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Geodesics of the Cheeger-Gromoll Metric

A. A. Salimov, S. Kazimova

Abstract

The main purpose of the paper is to investigate geodesics on the tangent bundle with respect to the Cheeger-Gromoll metric.

Key Words: Geodesics, Cheeger-Gromoll metric, Horizontal and vertical lift.

1. Introduction

In [1] Cheeger and Gromoll study complete manifolds of nonnegative curvature and suggest a construction of Riemannian metrics useful in that context. Inspired by a paper of Cheeger and Gromoll, in [4] Musso and Tricerri defined a new Riemannian metric $CGg$ on tangent bundle of Riemannian manifold which they called the Cheeger-Gromoll metric. The Levi-Civita connection of $CGg$ and its Riemannian curvature tensor are calculated by Sekizawa in [5] (for more details see [2],[3]). The main purpose of this paper is to investigate geodesics of the Cheeger-Gromoll metrics on tangent bundle.

Let $M\_n$ be a Riemannian manifold with metric $g$. We denote by $\mathfrak{S}^p\_q(M\_n)$ the set of all tensor fields of type $(p, q)$ on $M\_n$. Manifolds, tensor field and connections are always assumed to be differentiable and of class $C^\infty$.

Let $T(M\_n)$ be a tangent bundle of $M\_n$, and $\pi$ the projection $\pi : T(M\_n) \rightarrow M\_n$. Let the manifold $M\_n$ be covered by system of coordinate neighbourhoods $(U, x^i)$, where $(x^i), i = 1, ..., n$ is a local coordinate system defined in the neighbourhood $U$. Let $(y^i)$ be the Cartesian coordinates in each tangent spaces $T_p(M\_n)$ at $P \in M\_n$ with respect to the natural base $\{\frac{\partial}{\partial x^i}\}$. $P$ being an arbitrary point in $U$ whose coordinates are $x^i$. Then we can introduce local coordinates $(x^i, y^i)$ in open set $\pi^{-1}(U) \subset T(M\_n)$. We call them coordinates induced in $\pi^{-1}(U)$ from $(U, x^i)$. The projection $\pi$ is represented by $(x^i, y^i) \rightarrow (x^i)$. We use the notations $x^I = (x^i, x^i)$ and $x^i = y^i$. The indices $I, J, ...$ run from 1 to $2n$, the indices $\bar{i}, \bar{j}, ...$ run from $n+1$ to $2n$.

Let $X \in \mathfrak{S}^1\_0(M\_n)$, which locally are represented by $X = X^i \partial_i, \left(\partial_i = \frac{\partial}{\partial x^i}\right)$. Then the vertical and horizontal lifts $^VX$ and $^HX$ of $X$ (see [6]) are given, respectively by

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\[ V_X = X^i \partial_i, \left( \partial_i = \frac{\partial}{\partial x^i} \right) \] (1)
and
\[ H_X = X^i \partial_i - \Gamma^i_{jk} x^j X^k \partial_{\bar{i}} \] (2)

where \( \Gamma^i_{jk} \) are the coefficients of the Levi-Civita connection \( \nabla \).

Suppose that we are given on \( M_n \) a tensor field \( S \in \mathcal{I}^p_{q-1}(T(M_n)) \) where \( p > q > 1 \). We then define a tensor field \( \gamma S \in \mathcal{I}^{p-1}_{q-1}(U) \) by [6, p. 12]

\[ \gamma S = \left( x^i S^{i_1 \ldots i_p}_{e_1 \ldots e_q} \right) \partial_{i_1} \otimes \ldots \otimes \partial_{i_p} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_p} \]

with respect to the induced coordinates \((x^i, x_{\bar{i}})\). The tensor field \( \gamma S \) defined in each \( \pi^{-1}(U) \) determine global tensor field on \( T(M_n) \). We easily see that for any \( \varphi \in \mathcal{I}_1^1(M_n) \), \( \gamma \varphi \) has components \((\gamma \varphi) = \left( 0 x^i \varphi_i \right) \) with respect to the induced coordinates \((x^i, x_{\bar{i}})\) and \((\gamma \varphi)(Vf) = 0 \), \( f \in \mathcal{I}_0^0(M_n) \) i.e. \( \gamma \varphi \) is a vertical vector field on \( T(M_n) \).

