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# Partial open book decompositions and the contact class in sutured floer homology

*Tolga Etgü and Burak Özbağcı*

## Abstract

We demonstrate how to combinatorially calculate the EH-class of a compatible contact structure in the sutured Floer homology group of a balanced sutured three manifold which is associated to an *abstract* partial open book decomposition. As an application we show that every contact three manifold (closed or with convex boundary) can be obtained by gluing tight contact handlebodies whose EH-classes are nontrivial.

**Key word and phrases:** Partial open book decomposition, contact three-manifold with convex boundary, sutured manifold, sutured Floer homology, EH-contact class.

## 1. Introduction

A sutured manifold  $(M, \Gamma)$  is a compact oriented 3-manifold with nonempty boundary, together with a compact subsurface  $\Gamma = A(\Gamma) \cup T(\Gamma) \subset \partial M$ , where  $A(\Gamma)$  is a union of pairwise disjoint annuli and  $T(\Gamma)$  is a union of tori. Moreover each component of  $\partial M \setminus \Gamma$  is oriented, subject to the condition that the orientation changes every time we nontrivially cross  $A(\Gamma)$ . Let  $R_+(\Gamma)$  (resp.  $R_-(\Gamma)$ ) be the open subsurface of  $\partial M \setminus \Gamma$  on which the orientation agrees with (resp. is the opposite of) the boundary orientation on  $\partial M$ . A sutured manifold  $(M, \Gamma)$  is balanced if  $M$  has no closed components,  $\pi_0(A(\Gamma)) \rightarrow \pi_0(\partial M)$  is surjective, and  $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$  on every component of  $M$ . It follows that if  $(M, \Gamma)$  is balanced, then  $\Gamma = A(\Gamma)$  and every component of  $\partial M$  nontrivially intersects  $\Gamma$ . Since all our sutured manifolds will be balanced in this paper, we can think of  $\Gamma$  as a set of *oriented curves* on  $\partial M$  by identifying each annulus in  $\Gamma$  with its core circle. Here  $\Gamma$  is oriented as the boundary of  $R_+(\Gamma)$ .

Let  $\xi$  be a contact structure on a compact oriented 3-manifold  $M$  whose dividing set on the convex boundary  $\partial M$  is denoted by  $\Gamma$ . Then it is not too hard to see that  $(M, \Gamma)$  is a *balanced* sutured manifold with the identification we mentioned above. Recently, Honda, Kazez and Matić [12] introduced an invariant  $EH(M, \Gamma, \xi)$  of the contact structure  $\xi$  which lives in the sutured Floer homology group  $SFH(-M, -\Gamma)$  defined for the balanced sutured manifold  $(M, \Gamma)$  by Juhász [9]. This invariant generalizes the contact class in Heegaard Floer homology in the closed case as defined by Ozsváth and Szabó [15] and reformulated in [11].

In order to define  $EH(M, \Gamma, \xi)$ , Honda, Kazez and Matić first construct a partial open book decomposition of  $M$  compatible with the given contact structure  $\xi$  by generalizing the work of Giroux [7] in the closed case. Then they obtain an admissible balanced Heegaard diagram for  $(-M, -\Gamma)$  which not only leads to the calculation of the sutured Floer homology group  $SFH(-M, -\Gamma)$  but also includes the description (similar to the one in the closed case again due to Honda, Kazez and Matić [11]) of a certain cycle descending to the contact class  $EH(M, \Gamma, \xi)$  in  $SFH(-M, -\Gamma)$ , in fact in  $SFH(-M, -\Gamma)/\{\pm 1\}$ ; this  $\pm 1$  ambiguity is usually suppressed.

On the other hand, in [3] the authors gave an abstract definition of a partial open book decomposition of a compact 3-manifold with boundary; associated a balanced sutured manifold to a partial open book decomposition and constructed a compatible contact structure on this sutured manifold whose dividing set on the convex boundary agrees with the suture. In this paper we show that the sutured Floer homology group and the EH-contact class can be combinatorially calculated starting from an *abstract* partial open book decomposition.

We include here several sample calculations of the EH-contact class and as an application we show that every contact three manifold (closed or with convex boundary) can be obtained by gluing tight contact handlebodies whose EH-classes are nontrivial.

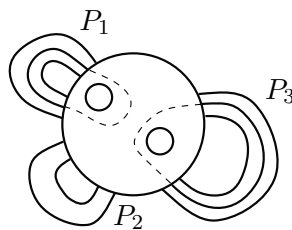
The reader is advised to turn to [12] and [3] for necessary background on partial open book decompositions, to Juhász's papers [9] and [10] for the definition and properties of the sutured Floer homology of balanced sutured manifolds and to Etnyre's notes [5] for the related material on contact topology of three-manifolds.

## 2. Partial Open Book Decompositions and Compatible Contact Structures

In this section we quickly review basics about partial open book decompositions and compatible contact structures as described in [12] and [3].

**Definition 2.1** ([3]) *A partial open book decomposition is a triple  $(S, P, h)$  satisfying the following conditions:*

- (1)  $S$  is a compact oriented connected surface with  $\partial S \neq \emptyset$ ;
- (2)  $P = P_1 \cup P_2 \cup \dots \cup P_r$  is a proper (not necessarily connected) subsurface of  $S$  such that  $S$  is obtained from  $\overline{S \setminus P}$  by successively attaching 1-handles  $P_1, P_2, \dots, P_r$ ;
- (3)  $h : P \rightarrow S$  is an embedding such that  $h|_A = \text{identity}$ , where  $A = \partial P \cap \partial S$ .



**Figure 1.** An example of  $S$  and  $P$  satisfying the conditions in Definition 2.1:  $\overline{S \setminus P}$  is a twice punctured disk,  $r = 3$ , and  $h$  is the embedding which is identity on  $P_2$  and maps the cores of  $P_1$  and  $P_3$  to the dashed arcs.

