Turkish Journal of Mathematics

Volume 32 | Number 2 Article 5

1-1-2008

Proximinality in L^1(I,X)

SHARIFA AL-SHARIF

R. KHALIL

Turk J Math
32 (2008) , 187 – 195.

Proximinality in $L^1(I, X)$

Sh. Al-Sharif and R. Khalil

Abstract

Let X be a Banach space and let (I, Ω, μ) be a measure space. For $1 \leq p < \infty$, let $L^p(I, X)$ denote the space of Bochner p-integrable functions defined on I with values in X. The object of this paper is to give sufficient conditions for the proximinality of $L^1(I, H) + L^1(I, G)$ in $L^1(I, X)$, where H and G are two proximinal subspaces of X which include as a special case the proximality of $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ in $L^1(I \times I)$.

Key Words: Proximinal, Banach spaces.

0. Introduction

Let (I, Ω, μ) be a measure space and let $L^p(I, X)$ denotes the space of Bochner p-integrable functions (equivalent classes) defined on (I, Ω, μ) with values in a Banach space X. It is known [2] that $L^p(I, X)$ is a Banach space under the norm

$$\|f\|_p = \left(\int \|f(t)\|^p \, d\mu(t)\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty.$$

A subspace E of a Banach space X is said to be proximinal if for each $x \in X$ there exists at least one $y \in E$ such that

$$\|x - y\| = d(x, E) = \inf \{\|x - z\| : z \in E\}.$$

The element y is called a best approximation of x in E.

If X and Y are Banach spaces, then $X \overset{\wedge}{\otimes} Y$ and $X \overset{\vee}{\otimes} Y$ denote the completions of the injective and projective tensor product of X with Y, [9] . Light and Cheney, (Theorem 2.26, [9]) ,

AMS Mathematics Subject Classification: 41A65, 41A50

proved that if G and H are finite dimensional subspaces of $L^1(S)$ and $L^1(I)$ respectively, then each element of $L^1(I \times S) = L^1(I) \overset{\vee}{\otimes} L^1(S)$ has a best approximation in the subspace $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(S)$. Deeb and Khalil, (Theorem 3.3 [1]), proved that if G and H are 1-summand subspaces of $L^1(I)$, then $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(S)$ is proximinal in $L^1(I \times I)$.

The object of this paper is to discuss the proximinality of $L^1(I, H) + L^1(I, G)$ in $L^1(I, X)$, where H and G are two proximinal subspaces of X. Further we conclude from our results the proximinality of $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ in $L^1(I \times I)$.

1. Distance Formula

For a Banach space X and two closed subspaces G and H of X, the distance formula from a point $f \in L^p(\mu, X)$ to the set $L^p(\mu, G)+ L^p(\mu, G)$ is computed by the following theorem.

Theorem 1.1. *Let (I, Ω, μ) be a measure space, X a Banach space and H, G be two closed subspaces of X. Then for each $f \in L^p(I, X)$*

$$\begin{aligned} \operatorname{dist}(f, L^p(I,H) + L^p(I,G)) &= \left(\int_I (\operatorname{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}} \\ &= \|\operatorname{dist}(f(.), H+G)\|_p \end{aligned}$$

Proof. Let $f \in L^p(I, X)$ and $u \in L^p(I, H) + L^p(I, G)$. Then $u = u_1 + u_2$ where $u_1 \in L^p(I, H)$ and $u_2 \in L^p(I, G)$ and

$$\begin{aligned} \|f - (u_1 + u_2)\|_p^p &= \int_I \|f(s) - (u_1(s) + u_2(s))\|^p \, ds \\ &\geq \int_I (\operatorname{dist}(f(s), H+G))^p \, ds. \end{aligned}$$

This implies that:

$$\begin{aligned}\|f-(u_1+u_2)\|_p &\geq \left(\int_I (\mathrm{dist}(f(s),H+G))^p\,ds\right)^{\frac{1}{p}}\\ &= \|\mathrm{dist}(f(.),H+G)\|_p\,. \qquad (1)\end{aligned}$$

