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Abstract

In this small note we characterize the essential spectra of a class of composition operators on $H^\infty(\mathbb{H})$ of the upper half-plane $\mathbb{H}$ and on $H^\infty(\mathbb{D})$ of the unit disc $\mathbb{D}$. These composition operators are induced by perturbations of translations.

1. Introduction

In this note we characterize the essential spectra of composition operators $C_\varphi : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$, where $\varphi(z) = z + b(z)$, the function b is bounded analytic with $\Im(b(z)) \geq M > 0$ $\forall z \in \mathbb{H}$ and $\lim_{z\to\infty} b(z) = b_0$ exists.

2. Translations

We begin with a special case where the function b is constant i.e. $b(z) \equiv b_0$ $\forall z \in \mathbb{H}$. To determine the spectra of these operators we use semigroup of operators techniques. We use the following Theorem 2.1 cited from [3, pp. 93]:

Let $D \subseteq \mathbb{C}$ be a domain in the complex plane and let X,Y be Banach spaces. Let $\Gamma \subset Y^*$ be a determining manifold for Y i.e. if $y^*(y) = 0$ $\forall y^* \in \Gamma$ then $y = 0$. Then

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$U : D \to \mathcal{B}(X,Y)$ is called holomorphic if the function $f(w) = y^*(U(w)x)$ is holomorphic on D $\forall x \in X$, $y^* \in \Gamma$.

Theorem 2.1 *Let $D \subset \mathbb{C}$ be a domain and $U : D \to \mathcal{B}(X,Y)$ be a function.*

If U is holomorphic on D then U is continuous and differentiable on D in the uniform operator topology of $\mathcal{B}(X,Y)$; i.e. for any $z_0 \in D$ and $\varepsilon > 0$ there exists $T \in \mathcal{B}(X,Y)$ and $\delta > 0$ such that $\forall w \in D$ with $| w - z_0 | < \delta$ we have

$$\| \frac{1}{w - z_0}(U(w) - U(z_0)) - T \|_{\mathcal{B}(X,Y)} < \varepsilon.$$

As a corollary, we have if U is holomorphic then $\forall F \in \mathcal{B}(X,Y)$ and the function $g(w) = F(U_w)$ is holomorphic.

For $w \in \mathbb{H}$ let $T_w : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$, $T_w f(z) = f(z+w)$. We consider the algebra of operators

$$\mathcal{B} = \overline{< \{T_w : w \in \mathbb{H}\} \cup \{I\} >},$$

will the closure of the linear span of $\{T_w : w \in \mathbb{H}\} \cup \{I\}$ in the operator norm of $\mathcal{B}(H^\infty)$. The algebra $\mathcal{B}$ is a commutative Banach algebra with identity. Let $\mathcal{M}$ be the maximal ideal space of $\mathcal{B}$. Then we have

$$\sigma(T_w) \subseteq \{\Lambda(T_w) : \Lambda \in \mathcal{M}\}$$

by Gelfand theory of commutative Banach algebras. The argument we follow is the H^∞ version of the argument done in [1, pp. 302].

Fix $\Lambda \in \mathcal{M}$ and consider $g(w) = \Lambda(T_w)$. We will see that g is holomorphic: for that, we use the Theorem 2.1. To apply the Theorem 2.1, we take $U_w = T_w$, $X = Y = H^\infty(\mathbb{H})$, $\Gamma = \{\delta_z : z \in \mathbb{H}\}$, where $\delta_z(f) = f(z)$. By Hahn-Banach theorem there exists $\tilde{\Lambda} \in \mathcal{B}(H^\infty)^*$ such that $\tilde{\Lambda} |_{\mathcal{B}} = \Lambda$. So by Theorem 2.1, if for any $z \in \mathbb{H}$ and $f \in H^\infty(\mathbb{H})$ the function $h(w) = \delta_z(T_w f)$ is holomorphic, then the function $g(w) = \tilde{\Lambda}(T_w) = \Lambda(T_w)$ is holomorphic. It is easy to see that h is holomorphic. Hence g is holomorphic. The function g also satisfies the relations

$$g(w_1 + w_2) = g(w_1)g(w_2) \qquad \forall w_1, w_2 \in \mathbb{H}.$$

and

$$| g(w) | \leq \| T_w \| = 1 \qquad \forall w \in \mathbb{H}$$

So we deduce that $g(w) = e^{it_0 w}$ *for some* $t_0 \in [0, \infty)$. *So we have for any* $w \in \mathbb{H}$

$$\sigma_e(T_w) \subseteq \sigma(T_w) \subseteq \{e^{itw} : t \in [0, \infty)\} \cup \{0\}$$

