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Diametral Dimension and Köthe Spaces

Tosun Terzioğlu

Abstract

We generalize some well-known results about the diametral dimension of classical Köthe spaces.

In this note we will call a Banach space $(\ell, \|\cdot\|)$ of scalar sequences *admissible*, if it satisfies the following conditions:

- (i) for $a \in \ell_\infty$, $x \in \ell$ the sequence $ax = (a_n x_n) \in \ell$ and $\|ax\| \leq \|a\|_\infty \|x\|$
- (ii) $\|e_n\| = 1$ for $n \in \mathbf{N} = \{0, 1, 2, \dots\}$.

As usual e_n denotes the sequence with 1 at n -th place and zero elsewhere. The classical spaces ℓ_p , $1 \leq p \leq \infty$ and c_0 are the best known examples of admissible sequence spaces. One can construct another class of admissible spaces by using Orlicz functions. For an admissible sequence space ℓ let us define its α -dual by $\ell^\alpha = \{u : ux \in \ell_1, \forall x \in \ell\}$. With the usual dual norm the space ℓ^α is also admissible.

It is easy to see that the norm of an admissible space is *monotone*. More precisely, if $x \in \ell$ and $|y_n| \leq |x_n|$, $n \in \mathbf{N}$, then $y \in \ell$ and $\|y\| \leq \|x\|$. Further, if $|x| = (|x_n|)$, then $\||x|\| = \|x\|$.

If $\|\cdot\|$ is a monotone norm with $\|e_n\| = 1$, $n \in \mathbf{N}$, defined on the space φ of sequences with finitely many non-zero terms, then the completion of $(\varphi, \|\cdot\|)$ is an admissible sequence space. In this case (e_n) is a basis of ℓ . However ℓ_∞ is an example of a non-separable admissible space.

A set of non-negative sequences A is called a *Köthe set* if

- (i) $\forall j \in \mathbf{N} \exists a \in A$ with $a_j > 0$;

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(ii) for $a, b \in A$ there is a $c \in A$ with $\max\{a_n, b_n\} \leq c_n$.

Let $(\ell, \|\cdot\|)$ be an admissible space. We define $K^\ell(A)$ to be the space of all sequences $x = (x_n)$ such that $xa = (x_n a_n) \in \ell$ for every $a \in A$. With the seminorms defined by $\|x\|_a = \|xa\|$, $a \in A$, $K^\ell(A)$ is a complete locally convex space and if A is countable, a Fréchet space. The usual definition of a Köthe space $K(A)$ is $K^{\ell_1}(A)$, but also the spaces $K^{\ell_p}(A)$, $1 \leq p \leq \infty$ or $K^{c_0}(A)$ have an extensive literature (see for example [6], [2], [7], [8]). Dragilev [1] considered Köthe spaces, where the underlying space ℓ is abstractly defined. Such general Köthe spaces reappeared recently in [4] and [5]. In this note we consider the diametral dimension of $K^\ell(A)$, where ℓ is an admissible space. All our results are well-known in classical case, that is when $\ell = \ell_1$ ([7], [8]). The author is grateful to V. P. Zahariuta for introducing him to Dragilev's work [1].

We recall first the definition of n -th diameter

$$d_n(V, U) = \inf \inf \{ \alpha > 0 : V \subset \alpha U + L \},$$

where $V \subset \rho U$, $\rho > 0$ and V, U are absolutely convex, closed neighborhoods of a locally convex space E . The second infimum is taken over of subspaces L of E with dimension not exceeding $n \in \mathbf{N}$. The set $\Delta(E)$ of all (ξ_n) , such that for every U there is a V with $\lim \xi_n d_n(V, U) = 0$, is called the *diametral dimension* of E ([8]).

Let A be a Köthe set, $a, b \in A$ with $a_n \leq b_n$ for all $n \in \mathbf{N}$. We will always assume $a_n/b_n = 0$ if $b_n = 0$. Let

$$U_a = \{x \in K^\ell(A) : \|x\|_a = \|xa\| \leq 1\}.$$

With this notation we have the following estimate which is already known in the case $\ell = \ell_1$ (cf. [8]).

Proposition 1 *Let $J \subset \mathbf{N}$ with $|J| = n + 1$ and $a_n > 0$ for all $n \in J$. Let $I \subset \mathbf{N}$ with $|I| \leq n$. Then*

$$\inf \left\{ \frac{a_j}{b_j} : j \in J \right\} \leq d_n(U_b, U_a) \leq \sup \left\{ \frac{a_i}{b_i} : i \notin I \right\}$$

Proof. Let us define $P_I : K^\ell(A) \rightarrow K^\ell(A)$ by

$$P_I(x) = \sum_{n \in I} x_n e_n.$$

If $x \in U_b$, then by monotonicity of the norm of ℓ we have $(x - P_I(x)) \in U_b$ also. It is easily seen that

$$\|x - P_I(x)\|_a \leq \sup\{a_i/b_i : i \notin I\} \|x - P_I(x)\|_b.$$

In particular, we have

$$U_b \subset \sup\{a_i/b_i : i \notin I\}U_a + P_I(K^\ell(A)).$$

This inclusion implies the right hand side. Let $s = \inf\{a_i/b_i : i \in J\}$ and assume there is a subspace L of dimension n and $0 < s_0 < s$ such that

$$U_b \subset s_0 U_a + L.$$

Let $M = \{x \in K^\ell(A) : x = P_J(x)\}$. If $x \in sU_a \cap M$, by monotonicity we have $\|x\|_q \leq 1$. So

$$U_a \cap M \subset (s_0/s)U_a + L.$$

Note that $\|\cdot\|_a$ is a norm on M and $P_J(U_a) \subset U_a$ by monotonicity. So this gives

$$U_a \cap M \subset (s_0/s)U_a \cap M + P_J(L).$$

Take any $x \in M$ with $\|x\|_a = 1$. From the above inclusion we find $y_1 \in P_J(L)$ with $\|x - y_1\|_a \leq s_0/s < 1$. Repeating this we find $y_k \in P_J(L)$ with

$$\|x - y_1 - \cdots - y_k\| \leq (s_0/s)^k.$$

Hence we have $x \in P_J(L)$ or $M = P_J(L)$, but dimension of M is $(n + 1)$. From this contradiction we get the left-hand side. \square

