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On Symmetric Monomial curves in \mathbb{P}^3

Mesut Şahin

Abstract

In this paper, we give an elementary proof of the fact that symmetric arithmetically Cohen-Macaulay monomial curves are set-theoretic complete intersections. The proof is constructive and provides the equations of the surfaces cutting out the monomial curve.

Key Words: Set-theoretic complete intersections, monomial curves

1. Introduction

Let K be an algebraically closed field and R be the polynomial ring $K[x_0, \dots, x_n]$. To any irreducible curve C in \mathbb{P}^n , one can associate a prime ideal $I(C) \subset R$ to be the set of all polynomials vanishing on C . The arithmetical rank of C , denoted by $\mu(C)$, is the least positive integer r for which $I(C) = \text{rad}(f_1, \dots, f_r)$, for some polynomials f_1, \dots, f_r or equivalently $C = H_1 \cap \dots \cap H_r$, where H_1, \dots, H_r are the hypersurfaces defined by $f_1 = 0, \dots, f_r = 0$, respectively. We denote by $\mu(I(C))$ the minimal number r for which $I(C) = (f_1, \dots, f_r)$, for some polynomials $f_1, \dots, f_r \in R$. These invariants are known to be bounded below by the codimension of the curve (or height of its ideal). So, one has the following relation:

$$n - 1 \leq \mu(C) \leq \mu(I(C))$$

Although $\mu(I(C))$ has no upper bound (see [1], for an example), an upper bound for $\mu(C)$ is provided to be n in [7] via commutative algebraic methods. Later in [2, 22] the equations of these n hypersurfaces that cut out the curve C were given explicitly by using elementary algebraic methods.

The curve C is called a *complete intersection* if $\mu(I(C)) = n - 1$. It is called an *almost complete intersection*, if instead, one has $\mu(I(C)) = n$. When the arithmetical rank of C takes its lower bound, that is $\mu(C) = n - 1$, the curve C is called a *set-theoretic complete intersection*, s.t.c.i. for short. It is clear that complete intersections are set-theoretic complete intersection. The corresponding question for almost complete intersection monomial curves is answered affirmatively in a series of papers by Eto [8, 9, 10].

Determining s.t.c.i. monomial curves is a classical and longstanding problem in algebraic geometry. Even more difficult is to give explicitly the equations of the hypersurfaces involved. It is known that all monomial

curves are s.t.c.i. in the projective n -space over a field of positive characteristic [3, 12, 14]. On the other hand, nobody knows whether or not the same question has an affirmative answer in the characteristic zero case. However, there are many partial results in this case [11, 13, 15, 16, 17, 19, 20, 21]. In fact, even the case of *symmetric* monomial curves is still mysterious.

The purpose of this note is to give an alternative proof of the fact that symmetric monomial curves in \mathbb{P}^3 which are arithmetically Cohen-Macaulay are s.t.c.i. by elementary algebraic methods inspired by [4]. The proof is constructive and provides the equations of the surfaces cutting out the curve.

2. Symmetric ACM Monomial Curves in \mathbb{P}^3

Let $p < q < r$ be some positive integers. A monomial curve $C(p, q, r)$ in \mathbb{P}^3 is given parametrically by

$$(w, x, y, z) = (u^r, u^{r-p}v^p, u^{r-q}v^q, v^r),$$

where $(u, v) \in \mathbb{P}^1$.

We say that the monomial curve $C(p, q, r)$ is *symmetric* if $p + q = r$. In this case the parametric representation of the curve $C(p, q, p + q)$ becomes $(u^{p+q}, u^q v^p, u^p v^q, v^{p+q})$.

It is known that all monomial curves are s.t.c.i. in \mathbb{P}^3 , if the base field K is of positive characteristic [12]. But, no one knows whether even the symmetric monomial curves are s.t.c.i. in \mathbb{P}^3 in the characteristic zero case. To address this case, we work with an algebraically closed field K of characteristic zero throughout the paper.