Let there be given in \( U \subset M_n \) a vector field \( X = X^i \partial_i \) and a covector field \( g_X = g_{ij} X^i dx^j \). Then we define a function \( \gamma g_X \in \mathcal{I}_0^0(M_n) \) in \( \pi^{-1}(U) \subset T(M_n) \) by \( \gamma g_X = x^i g_{ij} X^j \) with respect to the induced coordinates \((x^i, x_{\bar{i}})\). Now, let \( r \) be the norm a vector \( y = (y^i) = (x^i) \), i.e. \( r^2 = g_{ij} x^i x^j \). The Cheeger-Gromoll metric \( CG \) on tangent bundle \( T(M_n) \) is given by

\[ CG g(H_X, H_Y) = V(g(X, Y)), \] (3)
\[ CG g(H_X, V_Y) = 0, \] (4)
\[ CG g(V X, V Y) = \frac{1}{1 + r^2} \left[ V(g(X, Y)) + (\gamma g_X) + (\gamma g_Y) \right] \] (5)

for all vector field \( X, Y \in \mathcal{I}_0^0(M_n) \), where \( V(g(X, Y)) = (g(X, Y)) \circ \pi \).

It is obvious that the Cheeger-Gromoll metric \( CG \) is contained in the class of natural metrics (Recall that by a natural metric on tangent bundles we shall mean a metric which satisfies conditions (3) and (4)).

2. Expressions in Adapted Frames

In each local chart \( U \subset M_n \), we put \( X_{(j)} = \frac{\partial}{\partial x^j}, j = 1, \ldots, n \). Then from (1) and (2), we see that these vector fields have, respectively, local expressions
\[ H X_{(j)} = \delta^h_j \partial_h + (-\Gamma^h_{sjx^s}) \partial_h \]  
\[ V X_{(j)} = \delta^h_j \partial_h \]  
with respect to the natural frame \{ \partial_h, \partial_h \}, where \( \delta^h_j \)-Kronecker delta. These 2n vector fields are linear independent and generate, respectively, the horizontal distribution of \( \nabla \) and the vertical distribution of \( T(M_n) \).

We have call the set \( \{ H X_{(j)}, V X_{(j)} \} \) the frame adapted to the affine connection \( \nabla \) in \( \pi^{-1}(U) \subset T(M_n) \).

On putting \( e_{(j)} = H X_{(j)}, \) \( e_{(j)} = V X_{(j)} \), we write the adapted frame as \( \{ e_\beta \} = \{ e_{(j)}, e_{(j)} \} \). The indices \( \alpha, \beta, \ldots \) run over the range \{1, \ldots, 2n\} and indicate the indices with respect to the adapted frame.

Using (1), (2), (6) and (7) we have

\[ H X = \begin{pmatrix} X^j \delta^h_j \\ -X^j \Gamma^h_{sjx^s} \end{pmatrix} = X^j \begin{pmatrix} \delta^h_j \\ -\Gamma^h_{sjx^s} \end{pmatrix} = X^j e_{(j)} \]

\[ V X = \begin{pmatrix} 0 \\ X^h \end{pmatrix} = \begin{pmatrix} 0 \\ X^j \delta^h_j \end{pmatrix} = X^j \begin{pmatrix} 0 \\ \delta^h_j \end{pmatrix} = X^j e_{(j)} \]

i.e. the lifts \( H X \) and \( V X \) have respectively components

\[ H X = (H X^j) = \begin{pmatrix} H X^j \\ H X^j \end{pmatrix} = \begin{pmatrix} X^j \\ 0 \end{pmatrix} \]

\[ V X = (V X^j) = \begin{pmatrix} V X^j \\ V X^j \end{pmatrix} = \begin{pmatrix} 0 \\ X^j \end{pmatrix} \]

with respect to the adapted frame \( \{ e_\beta \} \). From (3)–(5) we see that the Cheeger-Gromoll metric \( CG g \) has components

\[ (CG g^j_{\beta j}) = \begin{pmatrix} CG g^j_{jl} & CG g^j_{jl} \\ CG g^j_{jl} & CG g^j_{jl} \end{pmatrix} = \begin{pmatrix} g^j_{jl} & 0 \\ 0 & \frac{1}{1+\epsilon} (g^j_{jl} + g^j_{jl} g^l x^s x^l) \end{pmatrix} \]

with respect to the adapted frame \( \{ e_\beta \} \).

For the Levi-Civita connection of the Cheeger-Gromoll metric we have the following.
Theorem 1 \([5]\) Let \((M_n, g)\) be a Riemannian manifold and equip its tangent bundle \(T(M_n)\) with the Cheeger-Gromoll metric \(CG\ g.\) Then the corresponding Levi-Civita connection \(CG\nu\) satisfies the following:

\[
\begin{align*}
CG\nabla_{X} Y &= H(\nabla X Y) - \frac{1}{2} \left( R(X, Y) g \right), \\
CG\nabla_{X} f &= \frac{1}{2\alpha} \left( R(y, X) Y \right) + V(\nabla X Y), \\
CG\nabla_{X} t &= \frac{1}{2\alpha} \left( R(y, X) Y \right), \\
CG\nabla_{X} h &= -\frac{1}{\alpha} \left( \langle CG\ g(\nabla X, \gamma\delta) Y + \langle CG\ g(\nabla Y, \gamma\delta) V \rangle X \right) + \frac{1}{\alpha} \gamma\delta - \frac{1}{\alpha} \left( \langle CG\ g(\nabla X, \gamma\delta) Y + \langle CG\ g(\nabla Y, \gamma\delta) V \rangle \gamma\delta. \right)
\end{align*}
\]

