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İSMAİL Ş. GÜLOĞLU

CEMAL KOÇ

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## Stack-Sortable Permutations and Polynomials

*İ. Ş. Güloğlu and C. Koç*

### Abstract

The Catalan numbers show up in a diverse variety of counting problems. In this note we give yet another characterization of the Catalan number  $C(n)$ . It is characterized as the dimension of a certain space of multilinear polynomials by exhibiting a basis.

**Key word and phrases:** Catalan numbers, stack-sortable permutations, permutations acting on polynomials

### 1. Introduction

The action of permutations on polynomials is one of the most indispensable techniques of algebra and it shows up in almost all considerations. In [1], to characterize differential forms which can be factorized as  $\mu \wedge \omega$  for a fixed 2-form  $\mu$  by means of a number of homogeneous exterior equations, a set of generators of  $Ann(\mu)$ , the annihilator ideal of  $(\mu)$  has been exhibited. In [2] this construction has been generalized to certain even and odd forms. However, the generating sets of even forms under consideration there are far from being minimal. In the construction of a minimal basis for  $Ann(\mu)$ , where  $\mu = \mu_1 + \dots + \mu_{2n}$ , whose terms are exterior products of vectors in a vector space for which  $\mu_1 \cdots \mu_{2n} \neq 0$ , the first step is to construct a basis for the subspace of the exterior algebra spanned by the products

$$(\mu_{\sigma(1)} - \mu_{\sigma(2)}) \cdots (\mu_{\sigma(2n-1)} - \mu_{\sigma(2n)}), \quad \sigma \in S_{2n}.$$

However, it is worthwhile to handle this problem in a more general context by considering the algebra of polynomials. This is the objective of this paper. We consider  $F[x_1, \dots, x_{2n}]$ , the ring of polynomials in the undeterminates  $x_1, x_2, \dots, x_{2n}$  over the field  $F$  on which permutations acts canonically,

$$(\sigma f)(x_1, \dots, x_{2n}) = f(x_{\sigma(1)}, \dots, x_{\sigma(2n)}),$$

and exhibit a basis for  $F[S_{2n}]p$ , the cyclic submodule generated by

$$p(x_1, \dots, x_{2n}) = (x_1 - x_2)(x_3 - x_4) \cdots (x_{2n-1} - x_{2n})$$

of the module  $F[x_1, \dots, x_{2n}]$  over the group algebra  $F[S_{2n}]$  where  $S_{2n}$  is the symmetric group on  $2n$  letters.

## 2. A Basis Corresponding to Stack-Sortable Permutations

In order to facilitate the presentation we consider the polynomial algebra  $F[x_1, \dots, x_n; y_1, \dots, y_n]$  and the action of permutations in  $S_n$  on this algebra defined by

$$f_\sigma = (\sigma f)(x_1, \dots, x_n; y_1, \dots, y_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}; y_1, \dots, y_n).$$

Now, we consider the polynomial

$$p(x_1, \dots, x_n; y_1, \dots, y_n) = (x_1 - y_1) \dots (x_n - y_n),$$

and the cyclic submodule  $F[S_n]p$  over the group algebra  $F[S_n]$ . We construct a basis for this submodule by using “231-avoiding permutations”, that is permutations for which  $\sigma(k) < \sigma(i) < \sigma(j)$  cannot occur when  $i < j < k$ . Such permutations are called stack-sortable permutations. Their set will be denoted by  $St_n$ . It is well known that the number of stack-sortable permutations of degree  $n$  is equal to the Catalan number  $C(n) = \frac{1}{n+1} \binom{2n}{n}$ . For this and many other characterizations of the Catalan numbers we refer to page 219 of the book [3]. Our construction furnishes yet another characterization of Catalan numbers as the dimension of a space of polynomials.