A *minimal* system of generators for the ideal of symmetric monomial curves in \mathbb{P}^3 is given in [6] as follows:

$$f = xy - wz \quad \text{and} \quad F_i = w^{q-p-i}y^{p+i} - x^{q-i}z^i, \quad \text{for all } 0 \leq i \leq q - p.$$

Recall that a monomial curve $C(p, q, r) \subset \mathbb{P}^3$ is called Arithmetically Cohen-Macaulay (ACM) if its projective coordinate ring is Cohen-Macaulay. In the same article [6], it is also proven that a monomial curve in \mathbb{P}^3 is ACM if and only if its ideal is generated by at most 3 polynomials. Now, if the ideal of a symmetric monomial curve $C(p, q, p + q)$ is generated by two polynomials it would follow that $p = q$. But, this contradicts with the assumption that $p < q < r$. So, the ideal of an ACM symmetric monomial curve $C(p, q, p + q)$ is generated by three polynomials and hence $p = q - 1$, where necessarily $q > 1$. Thus, all symmetric ACM monomial curves in \mathbb{P}^3 are of the form $C(q - 1, q, 2q - 1)$ and their defining ideals are generated minimally by the following three polynomials:

$$\begin{aligned} f &= xy - zw, \\ g &= -F_1 = x^{q-1}z - y^q, \\ h &= -F_0 = x^q - y^{q-1}w. \end{aligned}$$

The fact that $C(q - 1, q, 2q - 1)$ is a s.t.c.i. curve was shown in [17], but the equation of the second surface was not given. Here, we give an alternative proof that constructs the polynomial G such that the symmetric ACM monomial curve is the intersection of the surface $G = 0$ and a binomial surface defined by one of f, g

and h . We construct G by adding $x^q g$ to the q -th power of f and dividing the sum by z . Hence we get the following theorem.

Theorem. *Any symmetric Arithmetically Cohen-Macaulay monomial curve in \mathbb{P}^3 , which is given by $C(q - 1, q, 2q - 1)$ for some $q > 1$, is a set-theoretic complete intersection of the following two surfaces:*

$$g = x^{q-1}z - y^q = 0 \quad \text{and}$$

$$G = x^{2q-1} + \sum_{k=1}^q (-1)^k \frac{q!}{(q-k)!k!} x^{q-k} y^{q-k} z^{k-1} w^k = 0.$$

Proof. Note first that $zG = f^q + x^q g$. Take a point (w_0, x_0, y_0, z_0) from $Z(f, g, h)$. Then, by $z_0 G(w_0, x_0, y_0, z_0) = f^q(w_0, x_0, y_0, z_0) + x_0^q g(w_0, x_0, y_0, z_0) = 0$ we observe that either $G(w_0, x_0, y_0, z_0) = 0$ or $z_0 = 0$.

If $G(w_0, x_0, y_0, z_0) = 0$ then $(w_0, x_0, y_0, z_0) \in Z(g, G)$. If $z_0 = 0$ then by $g(w_0, x_0, y_0, z_0) = 0$ we get $y_0 = 0$, and by $h(w_0, x_0, y_0, z_0) = 0$ we get $x_0 = 0$. Thus $(w_0, x_0, y_0, z_0) = (1, 0, 0, 0)$ which is in $Z(g, G)$.

Let us now take a point $(w_0, x_0, y_0, z_0) \in Z(g, G)$. Then either $z_0 = 0$ or we can assume $z_0 = 1$. If $z_0 = 0$ then by $g(w_0, x_0, y_0, z_0) = 0$ we get $y_0 = 0$, and by $G(w_0, x_0, y_0, z_0) = 0$ we obtain $x_0 = 0$ in this case. Thus $(w_0, x_0, y_0, z_0) = (1, 0, 0, 0)$ which is in $Z(f, g, h)$. On the other hand, if $z_0 = 1$ then by $G = f^q + x_0^q g$ we see that $f(w_0, x_0, y_0, z_0) = 0$. Moreover, we have $x_0 y_0 = w_0$ and $x_0^{q-1} = y_0^q$ in this case. Hence we obtain the following $x_0^q = x_0 x_0^{q-1} = x_0 y_0^q = x_0 y_0 y_0^{q-1} = w_0 y_0^{q-1}$, meaning that $h(w_0, x_0, y_0, z_0) = 0$. \square

Remark. Note that the symmetric ACM monomial curves above are s.t.c.i. on the binomial surface $g = 0$. This is not true for symmetric non-ACM monomial curves; that is, they can never be a s.t.c.i. on a binomial surface [18, Theorem 5.1]. Thus it is very difficult to construct surfaces on which symmetric non-ACM monomial curves in \mathbb{P}^3 are s.t.c.i. with the simplest open case being the Macaulay's quartic curve $C(1, 3, 4)$.

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