Now take $\lambda = e^{it_0 w}$ *for some* $t_0 \in (0, \infty)$. *Then the function* $f(z) = e^{it_0 z}$ *is in* $H^\infty(\mathbb{H})$ *and satisfies*

$$T_w f(z) = e^{it_0 w} f(z).$$

So we have $f \in \ker(e^{it_0 w} I - T_w)$ *and hence* $\lambda = e^{it_0 w} \in \sigma(T_w)$. *Therefore*

$$\sigma(T_w) = \{e^{itw} : t \in [0, \infty)\} \cup \{0\}.$$

Now let $D_w = \{e^{2\pi i \frac{z}{w}} : z \in \mathbb{H}\}$, D_w *is the image of* $\mathbb{H}$ *under a holomorphic map and hence is open with nonempty interior. Consider the following subspace* K *of* H^∞*:*

$$K = \{f(z) = e^{it_0 z} k(e^{2\pi i \frac{z}{w}}) : k \in H^\infty(D_w)\}.$$

Observe that $K \subseteq \ker(e^{it_0 w} I - T_w)$. *Hence* $\ker(e^{it_0 w} I - T_w)$ *is infinite dimensional. This implies that* $e^{it_0 w} I - T_w$ *is not Fredholm and by Atkinson's theorem we have* $e^{it_0 w} \in \sigma_e(T_w)$. *So we have*

$$\sigma_e(T_w) = \sigma(T_w) = \{e^{itw} : t \in [0, \infty)\} \cup \{0\}.$$

3. The General Case

We begin this section by stating an important theorem about the compactness of the difference of certain composition operators with translations on the space of bounded analytic functions.

Theorem 3.1 *Let* $b : \mathbb{H} \to \mathbb{H}$ *be a bounded analytic function such that* $b(\mathbb{H}) \subset\subset \mathbb{H}$ *and let* $\lim_{z\to\infty} b(z) = b_0 \in \mathbb{H}$ *exists. Let* $\varphi(z) = z + b(z)$ *and* $T_{b_0} f(z) = f(z + b_0)$, $T_{b_0} : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$. *Then* $C_\varphi - T_{b_0}$ *is compact on* $H^\infty(\mathbb{H})$.

Let X *be a Banach space and* $K(X)$ *be the space of all compact operators on* X. *Take* $K \in K(X)$. *Since for any* $T \in \mathcal{B}(X)$ *and* $\lambda \in \mathbb{C}$, $[\lambda I - T] = [\lambda I - T - K]$ *in* $\mathcal{B}(X)/K(X)$*(where* $[.]$ *denotes the equivalence class in* $\mathcal{B}(X)/K(X)$*), we have*

$$\sigma_e(T + K) = \sigma_e(T),$$

i.e. the essential spectrum is invariant under compact perturbations. Using this fact and Theorem, 3.1 we conclude that if $\varphi(z) = z + b(z)$, the function b is bounded analytic with $\Im(b(z)) \geq M > 0 \quad \forall z \in \mathbb{H}$ and $\lim_{z\to\infty} b(z) = b_0$ then for $C_\varphi : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$ we have

$$\sigma_e(C_\varphi) = \sigma_e(T_{b_0}) = \{e^{itb_0} : t \in [0,\infty)\} \cup \{0\}.$$

Proof of Theorem 3.1 Take $\{f_n\}_{n=1}^{\infty} \subset H^\infty(\mathbb{H})$ such that $\| f_n \|_\infty \leq 1$. Consider $K_j = \{x + iy \in \mathbb{H} : | x | \leq j, \frac{1}{j} \leq | y | \leq j\}$, K_j's are compact, $K_{j+1} \supset K_j$ and $\bigcup_{j=1}^{\infty} K_j = \mathbb{H}$. Since $\{f_n\}$ is equibounded and equicontinuous on K_1, by Arzela Ascoli theorem $\{f_n\}$ has a subsequence $\{f_{n_j}\}$ that converges uniformly on K_1. Applying the same process on K_2 and $\{f_{n_j}\}$, going on iteratively we arrive at a subsequence $\{f_k\}$ that converges uniformly on each K_j and hence on each compact subset of $\mathbb{H}$.

Let $f(z) = \lim_{k\to\infty} f_k(z)$, then by Weierstrass theorem f is analytic on $\mathbb{H}$. Our aim is to show that, indeed, for $g(z) = f(z + b(z)) - f(z + b_0)$ we have $g \in H^\infty(\mathbb{H})$; or, in other words, for $g_k(z) = f_k(z + b(z)) - f_k(z + b_0)$, g_k converges uniformly on $\mathbb{H}$.