**Proposition 1** *The sets*

$$\{p_\sigma(x_1, \dots, x_n; y_1, \dots, y_n) \mid \sigma \in St_n\} \text{ and } \{p_\sigma(x_1, \dots, x_n; y_1, \dots, y_{n-1}, 0) \mid \sigma \in St_n\}$$

*are linearly independent.*

**Proof.** To each permutation  $\sigma$  we assign a sequence  $\varphi_\sigma = (s_1, s_2, \dots, s_n; s'_1, s'_2, \dots, s'_n)$ , where  $s_{\sigma(n)} = 1$ ,  $s'_n = 0$  and for  $k < n$  the terms  $s_{\sigma(k)}$  and  $s'_k$  are defined by

$$s_{\sigma(k)} = \begin{cases} 1 & \text{if } \sigma(k) < \sigma(k+1) \\ 0 & \text{if } \sigma(k) > \sigma(k+1) \end{cases} \quad \text{and} \quad s'_k = 1 - s_{\sigma(k)}.$$

We shall compute  $p_\tau(\varphi_\sigma)$  for stack-sortable permutations  $\sigma$  and  $\tau$ ; and using these values we shall obtain a linear system of equations for the coefficients  $a_\tau$  in the relation  $\sum_{\tau \in St_n} a_\tau p_\tau = 0$  from which it will be immediate that all these coefficients are 0. Since the last term of  $\varphi_\sigma$  is zero in our discussion we do not need any distinction of the sets given in the proposition.

Obviously we have  $p_\tau(\varphi_\tau) = \pm 1$ . We consider the ordering of permutations defined by

$$\tau > \sigma \iff \exists 1 \leq k \leq n \text{ such that } \tau(i) = \sigma(i) \text{ for all } i > k \text{ and } \tau(k) > \sigma(k).$$

we claim that when  $\sigma$  and  $\tau$  are stack-sortable permutations with  $\tau > \sigma$ , we have

$$p_\tau(\varphi_\sigma) = p_\tau(s_1, s_2, \dots, s_n; s'_1, s'_2, \dots, s'_n) = 0.$$

In fact, assuming  $\tau(k) > \sigma(k)$  for some  $k \leq n$ , and  $\tau(i) = \sigma(i)$  for all  $i > k$ , we observe that

$$\tau(k) = \sigma(r) \text{ for some } r < k \text{ and } \sigma(k) = \tau(l) \text{ for some } l < k.$$

Since for stack-sortable permutations  $\sigma$  and  $\tau$ , neither inequality  $\sigma(k) < \sigma(i) < \sigma(j)$  nor  $\tau(k) < \tau(i) < \tau(j)$  can occur when  $i < j < k$ . If we had  $s_{\sigma(k)} = 0$ , this would mean  $k < n$  and  $\sigma(k) > \sigma(k+1)$ , which would imply

$$\sigma(k+1) = \tau(k+1) < \sigma(k) = \tau(l) < \tau(k) \text{ for } l < k < k+1.$$

This would contradict the stack-sortability of  $\tau$ . Thus  $s_{\sigma(k)} = 1$  and hence  $s'_k = 0$ . The same argument repeated for  $s_{\sigma(r)} = 1$ , where  $r < k$ , would imply

$$\sigma(k) < \tau(k) = \sigma(r) < \sigma(r+1) \text{ for } r < r+1 \leq k.$$

This in turn shows that  $r+1 = k$  cannot happen, and  $r < r+1 < k$  contradicts the stack-sortability of  $\sigma$ . Thus we obtained

$$s_{\tau(k)} = s_{\sigma(r)} = 0 \text{ and } s'_k = 0.$$

to evaluate

$$p_{\tau}(x_1, \dots, x_n; y_1, \dots, y_n) = (x_{\tau(1)} - y_1) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n)$$

at  $\varphi_{\sigma} = (s_1, s_2, \dots, s_n; s'_1, s'_2, \dots, s'_n)$ , we substitute  $x_{\tau(k)} = s_{\tau(k)} = 0$  and  $y_k = s'_k = 0$  and obtain