Given a partial open book decomposition  $(S, P, h)$ , we construct a sutured manifold  $(M, \Gamma)$  as follows:  
Let

$$H = (S \times [-1, 0]) / \sim$$

where  $(x, t) \sim (x, t')$  for  $x \in \partial S$  and  $t, t' \in [-1, 0]$ . It is easy to see that  $H$  is a solid handlebody whose oriented boundary is the surface  $S \times \{0\} \cup -S \times \{-1\}$  (modulo the relation  $(x, 0) \sim (x, -1)$  for every  $x \in \partial S$ ). Similarly let

$$N = (P \times [0, 1]) / \sim$$

where  $(x, t) \sim (x, t')$  for  $x \in A$  and  $t, t' \in [0, 1]$ . Observe that each component of  $N$  is also a solid handlebody. The oriented boundary of  $N$  can be described as follows: Let the arcs  $c_1, c_2, \dots, c_n$  denote the connected components of  $\overline{\partial P \setminus \partial S}$ . Then, for  $1 \leq i \leq n$ , the disk  $D_i = (c_i \times [0, 1]) / \sim$  belongs to  $\partial N$ . Thus part of  $\partial N$  is given by the disjoint union of  $D_i$ 's. The rest of  $\partial N$  is the surface  $P \times \{1\} \cup -P \times \{0\}$  (modulo the relation  $(x, 0) \sim (x, 1)$  for every  $x \in A$ ).

Let  $M = N \cup H$  where we glue these manifolds by identifying  $P \times \{0\} \subset \partial N$  with  $P \times \{0\} \subset \partial H$  and  $P \times \{1\} \subset \partial N$  with  $h(P) \times \{-1\} \subset \partial H$ . Since the gluing identification is orientation reversing  $M$  is a compact oriented 3-manifold with oriented boundary

$$\partial M = (S \setminus P) \times \{0\} \cup -(S \setminus h(P)) \times \{-1\} \cup (\overline{\partial P \setminus \partial S}) \times [0, 1]$$

(modulo the identifications given above).

We define the suture  $\Gamma$  on  $\partial M$  as the set of closed curves obtained by gluing the arcs  $c_i \times \{1/2\} \subset \partial N$ , for  $1 \leq i \leq n$ , with the arcs in  $(\overline{\partial S \setminus \partial P}) \times \{0\} \subset \partial H$ , hence as an oriented simple closed curve and modulo identifications

$$\Gamma = (\overline{\partial S \setminus \partial P}) \times \{0\} \cup -(\overline{\partial P \setminus \partial S}) \times \{1/2\}.$$

In [3] we showed that the sutured manifold  $(M, \Gamma)$  associated to a partial open book decomposition  $(S, P, h)$  is balanced.

**Proposition 2.2 ([3])** *Let  $(M, \Gamma)$  be the balanced sutured manifold associated to a partial open book decomposition  $(S, P, h)$ . Then there exists a contact structure  $\xi$  on  $M$  satisfying the following conditions:*

- (1)  $\xi$  is tight when restricted to  $H$  and  $N$ ,
- (2)  $\partial H$  is a convex surface in  $(M, \xi)$  whose dividing set is  $\partial S \times \{0\}$ ,
- (3)  $\partial N$  is a convex surface in  $(M, \xi)$  whose dividing set is  $\partial P \times \{1/2\}$ .

Moreover such  $\xi$  is unique up to isotopy.

Let  $(M, \Gamma)$  be the balanced sutured manifold associated to a partial open book decomposition  $(S, P, h)$ . A contact structure  $\xi$  on  $(M, \Gamma)$  is said to be compatible with  $(S, P, h)$  if it satisfies conditions (1), (2) and (3) stated in Proposition 2.2. It follows from Proposition 2.2 that every partial open book decomposition has a unique compatible contact structure, up to isotopy, on the balanced sutured manifold associated to it, such that the dividing set of the convex boundary is isotopic to the suture.

Two partial open book decompositions  $(S, P, h)$  and  $(\tilde{S}, \tilde{P}, \tilde{h})$  are isomorphic if there is a diffeomorphism  $f : S \rightarrow \tilde{S}$  such that  $f(P) = \tilde{P}$  and  $\tilde{h} = f \circ h \circ (f^{-1})|_{\tilde{P}}$ . If  $(S, P, h)$  and  $(\tilde{S}, \tilde{P}, \tilde{h})$  are isomorphic partial

open book decompositions, then the associated compatible contact 3-manifolds  $(M, \Gamma, \xi)$  and  $(\widetilde{M}, \widetilde{\Gamma}, \widetilde{\xi})$  are isomorphic.

The following theorem is the key to obtaining a description of a partial open book decomposition of  $(M, \Gamma, \xi)$  in the sense of Honda, Kazez and Matić.

**Definition 2.3** *A standard contact 3-ball is a tight contact 3-ball with a convex boundary.*

**Theorem 2.4 ([12], Theorem 1.1)** *Let  $(M, \Gamma)$  be a balanced sutured manifold and let  $\xi$  be a contact structure on  $M$  with convex boundary whose dividing set  $\Gamma_{\partial M}$  on  $\partial M$  is isotopic to  $\Gamma$ . Then there exist a Legendrian graph  $K \subset M$  whose endpoints lie on  $\Gamma \subset \partial M$  and a regular neighborhood  $N(K) \subset M$  of  $K$  which satisfy the following:*

- (A) (i)  $T = \overline{\partial N(K) \setminus \partial M}$  is a convex surface with Legendrian boundary.
  - (ii) For each component  $\gamma_i$  of  $\partial T$ ,  $\gamma_i \cap \Gamma_{\partial M}$  has two connected components.
  - (iii) There is a system of pairwise disjoint compressing disks  $D_j^\alpha$  for  $N(K)$  so that  $\partial D_j^\alpha$  is a curve on  $T$  intersecting the dividing set  $\Gamma_T$  of  $T$  at two points and each component of  $N(K) \setminus \cup_j D_j^\alpha$  is a standard contact 3-ball, after rounding the edges.
- (B) (i) Each component  $H$  of  $\overline{M \setminus N(K)}$  is a handlebody (with convex boundary).
  - (ii) There is a system of pairwise disjoint compressing disks  $D_k^\delta$  for  $H$  so that each  $\partial D_k^\delta$  intersects the dividing set  $\Gamma_{\partial H}$  of  $\partial H$  at two points and  $H \setminus \cup_k D_k^\delta$  is a standard contact 3-ball, after rounding the edges.