Let $\varepsilon > 0$ be given. Then since $\lim_{z\to\infty} b(z) = b_0$ we have $j_0 \in \mathbb{N}$ such that

$$| b(z) - b_0 | < \varepsilon \qquad \forall z \in \mathbb{H} \backslash M_{j_0}$$

$$M_{j_0} = \{x + iy \in \mathbb{H} : | x | \leq j_0, 0 < y \leq j_0\}.$$

Now let $\alpha = \inf_{z\in\mathbb{H}} \Im(b(z))$. Since $\overline{b(\mathbb{H})}$ is compact in $\mathbb{H}$ we have $\alpha > 0$. And let $S_\alpha = \{x + iy \in \mathbb{H} : y > \alpha\}$. Take $z \in S_\alpha$ and let C be the circle of radius α and center z. Then by Cauchy Integral Formula and Cauchy estimates on derivatives we have

$$f_k'(z) = \frac{1}{2\pi i} \int_C \frac{f_k(\zeta) d\zeta}{(\zeta - z)^2} \Longrightarrow | f_k'(z) | \leq \frac{1}{\alpha} \| f_k \|_\infty,$$

hence

$$\sup_{z\in S_\alpha} | f_k'(z) | \leq \frac{1}{\alpha} \| f_k \|_\infty \leq \frac{1}{\alpha}.$$

Combining this with Mean Value Theorem, we have

$$| f_k(z + b(z)) - f_k(z + b_0) | \leq \frac{1}{\alpha} | b(z) - b_0 | \qquad \forall z \in \mathbb{H}.$$

Since $\| f_k \|_\infty \leq 1$, we have

$$\sup_{z\in M_{j_0}} | f_k(z+b(z)) - f_k(z+b_0) | \leq 2 \| f_k \|_\infty \leq 2 \qquad \forall k.$$

Hence for $g_k(z) = f_k(z+b(z)) - f_k(z+b_0)$ we have

$$\| g_k \| \leq \max\{2, \quad \frac{\varepsilon}{\alpha}\} \qquad \forall k.$$

So for $g(z) = \lim_{k\to\infty} g_k(z)$ we have $g \in H^\infty(\mathbb{H})$. The sequence $\{g_k\}$ converges uniformly on $\mathbb{H}$. Therefore for any sequence $\{f_n\}$ such that $\| f_n \|_\infty \leq 1$, $\{(C_\varphi - T_{b_0})f_n\}$ has a convergent subsequence in $H^\infty(\mathbb{H})$.

As a result $C_\varphi - T_{b_0}$ is compact on $H^\infty(\mathbb{H})$. □

We have the following two main results, one for composition operators on $H^\infty(\mathbb{H})$ and the other for composition operators on $H^\infty(\mathbb{D})$:

Theorem I. Let $\varphi : \mathbb{H} \to \mathbb{H}$ be an analytic self-map of the upper half plane satisfying

(a) $\varphi(z) = z+b(z)$ where $b : \mathbb{H} \to \mathbb{H}$ is a bounded analytic function satisfying $\Im(b(z)) \geq M > 0$ for all $z \in \mathbb{H}$ and for some M positive.

(b) The limit $\lim_{z\to\infty} b(z) = b_0$ exists and $b_0 \in \mathbb{H}$.

Let $T_{b_0} : H^\infty(\mathbb{H}) \to H^\infty(\mathbb{H})$ be the translation operator $T_{b_0}f(z) = f(z+b_0)$. Then we have

$$\sigma_e(C_\varphi) = \sigma_e(T_{b_0}) = \{e^{itb_0} : t \in [0,\infty)\} \cup \{0\}.$$

Let τ be the Cayley transform, the map that takes the upper half-plane conformally onto the unit disc in a one-to one manner. We observe that if for $\varphi : \mathbb{D} \to \mathbb{D}$ satisfies

$$\psi = \tau^{-1}\varphi\tau, \qquad \psi(z) = z + \eta(z)$$

where $\eta : \mathbb{H} \to \mathbb{H}$ is a bounded analytic function then φ has the form

$$\varphi(w) = \frac{2iw + \eta(\frac{i(1-w)}{1+w})(1-w)}{2i + \eta(\frac{i(1-w)}{1+w})(1-w)}.$$

So we formulate our result as the following theorem.

Theorem II. If $\varphi : \mathbb{D} \to \mathbb{D}$ is of the following form

$$\varphi(w) = \frac{2iw + b(\frac{i(1-w)}{1+w})(1-w)}{2i + b(\frac{i(1-w)}{1+w})(1-w)}$$

with $b : \mathbb{H} \to \mathbb{H}$ bounded analytic with $b(\mathbb{H}) \subset\subset \mathbb{H}$ and $\lim_{z\to\infty} b(z) = b_0$. Then for $C_\varphi : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ we have

$$\sigma_e(C_\varphi) = \{e^{itb_0} : t \in [0, \infty)\} \cup \{0\}.$$

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