$$p_{\tau}(\varphi_{\sigma}) = p_{\tau}(s_1, s_2, \dots, s_n; s'_1, s'_2, \dots, s'_n) = 0,$$

as claimed. To complete the proof, suppose that

$$\sum_{\tau \in St_n} a_{\tau} p_{\tau} = 0$$

and that  $\sigma$  is the least permutation for which  $a_{\sigma} \neq 0$ . Then we have

$$\sum_{\tau \in St_n} a_{\tau} p_{\tau}(\varphi_{\sigma}) = \pm a_{\sigma} = 0,$$

which is a contradiction. □

**Lemma 2** (i) If  $n \leq 2$ , every permutation is stack sortable.

(ii)  $\sum_{\tau \in St_3} \text{sgn}(\tau) p_{\tau} = 0$  and hence the polynomial  $p_{\sigma}$ , associated to the 3-cycle  $\sigma = (123)$ , is a linear combination of the  $p_{\tau_i}$ , where  $\tau_1 = (12)$ ,  $\tau_2 = (13)$ ,  $\tau_3 = (23)$ ,  $\tau_4 = (132)$  and  $\tau_5 = (1)$ .

(iii) If  $\sigma$  has a 231-pattern, say,  $\sigma = [\dots, \sigma(i), \dots, \sigma(j), \dots, \sigma(k), \dots]$ , with  $\sigma(k) < \sigma(i) < \sigma(j)$ , then  $p_{\sigma}$  is a linear combination of the  $p_{\sigma\tau_l}$ , where  $1 \leq l \leq 5$ ,  $\tau_1 = (ij)$ ,  $\tau_2 = (ik)$ ,  $\tau_3 = (jk)$ ,  $\tau_4 = (ijk)$  and  $\tau_5 = (kji)$ .

**Proof.** (i) is obvious, (ii) is obtained by a straightforward verification, and (iii) is an easy consequence of (ii). □

**Definition 3** An  $n$ -term sequence  $[t_1, t_2, \dots, t_k, \dots, t_m]$  is said to be tidy if

$$t_k \geq t_i \quad \text{for all } i < k, \text{ and } t_j > t_i \text{ for all } j \geq k, i < k.$$

A permutation is said to be tidy if the sequence  $[\sigma(1), \sigma(2), \dots, \sigma(n)]$  is tidy that is if  $\sigma(i) \geq \sigma^{-1}(n)$  for all  $i \geq \sigma^{-1}(n)$ .

**Lemma 4** If  $\sigma^{-1}(n) \in \{1, n\}$ , then  $\sigma$  is tidy.

**Proof.** Obvious. □

**Proposition 5** Every  $p_\sigma$  is a linear combination of the  $p_\tau$  corresponding to tidy permutations  $\tau$ .

**Proof.** Let

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n) \text{ with } \sigma(k) = n$$

Considering Lemma 2 (i) we can use induction on  $n$ . By assuming the assertion is true for  $n - 1$ , we consider the permutation  $\bar{\sigma} \in S_{n-1}$  defined by

$$\bar{\sigma}(i) = \begin{cases} \sigma(i) & \text{if } i < k \\ \sigma(i+1) & \text{if } i \geq k \end{cases}.$$

And applying Lemma 2(iii) to

$$(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k-1)} - y_{k-1})(x_{\sigma(k+1)} - y_{k+1}) \cdots (x_{\sigma(n)} - y_n)$$

we may assume without loss of generality that

$$\bar{\sigma}(i) \geq \bar{\sigma}^{-1}(n-1) \text{ for all } i \geq \bar{\sigma}^{-1}(n-1) \text{ in } \{1, 2, \dots, n-1\}.$$

In other words, the sequence  $[\sigma(1), \dots, \sigma(k-1), \sigma(k+1), \dots, \sigma(n)]$  is tidy with largest term  $\sigma(l) = n - 1$ . Now, we consider several particular cases separately.