Based on Theorem 2.4, Honda, Kazez and Matić describe a partial open book decomposition on  $(M, \Gamma)$  in Section 2 of their article [12]. In this paper, for the sake of simplicity and without loss of generality, we will assume that  $M$  is connected. As a consequence  $M \setminus N(K)$  in Theorem 2.4 is also connected.

In [3], we showed that the Honda-Kazez-Matić description gives an abstract partial open book decomposition  $(S, P, h)$ , the balanced sutured manifold associated to  $(S, P, h)$  is isotopic to  $(M, \Gamma)$ , and  $\xi$  is compatible with  $(S, P, h)$ . Conversely, let  $(S, P, h)$  be an abstract partial open book decomposition,  $(M, \Gamma)$  be the balanced sutured manifold associated to it, and  $\xi$  be a compatible contact structure. Then we showed [3] that  $(S, P, h)$  is given by the Honda-Kazez-Matić description.

### 3. The EH-contact Class is Combinatorial

The main result of [12] is the following theorem.

**Theorem 3.1 ([12], Theorem 0.1)** *Let  $(M, \Gamma)$  be a balanced sutured manifold and let  $\xi$  be a contact structure on  $M$  with convex boundary whose dividing set on  $\partial M$  is isotopic to  $\Gamma$ . Then there exists an invariant  $EH(M, \Gamma, \xi)$  of the contact structure  $\xi$  which lives in  $SFH(-M, -\Gamma)/\{\pm 1\}$ .*

**Remark 3.2** [4] *Given a balanced sutured manifold  $(M, \Gamma)$ , there exists a contact structure  $\xi$  on  $M$  which makes  $\partial M$  convex and realizes  $\Gamma$  as its diving set on  $\partial M$ . Conversely, given a contact 3-manifold  $(M, \xi)$  (with convex boundary) whose diving set is denoted by  $\Gamma$  on  $\partial M$ , then  $(M, \Gamma)$  is a balanced sutured manifold.*

**Remark 3.3** *The  $\pm 1$  ambiguity in the definition of  $EH(M, \Gamma, \xi)$  is usually suppressed. An alternative way is to work with  $\mathbb{Z}_2$  coefficients.*

Given a partial open book decomposition  $(S, P, h)$  consider the associated balanced sutured manifold  $(M, \Gamma)$  and the uniquely (up to isotopy) determined compatible contact structure  $\xi$  on  $M$ . In this section we will provide an algorithm to calculate the sutured Floer homology  $SFH(-M, -\Gamma)$  and the contact class  $EH(M, \Gamma, \xi)$  in  $SFH(-M, -\Gamma)$  starting from  $(S, P, h)$ .

We now review basic definitions and properties of Heegaard diagrams of sutured manifolds (cf. [9]). A sutured Heegaard diagram is given by  $(\Sigma, \alpha, \beta)$ , where the Heegaard surface  $\Sigma$  is a compact oriented surface with nonempty boundary and  $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$  and  $\beta = \{\beta_1, \beta_2, \dots, \beta_n\}$  are two sets of pairwise disjoint simple closed curves in  $\Sigma \setminus \partial\Sigma$ . Every sutured Heegaard diagram  $(\Sigma, \alpha, \beta)$ , uniquely defines a sutured manifold  $(M, \Gamma)$  as follows: Let  $M$  be the 3-manifold obtained from  $\Sigma \times [0, 1]$  by attaching 3-dimensional 2-handles along the curves  $\alpha_i \times \{0\}$  and  $\beta_j \times \{1\}$  for  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . The suture  $\Gamma$  on  $\partial M$  is defined to be the set of curves  $\partial\Sigma \times \{1/2\}$ .

In [9], Juhász proved that if  $(M, \Gamma)$  is defined by  $(\Sigma, \alpha, \beta)$ , then  $(M, \Gamma)$  is balanced if and only if  $|\alpha| = |\beta|$ , the surface  $\Sigma$  has no closed components and both  $\alpha$  and  $\beta$  consist of curves linearly independent in  $H_1(\Sigma, \mathbb{Q})$ . Hence a sutured Heegaard diagram  $(\Sigma, \alpha, \beta)$  is called balanced if it satisfies the conditions listed above. We will abbreviate balanced sutured Heegaard diagram as balanced diagram from now on.