Now, since simple functions are dense in $L^p(I,X)$, then given $\epsilon>0$, there exists a simple function φ in $L^p(I,X)$ such that $\|f-\varphi\|_p<\epsilon$. Write $\varphi=\sum_{i=1}^{n}\varkappa_{A_i}\,y_i$, where $\varkappa_{A_i}$ is the characteristic function of the set A_i in Ω and $y_i\in X$. We may assume that $\sum_{i=1}^{n}\varkappa_{A_i}=1$ and $\mu(A_i)>0$. Since $\varphi\in L^p(I,X)$ we have $\|y_i\|\,\mu(A_i)<\infty$ for $1\leq i\leq n$. For each $i=1,2,...,n$, if $\mu(A_i)<\infty$, select $h_i\in H$ and $g_i\in G$ such that:

$$\|y_i-(h_i+g_i)\|<\mathrm{dist}(y_i,H+G)+\frac{\epsilon}{(n\mu(A_i))^{\frac{1}{p}}}.$$

This could be done since $\mathrm{dist}(u,H+G)=\inf_{g\in H+G}\|u-g\|$ for all $u\in X$. If $\mu(A_i)=\infty$, put $h_i=g_i=0$. Let

$$w=\sum_{i=1}^{n}\varkappa_{A_i}\,(h_i+g_i)=\sum_{i=1}^{n}\varkappa_{A_i}\,h_i+\sum_{i=1}^{n}\varkappa_{A_i}\,g_i=w_1+w_2\,.$$

Clearly $w\in L^p(I,H)+L^p(I,G)$. Set $J=\mathrm{dist}(f,L^p(I,H)+L^p(I,G))$. Then

$$\begin{aligned}
J &\leq \|f-\varphi\|_p + \operatorname{dist}(\varphi, L^p(I,H)+L^p(I,G)) \\
&\leq \epsilon + \|\varphi - w\|_p \\
&= \epsilon + \|\varphi - (w_1 + w_2)\|_p \\
&= \epsilon + \left(\int_I \|\varphi(s) - (w_1(s) + w_2(s))\|^p \, ds \right)^{\frac{1}{p}} \\
&= \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} \|y_i - (g_i + h_i)\|^p \, ds \right)^{\frac{1}{p}} \\
&\leq \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} \left(\operatorname{dist}(y_i, H+G) + \frac{\epsilon}{(n\mu(A_i))^{\frac{1}{p}}} \right)^p ds \right)^{\frac{1}{p}} \\
&\leq \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} (\operatorname{dist}(y_i, H+G))^p \, ds \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \int_{A_i} \left(\frac{\epsilon}{(n\mu(A_i))^{\frac{1}{p}}} \right)^p ds \right)^{\frac{1}{p}} \\
&= \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} (\operatorname{dist}(y_i, H+G))^p \, ds \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \int_{A_i} \frac{\epsilon^p}{n\mu(A_i)} ds \right)^{\frac{1}{p}} \\
&= \epsilon + \left(\sum_{i=1}^{n} \int_{A_i} (\operatorname{dist}(y_i, H+G))^p \, ds \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \frac{\epsilon^p}{n\mu(A_i)} \mu(A_i) \right)^{\frac{1}{p}} \\
&= 2\epsilon + \left(\sum_{i=1}^{n} \int_{A_i} (\operatorname{dist}(y_i, H+G))^p \, ds \right)^{\frac{1}{p}} \\
&= 2\epsilon + \left(\int_I (\operatorname{dist}(\varphi, H+G))^p \, ds \right)^{\frac{1}{p}}
\end{aligned}$$

Since $\operatorname{dist}(\varphi(s), H+G) \leq \operatorname{dist}(f(s), H+G) + \|\varphi(s) - f(s)\|$, then:

$$\begin{aligned} J &\leq 2\epsilon + \left(\int_I (\text{dist}(f(s), H+G) + \|\varphi(s) - f(s)\|)^p \, ds \right)^{\frac{1}{p}} \\ &\leq 2\epsilon + \left(\int_I (\text{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}} + \left(\int_I \|\varphi(s) - f(s)\|^p \, ds \right)^{\frac{1}{p}} \\ &= 2\epsilon + \left(\int_I (\text{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}} + \|\varphi - f\|_p \\ &\leq 3\epsilon + \left(\int_I (\text{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}}. \end{aligned}$$