**Case 1.** Let  $\sigma^{-1}(n-1) = n$ . Then

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(n-1)} - y_{n-1})(x_{n-1} - y_n);$$

and by induction hypothesis applied to bijections from  $\{1, 2, \dots, n-1\}$  onto  $\{1, 2, \dots, n\} - \{n-1\}$ , it is a linear combination of polynomials of the form

$$p_\tau = ((x_{\tau(1)} - y_1) \cdots (x_{\tau(n-2)} - y_{n-2})(x_{\tau(n)} - y_{n-1}))(x_{n-1} - y_n),$$

where  $\tau(i) \geq \tau^{-1}(n)$  for all  $i \geq \tau^{-1}(n)$  in  $\{1, 2, \dots, n-1\}$ . Now, extending  $\tau$  to a permutation of  $\{1, 2, \dots, n\}$  with  $\tau(n-1) = n$ , we still have  $\tau(i) \geq \tau^{-1}(n)$  for all  $i \geq \tau^{-1}(n)$  in  $\{1, 2, \dots, n\}$ . Thus, each  $\tau$  is tidy.

**Case 2.** Let  $\sigma^{-1}(n-1) = 1$ . Then we may assume  $\sigma^{-1}(n) \neq n$  by Lemma 4, and the 231 pattern shows up with

$$\sigma(1) = n-1, \sigma(k) = n \text{ for some } k < n, \text{ and } \sigma(n) = r \text{ for some } r < n-1.$$

By using Lemma 2 (iii) we can express  $p_\sigma$  as a linear combination of the  $p_\tau$  where  $\tau \in \{\sigma \circ (1n), \sigma \circ (1k), \sigma \circ (kn), \sigma \circ (1kn), \sigma \circ (1nk)\}$ . For  $\tau = \sigma \circ (1n)$  we have  $\tau(n) = n - 1$  as in Case 1, and for all others we have either  $\tau(n) = n$  or  $\tau(1) = n$  as in Lemma 4.

**Case 3.** Let  $1 < \sigma^{-1}(n - 1) < \sigma^{-1}(n)$ , then

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(l)} - y_l) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n)$$

with  $\sigma(l) = n - 1$ ,  $\sigma(k) = n$  and  $\sigma(i) < l = \sigma^{-1}(n - 1)$  for all  $i < l$ . Since  $l > 1$ , the induction hypothesis allows us to express

$$(x_{\sigma(l)} - y_l) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n)$$

as a linear combination of polynomials

$$(x_{\tau(l)} - y_l) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n),$$

where each  $\tau$  is a tidy permutation of  $\{l, \dots, n\}$ , and also its extension to  $\{1, \dots, n\}$  defined by

$$\bar{\tau}(i) = \begin{cases} \sigma(i) & \text{if } i < l \\ \tau(i) & \text{if } i \geq l \end{cases}$$

is tidy. Thus  $p_\sigma$  is a linear combination of products

$$(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(l-1)} - y_{l-1}) (x_{\tau(l)} - y_l) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n),$$

each of which turns out to be a polynomial associated with a tidy permutation  $\bar{\tau}$ .

**Case 4.** Let  $\sigma^{-1}(n) < \sigma^{-1}(n - 1) < n$ . Then

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(l)} - y_l) \cdots (x_{\sigma(n)} - y_n)$$

with  $\sigma(l) = n - 1$  and  $\sigma(k) = n$ , and the sequence

$$[\sigma(1), \dots, \sigma(k - 1), \sigma(k + 1), \dots, \sigma(l), \dots, \sigma(n)]$$

is tidy with largest term  $\sigma(l) = n - 1$ . Since  $l < n$ , we can use the induction hypothesis to write  $(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(l-1)} - y_{l-1})$  as a linear combination of the products

$$(x_{\tau(1)} - y_1) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(l-1)} - y_{l-1}),$$

for which the sequence  $[\tau(1), \dots, \tau(k), \dots, \tau(l - 1)]$  is tidy and its terms are in  $\{1, 2, \dots, l - 1, n\}$ . Then each sequence