A partial open book decomposition of  $(M, \Gamma)$  gives a sutured Heegaard diagram  $(\Sigma, \alpha, \beta)$  of  $(M, -\Gamma)$  as follows: Let

$$\Sigma = P \times \{0\} \cup -S \times \{-1\} / \sim \subset \partial H$$

be the Heegaard surface. Observe that, modulo identifications,

$$\partial\Sigma = (\overline{\partial P \setminus \partial S}) \times \{0\} \cup -(\overline{\partial S \setminus \partial P}) \times \{-1\} \simeq -\Gamma .$$

Let  $a_j$  be a cocore of the 1-handle  $P_j \subseteq P$  for each  $j \in \{1, \dots, r\}$ , so that  $a_1, a_2, \dots, a_r$  are properly embedded pairwise disjoint arcs in  $P$  with endpoints on  $A$  such that  $S \setminus \cup_j a_j$  deformation retracts onto  $\overline{S \setminus P}$ . Then define two families  $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$  and  $\beta = \{\beta_1, \beta_2, \dots, \beta_r\}$  of simple closed curves in the Heegaard surface  $\Sigma$  by  $\alpha_j = a_j \times \{0\} \cup a_j \times \{-1\} / \sim$  and  $\beta_j = b_j \times \{0\} \cup h(b_j) \times \{-1\} / \sim$ , where  $b_j$  is an arc isotopic to  $a_j$  by a small isotopy such that the endpoints of  $a_j$  are isotoped along  $\partial S$ , in the direction given by the boundary orientation of  $S$ , and  $a_j$  intersect  $b_j$  transversely in one point  $x_j$  in the interior of  $S$ . It follows that if we orient  $a_j$ , and  $b_j$  is given the induced orientation from the isotopy, then the sign of the intersection of  $a_j$  and  $b_j$  at  $x_j$  is  $+1$ . Then  $(\Sigma, \alpha, \beta)$  is a sutured Heegaard diagram of  $(M, -\Gamma)$ , where the suture is  $-\Gamma$  since  $\partial\Sigma$  is isotopic to  $-\Gamma$ .

Next we would like to review the definition of the sutured Floer homology  $SFH(M, \Gamma)$  given by Juhász (for more details see [9]). Let  $(M, \Gamma)$  be a balanced sutured manifold and  $(\Sigma, \alpha, \beta)$  be an admissible balanced diagram defining it. Then  $SFH(M, \Gamma)$  is defined to be the homology of the chain complex  $(CF(\Sigma, \alpha, \beta), \partial)$ ,

where  $CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$  is the free abelian group generated by the points in

$$\mathbb{T}_\alpha \cap \mathbb{T}_\beta = (\alpha_1 \times \alpha_2 \times \cdots \times \alpha_r) \cap (\beta_1 \times \beta_2 \times \cdots \times \beta_r) \subset \text{Sym}^r(\Sigma).$$

For  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ , let  $\mathcal{M}_{\mathbf{x}, \mathbf{y}}$  denote the moduli space of pseudo-holomorphic maps

$$u : \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\} \rightarrow \text{Sym}^r(\Sigma)$$

satisfying,

1.  $u(1) = \mathbf{x}$  and  $u(-1) = \mathbf{y}$ ,
2.  $u(\partial\mathbb{D} \cap \{z \in \mathbb{C} : \text{Im}z \geq 0\}) \subset \mathbb{T}_\alpha$  and  $u(\partial\mathbb{D} \cap \{z \in \mathbb{C} : \text{Im}z \leq 0\}) \subset \mathbb{T}_\beta$ ,
3.  $u(\mathbb{D}) \cap (\partial\Sigma \times \text{Sym}^{r-1}(\Sigma)) = \emptyset$ .

Then the boundary map  $\partial$  is defined by

$$\partial\mathbf{x} = \sum_{\mu(\mathbf{x}, \mathbf{y})=1} \#(\mathcal{M}_{\mathbf{x}, \mathbf{y}})\mathbf{y},$$

where  $\mu(\mathbf{x}, \mathbf{y})$  is the relative Maslov index of the pair and  $\#(\mathcal{M}_{\mathbf{x}, \mathbf{y}})$  is a signed count of points in the 0-dimensional quotient (by the natural  $\mathbb{R}$ -action) of  $\mathcal{M}_{\mathbf{x}, \mathbf{y}}$ .

Let  $(S, P, h)$  be a partial open book decomposition and let  $(M, \Gamma)$  be the associated balanced sutured manifold. We described a balanced diagram  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$  of  $(M, -\Gamma)$  above. By changing the order of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  we obtain a balanced diagram of  $(-M, -\Gamma)$ . Such a balanced diagram is shown to be admissible in [12]. Hence the sutured Floer homology group  $SFH(-M, -\Gamma)$  can be defined using  $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ . The contact class  $EH(M, \Gamma, \xi)$  is an invariant of the contact structure  $\xi$  and defined [12] to be the homology class in  $SFH(-M, -\Gamma)$  which descends from the cycle  $\mathbf{x}$  in the complex  $CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ , where  $\mathbf{x} = (x_1, x_2, \dots, x_r) \in \text{Sym}^r(\Sigma)$ .

**Theorem 3.4** *Let  $(S, P, h)$  be a partial open book decomposition. Then the sutured Floer homology group  $SFH(-M, -\Gamma)$  and the contact class  $EH(M, \Gamma, \xi)$  in  $SFH(-M, -\Gamma)$  can be calculated combinatorially, where  $(M, \Gamma)$  is the balanced sutured manifold associated to  $(S, P, h)$  and  $\xi$  is a contact structure on  $M$  compatible with  $(S, P, h)$ .*

