if $W \neq W_1^{-1} W_2 W_1$ for other non-trivial words W_1, W_2 . Here we will only concentrate on cyclically reduced words. A cyclically reduced word in $\bar{\Gamma}$ is equivalent to $W(T, S, R)$ not beginning with T and ending with T , or beginning with S and ending with S^2 , or beginning with S^2 and ending with S , or beginning with R and ending with R . Certainly, every element of $\bar{\Gamma}$ is conjugate to a word in cyclically reduced form. If two words W_1, W_2 are cyclically reduced then they are conjugate if and only if W_1 is a cyclic permutation of W_2 [3].

The word $W(T, S, R)$ in $\bar{\Gamma}$ is called a *block reduced form*, abbreviated as *BRF*, if $W(T, S, R)$ begins with T and ends with S , or S^2 , or R . Also, a piece of the form (TS) or (TS^2) is called a *block*. If W is in *BRF* then its *block length*, denoted $BL(W)$, is the number of blocks in W . For example, if $W = (TS)^3 (TS^2)^2 (TS)$ then $BL(W) = 6$, and if $W = (TS)^2 (TS^2)^5 (TS)R$ then $BL(W) = 8$.

Firstly, we let us give some results about the conjugacy classes of the elements in $\bar{\Gamma}$.

Lemma 2.1 ([11]) *There are four conjugacy classes of finite order elements in $\bar{\Gamma}$; three for those of order 2 and one for those of order 3. Explicitly they are $\{S\}$ in order 3 with determinant 1, $\{T\}$ in order 2 with determinant 1 and $\{R\}, \{TR\}$ in order 2 with determinant -1.*

Lemma 2.2 *The blocks (TS) and (TS^2) are not conjugate in Γ but they are conjugate in $\bar{\Gamma}$ with R .*

Lemma 2.3 *Every element of $\bar{\Gamma}$ is conjugate to either T , or R , or TR , or S , or to a word in BRF .*

Proof. We know that every element of $\bar{\Gamma}$ is conjugate to a cyclically reduced word. Thus, we will concentrate on cyclically reduced words. Let $g = W(T, S, R)$ be cyclically reduced and not equal to T , or R , or TR , or S , or their conjugate. If $g = W(T, S, R)$ begins T then it must be end S , or S^2 , or R since $g = W(T, S, R)$ is cyclically reduced word. Thus $g = W(T, S, R)$ must be in block reduced word. If $g = W(T, S, R)$ begins with R then g must be followed by T , or S , or S^2 . Therefore there is a word W_1 which is cyclic permutation of W beginning with these: T , or S , or S^2 . Also, W_1 is conjugate to $g = W(T, S, R)$. Thus W_1 must be a cyclically reduced word. In the last case, if $g = W(T, S, R)$ begins S or S^2 , it must then be followed by T or R . Similarly, W is equivalent to a cyclically reduced word which begins T and must end S , or S^2 , or R . Therefore, every element $g = W(T, S, R)$ of $\bar{\Gamma}$ is conjugate to T , or R , or TR , or S , or a word in BRF . \square

We note that a block reduced word is of the form either

$$(TS)^{a_1}(TS^2)^{b_1}\dots(TS)^{a_k}(TS^2)^{b_k}$$

or

$$(TS)^{a_1}(TS^2)^{b_1}\dots(TS)^{a_k}(TS^2)^{b_k}R.$$

A word $W(T, S, R)$ is called a *standard block reduced form*, abbreviated as *SBRF*, if it has one of the following forms:

- (i) $W = (TS)^n$ for some integer n ;
- (ii) $W = (TS^2)^n$ for some integer n ;
- (iii) $W = ((TS)^n(TS^2)^k)^t$ for integers n, k, t ;
- (iv) $W = (TS)^{a_1}(TS^2)^{b_1}\dots(TS)^{a_k}(TS^2)^{b_k}$, where $a_1 = \max\{a_i\}$. If $a_1 = a_i$ for some i , then $b_1 \geq b_i$. If $b_1 = b_i$, then $b_2 \geq b_{i+1}$, and so on;
- (v) $W = (TS)^nR$ for some integer n ;
- (vi) $W = (TS^2)^nR$ for some integer n ;

(vii) $W = ((TS)^n(TS^2)^k)^t R$ for integers n, k, t ;

(viii) $W = (TS)^{a_1}(TS^2)^{b_1} \dots (TS)^{a_k}(TS^2)^{b_k} R$, where $a_1 = \max\{a_i\}$. If $a_1 = a_i$ for some i then $b_1 \geq b_i$. If $b_1 = b_i$, then $b_2 \geq b_{i+1}$ and so on.