for any \(X, Y \in \mathfrak{X}(M_n)\), where \(R\) and \(\gamma\delta\) denotes respectively the curvature tensor of \(\nabla\) and the canonical vertical vector field on \(T(M_n)\) with components

\[
\gamma\delta = \begin{pmatrix} 0 \\ x^i \delta^j \end{pmatrix} = \begin{pmatrix} 0 \\ x^j \end{pmatrix} = x^j \partial_j = x^j e_{(j)}.
\]

With respect to the adapted frame \(\{e_{\alpha}\}\) of \(T(M_n)\), we write \(CG\nabla_{e_{\alpha}} e_{\beta} = CG\Gamma_{\alpha\beta} e_{\gamma}\) where \(CG\Gamma_{\alpha\beta}\) denote the Christoffel symbols constructed by \(CG\ g.\) The particular values of \(CG\Gamma_{\alpha\beta}\) for different indices, on taking account of (8) are then found to be

\[
\begin{align*}
CG\Gamma^h_{ji} &= \Gamma^h_{ji}, & CG\Gamma^h_{ji} &= \frac{1}{2} R^h_{jik} x^k \\
CG\Gamma^h_{ji} &= \frac{1}{2\alpha} R^h_{jki} x^k, & CG\Gamma^h_{ji} &= \Gamma^h_{ji} \\
CG\Gamma^h_{ji} &= \frac{1}{2\alpha} R^h_{ikj} x^k, & CG\Gamma^h_{ji} &= 0 \\
CG\Gamma^h_{ji} &= 0 \\
CG\Gamma^h_{ji} &= \frac{1}{\alpha}(x_j \delta_i^h + x_i \delta_j^h) + \frac{1}{\alpha} g_{ji} x^h - \frac{1}{\alpha} x_j x_i x^h
\end{align*}
\]

with respect to the adapted frame, where \(x_j = g_{ji} x^i\), \(R^h_{iJK} = g^{ht} g_{js} R^h_{tik}.\)

3. Results

Let \(\tilde{C} : [0, 1] \rightarrow T(M_n)\) be a curve on \(T(M_n)\) and suppose that \(\tilde{C}\) is expressed locally by \(x^A = x^A(t),\) i.e., \(x^h = x^h(t),\) \(x^h = x^h(t) = y^h(t)\) with respect to induced coordinates \((x^h, x^h)\) in \(\pi^{-1}(U) \subset T(M_n),\) \(t\) being a parameter. Then the curve \(C = \pi \circ \tilde{C}\) on \(M_n\) is called the projection of the curve \(\tilde{C}\) and denoted by \(\pi \tilde{C}\) which is expressed locally by \(x^h = x^h(t).\) Let \(X^h(t)\) be a vector field along \(C.\) Then , on \(T(M_n)\) we define a curve \(\tilde{C}\) by

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\begin{align*}
\begin{cases}
x^h = x^h(t) \\
x^\bar{h} = X^\bar{h}(t).
\end{cases}
\end{align*}

(10)

If the curve (10) satisfies at all points the relation
\[ \frac{\delta X^h}{dt} = \frac{dX^h}{dt} + \Gamma^h_{ji} \frac{dx^j}{dt} X^i = 0, \]
then the curve \( \tilde{C} \) is said to be a horizontal lift of the curve \( C \) and denoted by \( H^C \) [6,p.172]. If \( X^h \) is the tangent vector field \( \frac{dx^h}{dt} \) to \( C \), then the curve \( \tilde{C} \) defined by (10) is called the natural lift of the curve \( C \) and denoted by \( C^* \).

The geodesics of the connection \( CG^\nabla \) is given by the differential equations
\[ \frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + CG^\nabla \Gamma^A_{CB} \frac{dx^C}{dt} \frac{dx^B}{dt} = 0, \]
(11)

with respect to induced coordinates \((x^h, x^\bar{h})\), where \( t \) is the arc length of a curve on \( T(M_n) \).