We recall that a locally convex space E is a Schwartz space if and only if for each U there is a V with $\lim d_n(V, U) = 0$.

For $k \in \mathbb{N}$ let $P_k(x) = \sum_{n=0}^k x_n e_n$, $x \in K^\ell(A)$. With this notation we have the following result.

Proposition 2 *The following are equivalent*

- (i) $K^\ell(A)$ is a Schwartz space

(ii) for every $a \in A$ there is $b \in A$ with $\lim a_n/b_n = 0$

(iii) for each $a \in A$ there is $b \in A$ such that for every $\epsilon > 0$ we can find k_0 with

$$\|x - P_k(x)\|_a \leq \epsilon \|x\|_b, \quad x \in K^\ell(A), \quad k \geq k_0.$$

Proof. Assuming $K^\ell(A)$ is a Schwartz space for $a \in A$ we choose $b \in A$ with $a_n \leq b_n$ and $\lim d_n(U_b, U_a) = 0$. Define a linear map $T : K^\ell(A) \rightarrow \ell$ by $T(x) = xa$. Then $T(U_a)$ is contained in the closed unit ball B of ℓ . Therefore $d_n(T(U_b), B) \leq d_n(U_b, U_a)$ and hence $T(U_b)$ is a relatively compact subset of ℓ . If $\alpha_n = a_n/b_n$, the sequence $(\alpha_n e_n)$ lies in $T(U_b)$ and therefore has a subsequence which converges to some limit, which can only be zero. This means $\lim \alpha_n = 0$.

In the proof of the previous result we have observed

$$\|x - P_k(x)\|_a \leq \sup\{a_j/b_j : j \geq k\} \|x\|_b.$$

So if $\lim a_n/b_n = 0$, this gives (iii). Finally (iii) implies (i) is immediate. \square

The diametral dimension $\Delta(E)$ of any locally convex space E is equal to c_0 if E is not a Schwartz space. On the other hand, if $K^\ell(A)$ is a Schwartz space for $a \in A$ we find $b \in A$ with $\lim a_n/b_n = 0$. So if $N_0 = \{n : a_n > 0\}$ and $|N_0| = k$ then $d_n(U_b, U_a) = 0$ for $n > k$. If $|N_0| = \infty$, we find a map $\rho : N \rightarrow N_0$ such that $(a_{\rho(n)}/b_{\rho(n)})$ is non-increasing and by Prop. 1 we easily obtain

$$d_n(U_b, U_a) = a_{\rho(n)}/b_{\rho(n)}.$$

We have several corollaries of this result. Recalling that the diametral dimension of a non-Schwartz locally convex space is always equal to c_0 , we have the first result from above discussion.

Proposition 3 $\Delta(K^\ell(A))$ is independent of the admissible sequence space ℓ .

Let A be a countable Köthe set, where $0 < a_n^k \leq a_n^{k+1}$. We call $K^\ell(A)$ regular ([2]) if

$$a_{n+1}^k/a_{n+1}^{k+1} \leq a_n^k/a_n^{k+1}.$$

Proposition 4 *If $K^\ell(A)$ is a regular Köthe space, then either $K^\ell(A)$ is a Schwartz space or $K^\ell(A)$ is isomorphic to ℓ .*

Proof. We have $d_n(U_{k+1}, U_k) = a_n^k/a_n^{k+1}$ by Prop.1. If $K^\ell(A)$ is not a Schwartz space, then for some k_0 we have $\inf_n d_n(U_k, U_{k_0}) > 0$ for all $k > k_0$. So for each $k \geq k_0$ there is a $\rho_k > 0$ with $\rho_k a_n^k \leq a_n^{k_0}$, $n \in N$. This shows that $K^\ell(A)$ is isomorphic to ℓ by a diagonal transformation. \square

If $K^\ell(A)$ is nuclear, for $a \in A$ we find $b \in A$ with

$$\sum_{n=0}^{\infty} d_n(U_b, U_a) < \infty.$$

([3], [7]). By what we have already shown this implies $\sum a_n/b_n < \infty$. Conversely, if for each $a \in A$ we can find $b \in A$ such that $(a_n/b_n) \in \ell_1$ then in particular $K^\ell(A)$ is a Schwartz space and so $\lim_{k \rightarrow \infty} P_k(x) = x$, $x \in K^\ell(A)$ by Prop.2. If $y \in \varphi$, by monotonicity we have

$$\sup |y_n|a_n \leq \|y\|_p \leq \sum_{n \in N} |y_n|a_n, \quad a \in A$$

and so $K^1(A) \subset K^\ell(A) \subset K^{co}(A)$. On the other hand we have

$$\sum |x_n|a_n \leq \rho \sup |x_n|b_n$$

and therefore $K^\ell(A) = K(A)$. This result was also proved by Dragilev in [1].

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