$$[\tau(1), \dots, \tau(k), \dots, \tau(l - 1), \sigma(l), \dots, \sigma(n)]$$

becomes a tidy sequence associated to a permutation in  $S_n$ ; and thus  $p_\sigma$  becomes a linear combination of polynomials

$$(x_{\tau(1)} - y_1) \cdots (x_{\tau(k)} - y_k) \cdots (x_{\tau(l-1)} - y_{l-1}) (x_{\sigma(l)} - y_l) \cdots (x_{\sigma(n)} - y_n)$$

associated to tidy permutations. □

Combining the above propositions we establish our main result.

**Theorem 6** *The set*

$$\{p_\tau | \tau \in St_n\}$$

*is an  $F$ -basis for the cyclic submodule  $F[S_n]p$  and hence its dimension is the Catalan number  $C(n)$ .*

**Proof.** Linear independence follows from Proposition 1. To complete the proof we use induction on  $n$ . Suppose that when  $m < n$ , for every permutation  $\rho \in S_m$ ,  $p_\rho$  is a linear combination of the polynomials  $p_\omega$  where  $\omega$  runs over stack-sortable permutations in  $S_m$ . Take any  $p_\sigma$  and use Proposition 5 to express it as a linear combination of polynomials  $p_\tau$  corresponding to tidy permutations  $\tau$ . Now, pick-up a tidy permutation  $\tau$  with  $k = \tau^{-1}(n)$ . Then,  $\tau(i) \geq k$  for  $i \geq k$ , and  $\tau(i) < k$  for  $i < k$ . We consider two cases separately.

**Case1.**  $k = 1$ , we use Proposition 5 and the induction hypothesis to write

$$(x_{\tau(2)} - z_1) \cdots (x_{\tau(n)} - z_{n-1})$$

as a linear combination of the products

$$(x_{\rho(1)} - z_1) \cdots (x_{\rho(n-1)} - z_{n-1}),$$

where each  $\rho \in S_{n-1}$  is stack-sortable. By letting  $z_1 = y_2, \dots, z_{n-1} = y_n$  we see that

$$\begin{aligned} (x_n - y_1)(x_{\rho(1)} - z_1) \cdots (x_{\rho(n-1)} - z_{n-1}) &= (x_n - y_1)(x_{\rho(1)} - y_2) \cdots (x_{\rho(n-1)} - y_n) \\ &= (x_{\bar{\rho}(1)} - y_1)(x_{\bar{\rho}(2)} - y_2) \cdots (x_{\bar{\rho}(n)} - y_n), \end{aligned}$$

where the permutation  $\bar{\rho}$  is defined by

$$\bar{\rho}(i) = \begin{cases} \rho(i-1) & \text{if } i > 1 \\ n & \text{if } i = 1 \end{cases}$$

and it is a stack-sortable permutation in  $S_n$ .

**Case 2.**  $k = \tau^{-1}(n) > 1$ . Then we have

$$p_\tau = (x_{\tau(1)} - y_1) \cdots (x_{\tau(k-1)} - y_{k-1})(x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n)$$

with

$$\{\tau(1), \dots, \tau(k-1)\} = \{1, \dots, k-1\} \text{ and } \{\tau(k), \dots, \tau(n)\} = \{k, \dots, n\}$$

and applying the induction hypothesis to the restrictions of  $\tau$  to  $\{1, \dots, k-1\}$  and  $\{k, \dots, n\}$  we can write  $(x_{\tau(1)} - y_1) \cdots (x_{\tau(k-1)} - y_{k-1})$  as a linear combination of the products  $(x_{\eta(1)} - y_1) \cdots (x_{\eta(k-1)} - y_{k-1})$  with  $\eta \in St_{k-1}$ . Moreover, by using Case1 we can write  $(x_{\tau(k)} - y_k) \cdots (x_{\tau(n)} - y_n)$  as a linear combination of products  $(x_{\zeta(k)} - y_k) \cdots (x_{\zeta(n)} - y_n)$  where each  $\zeta$  is a stake sortable permutation of  $\{k, \dots, n\}$  with  $\zeta(k) = n$ . Then for  $p_\tau$  we obtain a linear combination of products