This implies that

$$\text{dist}\,(f, L^p(I,H) + L^p(I,G)) \leq \left(\int_I (\text{dist}(f(s), H+G))^p \, ds \right)^{\frac{1}{p}} \tag{2}$$

From (1) and (2) the result holds. □

The following Corollary is an application of Theorem 1.1.

Corollary 1.2. *Let H and G be subspaces of a Banach space X and (I, Ω, μ) be a measure space such that $\mu(I) < \infty$. Then $g \in L^p(I,H) + L^p(I,G)$ is a best approximation for $f \in L^p(I,X)$ if and only if for almost all $t \in I$, $g(t)$ is a best approximation in $H+G$ for $f(t)$.*

Proof. Let $g = g_1 + g_2$ be a best approximation in $L^p(I,H) + L^p(I,G)$ for $f \in L^p(I,X)$. Then $\|f-g\|_p =$dist$(f, L^p(I,H) + L^p(I,G))$. By Theorem 1.1 we have:

$$\left(\int_I \|f(t) - (g_1(t) + g_2(t))\|^p \, dt \right)^{\frac{1}{p}} = \left(\int_I (\text{dist}(f(t), H+G))^p \, dt \right)^{\frac{1}{p}}.$$

Since $\mathrm{dist}(f(t), H+G) \leq \|f(t)-(y+z)\|$ for any $y \in H$ and $z \in G$, it follows that

$$\mathrm{dist}(f(t), H+G) \leq \|f(t)-(g_1\ (t)+g_2\ (t))\|.$$

Since x^p is an increasing function for $p \geq 1$, we have:

$$(\mathrm{dist}(f(t), H+G))^p \leq \|f(t)-(g_1\ (t)+g_2\ (t))\|^p.$$

Thus $\|f(t)-(g_1\ (t)+g_2\ (t))\|^p = (\mathrm{dist}(f(t), H+G))^p$ for almost all $t \in I$ and so

$$\|f(t)-(g_1\ (t)+g_2\ (t))\| = \mathrm{dist}(f(t), H+G) \text{ for almost all } t \in I.$$

Hence $g_1\ (t)+g_2\ (t)$ is a best approximation for $f(t) \in X$ for almost all $t \in I$. □

2. Proximinality in $L^1(I, X)$.

Let X be a Banach space and (I, Ω, μ) be a finite measure space. The Main result of this section is to give sufficient conditions for the proximinality of $L^1(I,H)+L^1(I,G)$ in $L^1(I,X)$, where H and G are two proximinal subspaces of X which include as a special case the proximality of $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ in $L^1(I \times I)$. We start with the following definition.

Definition 2.1 *Two subspaces H and G of a Banach space X are said to be p-orthogonal if $\|h\ +g\|^p = \|h\|^p + \|g\|^p$ for every $h \in H$ and $g \in G$.*

It is easy to see that if H and G are p-orthogonal, then a function f is in $L^p(I, H+G)$ whenever f is in $L^p(I,H)+L^p(I,G)$.

Now we prove the following Main result

Theorem 2.2. *Let G and H be closed subspaces in X such that H and G are p-orthogonal. Then the following are equivalent:*

(1) *$L^1(I,H)+L^1(I,G)$ is proximinal in $L^1(I,X)$,*

(2) *$L^p(I,H)+ L^p(I,G)$ is proximinal in $L^P(I,X)$, $1 \leq p < \infty$.*

Proof. (1) → (2). Let $f \in L^p(I,X)$. Since (I, Ω, μ) is a finite measure space, then $f \in L^1(I,X)$. By assumption there exists $g = g_1\ + g_2\ \in L^1(I,H)+L^1(I,G)$ such that: $\|f-(g_1\ +g_2\ )\| \leq \|f-U\|$ for every $U \in L^1(I,H)+L^1(I,G)$. So