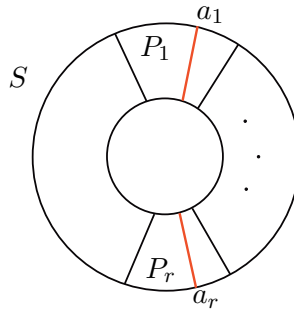
**Proof.** A balanced diagram  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$  is called *simple* if every component of  $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$  whose closure is disjoint from  $\partial\Sigma$  is a bigon or a square. In [10], Juhász proves, by modifying the procedure of Sarkar and Wang [17], that any balanced diagram can be turned into a simple using some isotopies and handle slides of the  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  curves on  $\Sigma$ . This is one of the main steps of an algorithm to calculate the sutured Floer homology combinatorially since the boundary homomorphism in the chain complex defining the homology induced by a simple balanced diagram can be calculated combinatorially. To prove that this general procedure fits in our discussion where balanced diagrams are supposed to be induced by partial open book decompositions, we adapt the proof of Theorem 2.1 in [16] on the Sarkar-Wang algorithm to the analogous algorithm of Juhász for balanced diagrams. Juhász defines the distance to  $\partial\Sigma$  and badness of a component of  $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$  in a natural way so that distance is positive if and only if its closure intersects  $\partial\Sigma$ , and badness is positive if and

only if the component is neither a bigon nor a square. As he explains in [10], to obtain a simple from a given balanced diagram, one can start from the bad regions, i.e., regions with positive badness and positive distance, with maximum distance, and use isotopies (finger moves) and handle slides of  $\alpha$  and  $\beta$  curves to gradually decrease maximum distance of bad regions, and eventually eliminate them all. In the general case, handle slides are necessary only in a very special case (Case A in [10] p. 326) that requires the existence of a  $\beta$  curve which is the union of some arcs on the boundary of square regions of positive distance only, but this is impossible in our situation, i.e., when the balanced diagram is induced by a partial open book decomposition, since each  $\beta_j$  forms part of the boundary of a region of distance 0, e.g. the component of  $\Sigma \setminus (\alpha \cup \beta)$  which contains either component of  $(P_j \setminus I_j) \times \{0\}$ , where  $I_j$  is the image of the small isotopy from the cocore  $a_j$  of  $P_j$  to  $b_j$ . On the other hand, all the necessary isotopies are on  $S \times \{-1\} \subset \Sigma$ , away from  $A$ . Denote the composition of these isotopies by  $\phi$  and observe that  $\phi$  is a diffeomorphism fixing  $A$  and isotopic to identity. The resulting simple diagram corresponds to the partial open book decomposition  $(S, P, h')$ , where  $h' = \phi \circ h$ . Hence the cycle  $\mathbf{x} \in \text{Sym}^r(P \times \{0\}) \subset \text{Sym}^r(\Sigma)$  considered in this simple balanced diagram still descends to  $EH(M, \Gamma, \xi)$ .

Once we have a simple diagram, by [10], it is combinatorial to calculate the boundary map of the sutured Floer chain complex. We just make a list of all the generators and count all the empty embedded bigons and squares on the Heegaard surface connecting these generators by examining the diagram. Finally by using simple linear algebra we can compute  $SFH(-M, -\Gamma)$  and identify  $[\mathbf{x}] = EH(M, \Gamma, \xi) \in SFH(-M, -\Gamma)$ .  $\square$

In the rest of this section we present a few examples to demonstrate the procedure explained in the proof of Theorem 3.4 above.

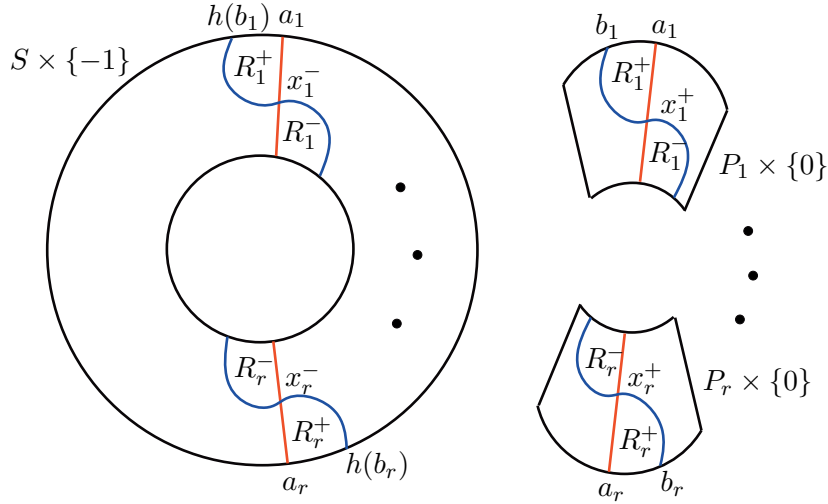
**Example 1.** Let  $S$  be an annulus,  $P$  be a regular neighborhood of  $r$  disjoint arcs connecting the two distinct boundary components of  $S$  as in Figure 2, and the monodromy  $h$  be the inclusion of  $P$  into  $S$ .



**Figure 2.** The annulus  $S$ ,  $r$  components  $P_1, \dots, P_r$  of  $P$ , and a basis  $\{a_1, \dots, a_r\}$  conveyed in Example 1.

According to the notation in Figure 3, the chain complex  $CF(\Sigma, \beta, \alpha)$  is generated by the  $2^r$  generators  $\{(x_1^{\epsilon_1}, x_2^{\epsilon_2}, \dots, x_r^{\epsilon_r})\}$ , where  $\epsilon_j = \pm$  for  $j = 1, 2, \dots, r$ . On the other hand, the regions that contribute to the boundary homomorphism  $\partial$  are  $R_1^\pm, R_2^\pm, \dots, R_r^\pm$ . Each bigon  $R_j^\pm$  effects only the generators of the form  $(x_1^{\epsilon_1}, x_2^{\epsilon_2}, \dots, x_j^\pm, \dots, x_r^{\epsilon_r})$  and the contribution is  $\pm 1$  times the generator which differs only in the  $j^{\text{th}}$  component. The fact that the contribution has absolute value 1 follows from Theorem 7.4 in [10], and for each  $j$  the signs of the contributions of  $R_j^\pm$  are opposite of each other by Lemma 9.1 (and especially the part of its





**Figure 3.** The surfaces  $S \times \{-1\}$  and  $P \times \{0\}$ ,  $\alpha$  and  $\beta$  curves, their intersections  $x_j^\pm$ , and the regions  $R_j^\pm$  mentioned in Example 1.

proof regarding the choice of a coherent system of orientations) in [14]. For example,  $\partial(x_1^-, x_2^+, x_3^+, \dots, x_r^+) = (x_1^+, x_2^+, x_3^+, \dots, x_r^+) - (x_1^+, x_2^+, x_3^+, \dots, x_r^+)$ , where the first term is induced by  $R_1^+$  and the second term is induced by  $R_1^-$ . Consequently, the boundary map is trivial, hence  $SFH(-M, -\Gamma) \cong CF(\Sigma, \beta, \alpha) \cong \mathbb{Z}^{2^r}$ , and

$$EH(M, \Gamma, \xi) = [\mathbf{x}] = [(x_1^+, x_2^+, \dots, x_r^+)]$$

is a generator of one of the  $\mathbb{Z}$  summands.