$$\begin{aligned} &(x_{\eta(1)} - y_1) \cdots (x_{\eta(k-1)} - y_{k-1})(x_{\zeta(k)} - y_k) \cdots (x_{\zeta(n)} - y_n) \\ &= (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n), \end{aligned}$$

where the sequence  $[\eta(1), \dots, \eta(k-1), \varsigma(k), \dots, \varsigma(n)]$  defines the stack-sortable permutation  $\sigma$  in  $S_n$  as

$$\sigma(i) = \begin{cases} \eta(i) & \text{if } i < k-1 \\ \varsigma(i) & \text{if } i \geq k, \end{cases}$$

and thus each product under consideration becomes

$$p_\sigma = (x_{\sigma(1)} - y_1) \cdots (x_{\sigma(k)} - y_k) \cdots (x_{\sigma(n)} - y_n) \text{ for some } \sigma \in St_n.$$

□

**Corollary 7** *Let*

$$p(x_1, \dots, x_{2n}) = (x_1 - x_2)(x_3 - x_4) \cdots (x_{2n-1} - x_{2n}).$$

*Then the set*

$$\{p_\sigma = (x_{\sigma(1)} - x_{n+1}) \cdots (x_{\sigma(n)} - x_{2n}) \mid \sigma \in St_n\}$$

*forms a basis for the cyclic submodule  $F[S_{2n}]p$ .*

**Proof.** Letting  $y_k = x_{n+k}$  for  $k = 1, \dots, n$  it is sufficient to note that

$$(x_i - x_j)(y_k - y_l) = (x_i - y_k)(x_j - y_l) - (x_i - y_l)(x_j - y_k),$$

because this allows us to write every element of  $F[S_{2n}]p$  as a linear combination of polynomials in the form  $(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(n)} - y_n)$ , where  $\sigma$  is a permutation of  $\{1, \dots, n\}$ .

□

**Corollary 8** *Let*

$$p(x_1, \dots, x_{2n}) = (x_1 - x_2)(x_3 - x_4) \cdots (x_{2n-1} - x_{2n})x_{2n+1}.$$

*Then the set*

$$\{p_\sigma = (x_{\sigma(1)} - x_{n+2}) \cdots (x_{\sigma(n)} - x_{2n+1})x_{\sigma(n+1)} \mid \sigma \in St_{n+1}\}$$

*forms a basis for the cyclic submodule  $F[S_{2n+1}]p$  and hence its dimension is  $C(n)$ .*

**Proof.** We let  $y_k = x_{n+k+1}$  for  $k = 1, \dots, n$  and  $y_{n+1} = 0$  and we see by the theorem that each

$$p_\sigma = (x_{\sigma(1)} - x_{n+2}) \cdots (x_{\sigma(n)} - x_{2n+1})(x_{\sigma(n+1)} - 0),$$

which is considered to be the evaluation of the polynomial

$$(x_{\sigma(1)} - y_1) \cdots (x_{\sigma(n)} - y_n)(x_{\sigma(n+1)} - y_{n+1}),$$

which at  $y_{n+1} = 0$  can be expressed as a linear combination of the polynomials

$$(x_{\tau(1)} - y_1) \cdots (x_{\tau(n)} - y_n)(x_{\tau(n+1)} - y_{n+1}), \tau \in St_n$$

evaluated at  $y_{n+1} = 0$  as desired. Linear independence follows from Proposition 1.

□



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İ. Ş. GÜLOĞLU, C. KOÇ  
Department of Mathematics  
Doğuş University,  
Acıbadem, Zeamet Sokak  
34722, Kadıköy, İstanbul, TURKEY

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