$\|f(t) - (g_1(t) + g_2(t))\| \leq \|f(t) - y\|$, for every $y = y_1 + y_2 \in H + G$ and for almost all $t \in I$. Thus

$$\|f(t) - (g_1(t) + g_2(t))\| \leq \|f(t) - w(t)\|,$$

for all $w \in L^p(I, H+G)$. Since $0 \in H+G$ it follows that: $\|(g_1(t) + g_2(t))\| \leq 2\|f(t)\|$. Hence $g_1 + g_2 \in L^p(I, H+G)$. Since H and G are p-orthogonal it follows that $g_1 + g_2 \in L^p(I, H) + L^p(I, G)$.

(2) $\rightarrow$ (1). Consider the map, $J : L^1(I, X) \rightarrow L^p(I, X)$, where, $J(f)(t) = \|f(t)\|^{\frac{1}{p}-1} f(t)$. As in the proof of Theorem 1.1, [1], J is one-one, onto and $J(L^1(I, G)) = L^p(I, G)$, $J(L^1(I, H)) = L^p(I, H)$.

Now, let $f \in L^1(I, X)$, $f(t) \neq 0$. Since $J(f) \in L^p(I, X)$, then by assumption there exists $f_1 + f_2 \in L^p(I, H) + L^p(I, G)$ such that

$$\|J(f) - J(f_1 + f_2)\|_p \leq \|J(f) - J(u_1 + u_2)\|_p$$

for all $u_1 \in L^p(I, H)$ and $u_2 \in L^p(I, G)$. Since $f_1 + f_2$, $u_1 + u_2$ are in $L^p(I, H) + L^p(I, G)$, it follows that $f_1 + f_2$, $u_1 + u_2 \in L^p(I, H+G)$. Hence $f_1 + f_2 = J(g_1 + g_2)$ and $u_1 + u_2 = J(h_1 + h_2)$, where $g_1 + g_2, h_1 + h_2 \in L^1(I, H+G)$. But H and G are p-orthogonal. Hence $g_1 + g_2, h_1 + h_2 \in L^1(I, H) + L^1(I, G)$. Thus

$$\|J(f) - J(g_1 + g_2)\|_p \leq \|J(f) - J(h_1 + h_2)\|_p$$

for all $h_1 \in L^1(I, H)$, $h_2 \in L^1(I, G)$. This implies that

$$\|J(f)(t) - J(g_1 + g_2)(t)\| \leq \|J(f)(t) - (y_1 + y_2)\|,$$

for every $y_1 \in H$, $y_2 \in G$ for almost all $t \in I$. Thus

$$\|J(f)(t) - J(g_1 + g_2)(t)\| \leq \left\|J(f)(t) - \|f(t)\|^{\frac{1}{p}-1}(y_1 + y_2)\right\|.$$

Multiply both sides by $\|f(t)\|^{1-\frac{1}{p}}$ we get:

$$\left\|f(t) - \|f(t)\|^{1-\frac{1}{p}} \|g_1(t) + g_2(t)\|^{\frac{1}{p}-1} (g_1(t) + g_2(t)\right\| \leq \|f(t) - (y_1 + y_2)\|$$

for every $y_1 \in H$, $y_2 \in G$. Since $J(g_1 + g_2)(t)$ is a best approximation of $J(f)(t)$ in $H+G$ (Corollary 1.2) and $0 \in H+G$ it follows that $\|J(g_1 + g_2)(t)\| \leq 2\|J(f)(t)\|$ and hence $J(g_1 + g_2) \in L^P(I, H + G)$. This implies that $g_1 + g_2 \in L^1(I, H + G)$. Thus w, $w(t) = \|f(t)\|^{1-\frac{1}{p}} \|g_1(t) + g_2(t)\|^{\frac{1}{p}-1} (g_1(t) + g_2(t)$ is in $L^1(I, H + G)$. Hence $w \in L^1(I, H) + L^1(I, G)$ and $\|f - w\|_1 \leq \|f - (\theta_1 + \theta_2)\|_1$ for every $\theta_1 \in L^1(I, H)$, $\theta_2 \in L^1(I, G)$. □