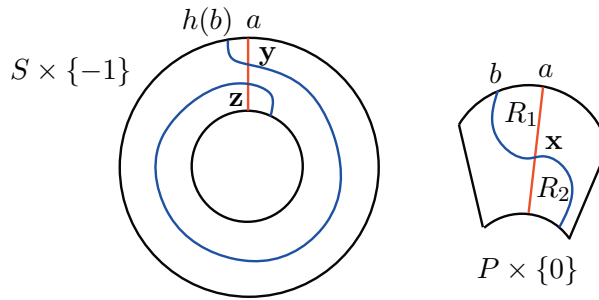
Next we describe the balanced sutured manifold  $(M, \Gamma)$  associated to this partial open book decomposition  $(S, P, h)$ , where we fix a positive integer  $r$  for the rest of the discussion. For each positive integer  $m$ , let  $Y(m)$  denote the balanced sutured manifold obtained by taking out  $m$  disjoint open 3-balls from a closed 3-manifold  $Y$  and declaring the suture to have exactly one connected component on each component of  $\partial Y(m)$ , as in [9]. Then we claim that  $(M, \Gamma) \cong Y(r)$  for  $Y = S^1 \times S^2$ . To prove our claim we observe that the closed 3-manifold which corresponds to the open book decomposition with an annulus page and identity monodromy is  $S^1 \times S^2$ . Thus  $M$  can be obtained from  $Y$  by taking out  $r$  disjoint open 3-balls corresponding to  $r$  connected components of  $S \setminus P$ . Moreover, by our construction in Section 2, the suture  $\Gamma$  has  $r$  connected components each of which belongs to a different component of  $\partial M$ . In the light of this observation, the sutured Floer homology can be calculated alternatively by using Proposition 9.14 in [9] which, states that  $SFH(Y(m)) \cong \bigoplus_{2^{m-1}} \widehat{HF}(Y)$  and the fact that  $\widehat{HF}(S^1 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ .

Furthermore, we can identify the contact structure  $\xi$  on  $M$ , which is compatible with  $(S, P, h)$ , as the contact structure obtained by removing  $r$  disjoint standard contact open 3-balls from the unique (up to isotopy) tight contact structure  $\xi_{std}$  on  $S^1 \times S^2$ . Hence the nontriviality of  $EH(M, \Gamma, \xi)$  also follows from Theorem 4.5 in [12] combined with the facts that

$$EH(S^1 \times S^2, \xi_{std}) = c(S^1 \times S^2, \xi_{std}) \text{ and } c(S^1 \times S^2, \xi_{std}) \neq 0,$$

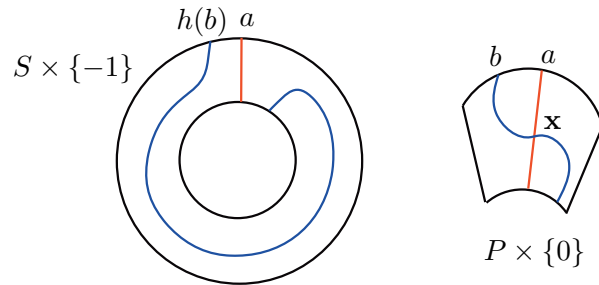
where  $c(S^1 \times S^2, \xi_{std})$  denotes the contact Ozsváth-Szabó invariant.

**Example 2.** Let  $S$  and  $P$  be as in the previous example for  $r = 1$  and the monodromy  $h$  be the restriction (to  $P$ ) of a *left-handed* Dehn twist along the core of  $S$ . Using the notation in Figure 4, the generators of the chain complex are  $\mathbf{x}, \mathbf{y}$  and  $\mathbf{z}$ . Moreover,  $\partial\mathbf{x} = 0$ ,  $\partial\mathbf{y} = \mathbf{x}$  (by  $R_1$ ) and  $\partial\mathbf{z} = \mathbf{x}$  (by  $R_2$ ). Hence  $SFH(-M, -\Gamma) = \mathbb{Z}$  and  $EH(M, \Gamma, \xi) = 0$ . This is consistent with the fact that the open book decomposition with annulus page and left-handed Dehn twist monodromy is compatible with an overtwisted contact  $S^3$ . In addition, by Proposition 4.2 in [12], if the monodromy of a partial open book decomposition is not *right-veering*, i.e., if there is a properly embedded arc  $l \subset P$  with endpoints on  $A$  such that  $h(l)$  is not to the right of  $l$ , then the contact invariant of the compatible contact structure is zero.



**Figure 4.** The surfaces  $S \times \{-1\}$  and  $P \times \{0\}$ , curves  $\alpha$  and  $\beta$ , their intersections  $\mathbf{x}, \mathbf{y}, \mathbf{z}$ , and the regions  $R_1$  and  $R_2$  in Example 2.