Lemma 2.4 *The trace classes in $\bar{\Gamma}$ are in one to one correspondence with words in SBRF words as well as $\{T\}, \{R\}, \{TR\}, \{S\}$.*

Lemma 2.5 *If $W(T, S, R)$ in BRF is a word in $\bar{\Gamma}$ with $BL(W) \geq 1$, then the transformation for W has only positive entries.*

Proof. If the sum of the exponents of R in $W(T, S, R)$ is even (i.e. $W(T, S, R) = W(T, S)$) it is proved in [1]. Suppose the sum of the exponents of R is odd. Then the form of $W(T, S, R)$ is $W = W_1 R$, where W_1 is one of the above forms (i), (ii), (iii) and (iv). In [1] it is shown that W_1 has only positive entries, i.e., it has a matrix representation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d > 0$. Since $W = W_1 R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$, W has only positive entries. \square

Theorem 2.6 ([1]) *Let $W(T, S, R)$ be a word in $\bar{\Gamma}$ such that the sum of exponents of R is even and different from $(TS)^n, (TS^2)^n$ is in BRF, and if $BL(W) = n$, then $tr(W) \geq n+1$.*

Theorem 2.7 *Let $W(T, S, R)$ be a word in $\bar{\Gamma}$ such that the sum of exponents of R is odd in BRF and if $BL(W) = n$, then $tr(W) \geq n$.*

Proof. The proof is done by induction on the block length. It is clear that the form of $W = W(T, S, R)$ is $W_1 R$ where W_1 is one of the above forms (i), (ii), (iii) and (iv). If $BL(W) = 1$, then $W = (TS)R$ or $W = (TS^2)R$. Thus we obtain as $(TS)R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

and $(TS^2)R = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Therefore we find $tr(W) = 1$.

Suppose that $W = W_1 R = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$ in BRF has block length n , where

$W_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a, b, c, d are positive entries (from Lemma 2.5 there are such positive

entries) and $tr(W) = b + c \geq n$.

Let the block length of W' be $n+1$. The element W' is obtained by appending (TS) or (TS^2) to W_1R . The form of word W' is either $W_1R(TS)$ or $W_1R(TS^2)$. These words are $W_1(TS^2)R$ and $W_1(TS)R$, respectively. Thus, from the relations in $\bar{\Gamma}$ and the inductive hypothesis, we have

$$W' = W_1R(TS) = \begin{pmatrix} b & a \\ d & c \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a+b \\ d & c+d \end{pmatrix},$$

$$tr(W') = b + c + d \geq n + d \geq n + 1$$

and

$$W' = W_1R(TS^2) = \begin{pmatrix} b & a \\ d & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a+b & a \\ c+d & c \end{pmatrix}$$

$$tr(W') = a + b + c \geq n + a \geq n + 1.$$

□

Now each element of the extended modular group $\bar{\Gamma}$ belongs to only one trace class. Thus the trace classes are determined in the next two theorems. In these theorems, we will give the trace classes for the words $W(T, S, R)$ in which the sum of exponents of R is even, i.e., $W(T, S, R) = W(T, S)$ and the trace classes for the words $W(T, S, R)$ in which the sum of exponents of R is odd.

For a given positive trace, the procedure is as follows.

Theorem 2.8 1) If $tr(W) = 0$ the representative is $\{T\}$,

2) If $tr(W) = 1$ the representative is $\{S\}$,

3) If $tr(W) = 2$ there are infinite trace classes. The distinct words $(TS)^n$ as n runs over the positive integers give the representatives.

4) If $tr(W) > 2$ then: List all words in SBRF of block length $(tr(W) - 1)$ or less. (Equivalently, list all standard block reduced sequences whose sum is $(tr(W) - 1)$ or less.)

Theorem 2.9 1) If $\text{tr}(W) = 0$, the representatives are $\{R\}$ and $\{TR\}$;
 2) If $\text{tr}(W) = 1$, the representative is $\{(TS)R\}$,
 3) If $\text{tr}(W) > 1$, the representatives are the words in $SBRF$ of block length $\text{tr}(W)$ or less.

The following Corollaries give the type of the word $W(T, S, R)$. If the sum of exponents of R is even, then we have the following corollary.

Corollary 2.10 (i) If an element of the extended modular group $\bar{\Gamma}$ in the trace classes is $\{T\}$ or $\{S\}$ then it is an elliptic element.

(ii) If an element of the extended modular group $\bar{\Gamma}$ in the trace class is $\{(TS)^n\}$ then it is a parabolic element.

(iii) If an element of the extended modular group $\bar{\Gamma}$ belongs to a trace class different from the above (i) and (ii), then it is a hyperbolic element.

If the sum of exponents of R is odd, then we have this corollary:

Corollary 2.11 If an element is in the trace classes $\{R\}$ or $\{TR\}$ then it is a reflection, in other case it is a glide reflection.

Now, as a result of the above theorems, we can give the following example .

Example 2.1 From [8], the presentation of the second commutator subgroup $\bar{\Gamma}''$ of $\bar{\Gamma}$ is

$$\bar{\Gamma}'' = \langle [S, TST], [S, TS^2T], [S^2, TST], [S^2, TS^2T] \rangle,$$

where $[a, b] = aba^{-1}b^{-1}$. Therefore, it can be seen that the length of all the generators of $\bar{\Gamma}''$ is 4 and also there is no relation between the elements T and S . Since every element of $\bar{\Gamma}''$ is obtained from the generators of $\bar{\Gamma}''$, we find the block length of every element of $\bar{\Gamma}''$ greater than or equal 4. Therefore, $\bar{\Gamma}''$ does not contain an elliptic element or a parabolic element. So equivalently, $\bar{\Gamma}''$ contains the only hyperbolic elements.

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