We find it more convenient to refer equations (11) to the adapted frame \( \{e_\alpha\} \). From (6) and (7) we see that the matrix of change of frames \( e_\beta = A_\beta^\alpha \partial_A \) has components of the form
\[ A = (A_\beta^B) = \begin{pmatrix}
\delta^k_h & 0 \\
-\Gamma^k_{sj}x^s & \delta^k_j
\end{pmatrix}.\]

The inverse of the matrix \( A \) is given by
\[ \tilde{A} = (\tilde{A}^\alpha_A) = \begin{pmatrix}
\delta^h_i & 0 \\
\Gamma^h_{si}x^s & \delta^h_i
\end{pmatrix}.\]

Using \( \tilde{A} \), we now write
\[ \theta^\alpha = \tilde{A}^\alpha_A dx^A \]
or
\[ \theta^h = \tilde{A}^h_A dx^A = \delta^h_i dx^i = dx^h, \]
for \( \alpha = h \)
\[ \theta^\bar{h} = \tilde{A}^\bar{h}_A dx^A = \Gamma^\bar{h}_{si} x^s dx^i + \delta^\bar{h}_i dx^i = dy^h + \Gamma^\bar{h}_{si} y^s dx^i = \delta y^h, \]
for \( \alpha = \bar{h} \) and put
Theorem 2
Let \( \frac{\vartheta^h}{dt} = A^h \wedge \frac{dx^A}{dt} = \frac{dx^h}{dt} \),
\[ \frac{\vartheta^h}{dt} = A^h \wedge \frac{dx^A}{dt} = \frac{\delta y^h}{dt} \]
along a curve \( x^A = x^A(t) \) on \( T(M_n) \).

If we therefore write down the form equivalent to (11), namely,
\[ \frac{d}{dt}(\theta^a_{\gamma\beta} \vartheta^\gamma \vartheta^\beta) = 0 \]
with respect to adapted frame and taking account of (9), then we have
\[
\begin{align*}
\left\{ (a) \right. & \quad \frac{\delta^2 x^h}{dt^2} + \frac{1}{\alpha} R^h_{kji} y^i \frac{\delta y^j}{dt} \frac{dx^i}{dt} = 0, \\
(b) & \quad \frac{\delta^2 y^h}{dt^2} + \left[ -\frac{1}{\alpha}(y_j \delta^h_i + y_i \delta^h_j) + \frac{1}{\alpha} \gamma_{jyi} y^h - \frac{1}{\alpha} y_j y_i y^h \right] \frac{\delta y^j}{dt} \frac{\delta y^i}{dt} = 0,
\end{align*}
\]
where \( y^i = x^i \). Thus we have the following theorem.

**Theorem 2** Let \( \tilde{C} \) be a curve on \( T(M_n) \) and locally expressed by \( x^h = x^h(t), \ x^k = y^h(t) \) with respect to induced coordinates \( (x^h, x^k) \) in \( \pi^{-1}(U) \subset T(M_n) \). The curve \( \tilde{C} \) is a geodesic of \( CG \), if it satisfies the equations (12).

If a curve \( \tilde{C} \) satisfying (12) lies on a fibre given by \( x^h = \) const, then by virtue of \( \frac{dx^h}{dt} = 0 \) and \( \frac{\delta y^h}{dt} = \frac{dy^h}{dt} \), the equations (12) reduces to
\[ \frac{d^2 y^h}{dt^2} + \left[ -\frac{1}{\alpha}(y_j \delta^h_i + y_i \delta^h_j) + \frac{1}{\alpha} \gamma_{jyi} y^h - \frac{1}{\alpha} y_j y_i y^h \right] \frac{dy^j}{dt} \frac{dy^i}{dt} = 0. \quad (13) \]

Hence we have this final theorem

**Theorem 3** If a geodesic lies on a fibre of \( T(M_n) \) with metric \( CG \), the geodesic is expressed by equation (13).

Let \( C = \pi \circ C^H \) be a geodesic of \( \triangledown \) on \( M_n \). Then \( \frac{\delta^2 x^h}{dt^2} = 0 \). Using this condition and condition
\[ \frac{\delta y^i}{dt} = \delta X^h_i = 0, \]
we have

**Theorem 4** The horizontal lift of a geodesic on \( M_n \) is always geodesic on \( T(M_n) \) with the metric \( CG \).

Let now \( C = \pi \circ C^* \) be a geodesic of \( \triangledown \) on \( M_n \), i.e. \( \frac{\delta^2 x^h}{dt^2} = \frac{\delta}{dt} \left( \frac{dx^h}{dt} \right) = 0 \). On the other hand, from definition of the natural lift of the curve, we obtain
\[
\frac{\delta y^h}{dt} = \frac{\delta}{dt} \left( \frac{dx^h}{dt} \right) = 0.
\]  
(14)

Then from (12) and (14) we easily see that the natural lift of a curve on \( M_n \) defined \( x^h = x^h(t) \) is geodesic on \( T(M_n) \) with the metric \( CGg \). Thus we have

**Theorem 5** The natural lift \( C^* \) of a any geodesic on \( M_n \) is a geodesic on \( T(M_n) \) with the metric \( CGg \).

**References**