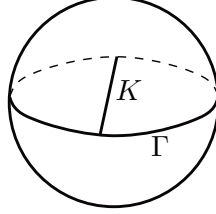
**Example 3. (Standard contact 3-ball)** Let  $S$  and  $P$  be as in the first example for  $r = 1$ , and the monodromy  $h$  be the restriction (to  $P$ ) of a *right-handed* Dehn twist along the core of  $S$ . Then using the notation in Figure 5, there is a single generator  $\mathbf{x}$  in the chain complex  $CF(\Sigma, \beta, \alpha)$  and hence the boundary homomorphism is trivial. It follows that  $SFH(-M, -\Gamma) \cong \mathbb{Z}$  and  $EH(M, \Gamma, \xi)$  is a generator.



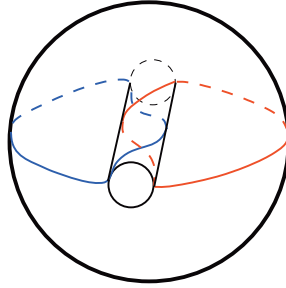
**Figure 5.** The surfaces  $S \times \{-1\}$  and  $P \times \{0\}$ , curves  $\alpha$  and  $\beta$ , and the intersection  $\mathbf{x}$  for  $r = 1$  in Example 3.

In fact, we can identify the contact 3-manifold  $(M, \Gamma, \xi)$  as the standard contact 3-ball (cf. Definition 2.3). Here the Legendrian graph  $K$  which satisfies the conditions in Theorem 2.4 is a single arc in  $B^3$  connecting two distinct points on  $\Gamma$  as depicted in Figure 6. The complement  $H$  of a regular neighborhood  $N = N(K)$  in the standard contact 3-ball  $B^3$  is a solid torus with two parallel dividing curves (see Figure 7) on  $\partial H$  which are homotopically nontrivial inside  $H$ . Here a meridional disk in  $H$  will serve as the required compressing disk  $D_1^\partial$  for  $H$  in Theorem 2.4 (B). On the other hand,  $N$  is already a standard contact 3-ball.

This shows in particular that the standard contact 3-ball can be obtained from a tight solid torus  $H$  by attaching a tight 2-handle  $N$ . Note that, in Example 2, we attached a tight 2-handle to a tight solid torus to obtain an overtwisted contact structure.



**Figure 6.** The Legendrian arc  $K$  in the standard contact 3-ball.

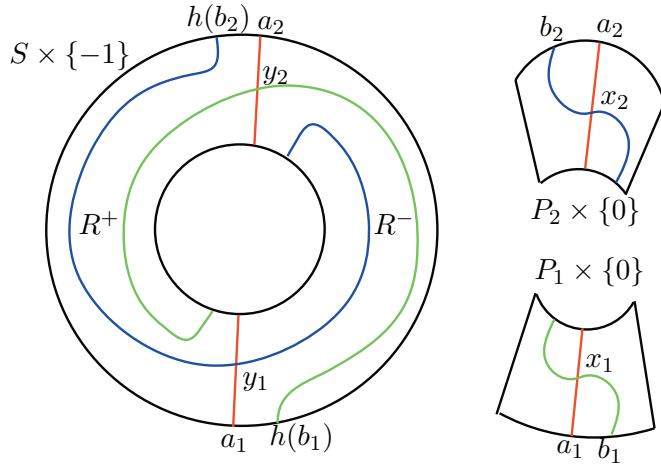


**Figure 7.** The dividing curves on  $\partial H$ .

With a different point of view, one can see that  $(M, \Gamma) \cong S^3(1)$ , which is obtained by removing an open 3-ball from the standard tight contact  $S^3$ . Observe that the open book decomposition with annulus page and right-handed Dehn twist monodromy is compatible with the standard tight contact  $S^3$  and hence has nonzero contact class. Therefore Theorem 4.5 in [12] already implies that the contact invariant in this example is not zero. Moreover, the sutured Floer homology calculations are consistent with the aforementioned result of Juhász.

**Example 4.** Let  $S$  and  $P$  be as in the first example for  $r \in \{2, 3\}$ , and the monodromy  $h$  be the restriction (to  $P$ ) of a *right*-handed Dehn twist along the core of  $S$ . First consider the case  $r = 2$ . Using the notation in Figure 3, the generators of the chain complex are  $\mathbf{x} = (x_1, x_2)$  and  $\mathbf{y} = (y_1, y_2)$  with  $\partial \mathbf{x} = 0$  and  $\partial \mathbf{y} = \mathbf{x} - \mathbf{x} = 0$  (by  $R^\pm$ ), where the opposite signs for the contributions of  $R^\pm$  follow from Lemma 9.1 in [14] as in Example 1. Hence  $SFH(-M, -\Gamma) \cong \mathbb{Z} \oplus \mathbb{Z}$  and  $EH(M, \Gamma, \xi)$  is a generator of one of the  $\mathbb{Z}$  summands. Note that  $(M, \Gamma) \cong S^3(2)$ .

Next consider the case  $r = 3$ . Using the notation in Figure 9, the six generators of the chain complex  $CF(\Sigma, \beta, \alpha)$  are  $\{\mathbf{x}_{ijk} = (x_{1i}, x_{2j}, x_{3k}) : \{i, j, k\} = \{1, 2, 3\}\}$ , where  $x_{ij}$  is the single intersection point in  $\alpha_i \cap \beta_j$ , and the contact class  $\mathbf{x}$  is  $\mathbf{x}_{123} = (x_{11}, x_{22}, x_{33}) \in Sym^3(\Sigma)$ . The boundary homomorphism is given by  $\partial \mathbf{x}_{123} = 0$ ,  $\partial \mathbf{x}_{213} = \mathbf{x}_{123} - \mathbf{x}_{123} = 0$  (by  $R_1 \cup R_2$  and  $R_4 \cup R_5$ ),  $\partial \mathbf{x}_{321} = \mathbf{x}_{123} - \mathbf{x}_{123} = 0$  (by  $R_5 \cup R_6$  and  $R_2 \cup R_3$ ),  $\partial \mathbf{x}_{132} = \mathbf{x}_{123} - \mathbf{x}_{123} = 0$  (by  $R_3 \cup R_4$  and  $R_1 \cup R_6$ ),  $\partial \mathbf{x}_{231} = \mathbf{x}_{321} + \mathbf{x}_{213} + \mathbf{x}_{132}$  (by  $R_1, R_3$  and



**Figure 8.** The surfaces  $S \times \{-1\}$  and  $P \times \{0\}$ , curves  $\alpha$  and  $\beta$ , the intersections  $\mathbf{x}$  and  $\mathbf{y}$ , and the regions  $R^\pm$  for  $r = 2$  in Example 4.