Theorem 2.3. *Let G and H be two reflexive subspaces of a Banach space X such that G and H are p-orthogonal. Then $L^p(I, H)$+ $L^p(I, G)$ is proximinal in $L^P(I, X)$.*

Proof. Since H and G are reflexive then, by Theorem 2.13, [9], the subspaces $L^1(I, G)$ and $L^1(I, H)$ are proximinal in $L^1(I, X)$. By Theorem 1.1, [8], $L^p(I, G)$ and $L^p(I, H)$ are proximinal in $L^p(I, X)$. Since H and G are reflexive then, $L^p(I, G)$ and $L^p(I, H)$ are reflexive for $1 < p < \infty$ ([2], p, 82, 98) and hence their intersection is reflexive ([5], p, 126). By Theorem 16.12, [5], $L^p(I, H)/(L^p(I, G) \cap L^p(I, H))$ is reflexive. But $(L^p(I, G) + L^p(I, H))/L^p(I, G)$ is isomorphic to $L^p(I, H)/(L^p(I, G) \cap L^p(I, H))$, ([5], p. 123). Consequently $(L^p(I, G) + L^p(I, H))/L^p(I, G)$ is reflexive ([3], p. 9). Hence by Corollary 2.1, [10], $(L^p(I, G) + L^p(I, H))/L^p(I, G)$ is proximinal and so by Theorem 2.1, [1], $L^p(I, G) + L^p(I, H)$ is proximinal in $L^p(I, X)$. □

Corollary 2.5 *Let H and G be two reflexive subspaces of $L^1(I)$ such that H and G are p-orthogonal. Then $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ is proximinal in $L^1(I \times I)$.*

Proof. Since H and G are two reflexive p-orthogonal subspaces of $L^1(I)$, by Theorem 2.3 and 2.1 $L^1(I, G) + L^1(I, H)$ is proximinal in $L^1(I \times I)$. By Theorem 1.15, [9], $L^1(I) \overset{\wedge}{\otimes} G + H \overset{\wedge}{\otimes} L^1(I)$ is proximinal in $L^1(I \times I)$. □

References

[1] Deeb, W. and Khalil, R.: The Sum of Proximinal Subspaces. Soochow Journal of Mathematics. 18, No. 2 (1992), 163-167.

[2] Diestel, J. and Uhl, J.: Vector Measures. Amer. Math. Soc. 1977.

[3] Dulst, D.: Reflexive and Super Reflexive Banach Spaces. Mathematical Centre Tracts. Amesterdam. 102 (1978).

[4] Holmos, R.: A course on Optimization and Best Approximation. Springer-Verlag, Berlin, New York, 1972.

[5] Holmos, R.: Geometric Functional Analysis and its Applications. GTM, 24, Springer-Verlag, 1975.

[6] Holland, S., Light, W. and Sulley, L.: On Proximinality in $L^1(T \times S)$. Proc. of the Amer. Math. Soc. 86, No. 2 (1982), 279-282.

[7] Khalil, R.: Best Approximation in $L^P(I, X)$. Math. Proc. Cambridge Philos. Soc. 94 (1983), 277-279.

[8] Khalil R. and Deeb, W.: Best Approximation in $L^P(\mu, X)\ II$. J. Approx. Theory. 59(1989), 296-299.

[9] Light, W. and Cheney, E.: Approximation Theory in Tensor Product Spaces. Lecture Notes in Mathematics. 1169. Springer-Verlag, Berlin, New York, 1985.

[10] Singer, I.: Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces. Springer-Verlag, Berlin, New York, 1970.

Sh. AL-SHARIF, R. KHALIL
Mathematics Department,
Yarmouk University,
Jordan University, Irbed-JORDAN
e mail: Sharifa@yu.edu.jo
e mail: Roshdi@ju.edu.jo

Received 29.01.2007