$R_5$ ), and  $\partial \mathbf{x}_{312} = \mathbf{x}_{321} + \mathbf{x}_{213} + \mathbf{x}_{132}$  (by  $R_2, R_4$  and  $R_6$ ). As a result  $SFH(-M, -\Gamma) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$  and  $EH(M, \Gamma, \xi)$  is a generator of one of the  $\mathbb{Z}$  summands. Note that  $(M, \Gamma) \cong S^3(3)$ .

**Example 5. (Standard neighborhood of an overtwisted disk)** Let  $(S, P, h)$  be the partial open book decomposition shown in Figure 10.

Then using the notation in Figure 3, the chain complex  $CF(\Sigma, \beta, \alpha)$  has two generators  $\mathbf{x}$  and  $\mathbf{y}$ , and the boundary homomorphism is given by  $\partial \mathbf{x} = 0$ , and  $\partial \mathbf{y} = \mathbf{x}$  by the bigon  $R$ . Hence  $SFH(-M, -\Gamma) = 0$  and obviously  $EH(M, \Gamma, \xi) = 0$ . In fact, this is the partial open book considered in Example 1 of [12] which is compatible with the standard neighborhood of an overtwisted disk.

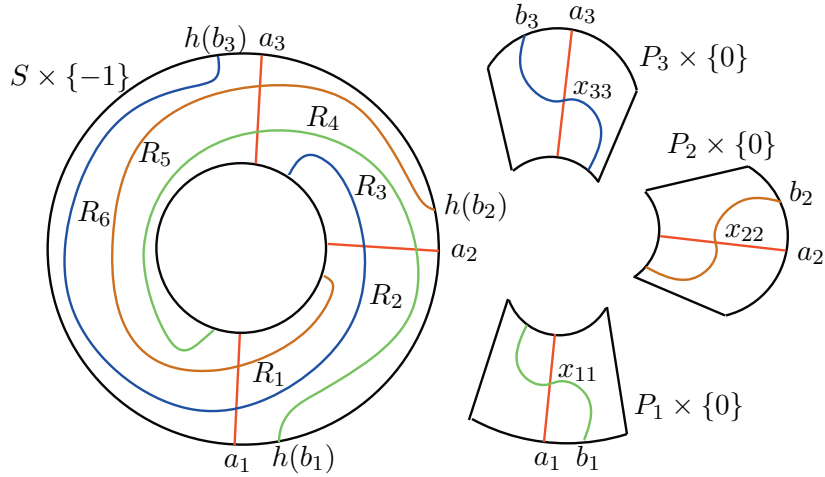
Here we observe that by Proposition 2.2,  $(M, \Gamma, \xi)$  is obtained by gluing two compact connected contact 3-manifolds with convex boundaries, namely  $(H, \Gamma_{\partial H}, \xi|_H)$  and  $(N, \Gamma_{\partial N}, \xi|_N)$  along parts of their boundaries. In the following we will compute the sutured Floer homology groups and the  $EH$ -classes of these contact submanifolds.

We know that

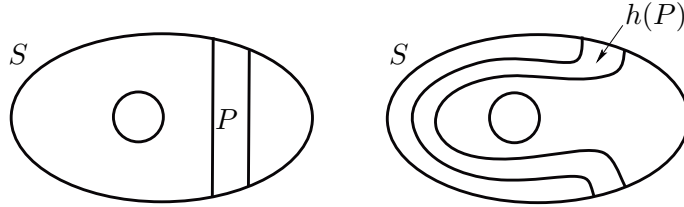
$$H = (S \times [-1, 0]) / \sim,$$

where  $S$  is an annulus and  $(x, t) \sim (x, t')$  for  $x \in \partial S$  and  $t, t' \in [-1, 0]$ . There is a unique (up to isotopy) compatible tight contact structure on  $H$  whose dividing set  $\Gamma_{\partial H}$  on  $\partial H$  is  $\partial S \times \{0\}$  (cf. Proposition 2.2). Hence  $(H, \Gamma_{\partial H}, \xi|_H)$  is a solid torus carrying a tight contact structure where  $\Gamma_{\partial H}$  consists of two parallel curves on  $\partial H$  which are homotopically nontrivial in  $H$ . We observe that when we cut  $H$  along a compressing disk we get a standard contact 3-ball  $B^3$  with its connected dividing set  $\Gamma_{\partial B^3}$  on its convex boundary. Note that  $\Gamma_{\partial B^3}$  is obtained by “gluing”  $\Gamma_{\partial H}$  and the dividing set on the compressing disk. Let  $K$  be a properly embedded Legendrian arc (as depicted in Figure 12) in  $B^3$  connecting two points on  $\Gamma_{\partial H}$ .

Then  $K$  can be viewed as a Legendrian arc (disjoint from the compressing disk) in  $H$  connecting two distinct points of  $\Gamma_{\partial H}$  which satisfies the conditions in Theorem 2.4, just as we discussed in Example 3. This gives a partial open book decomposition  $(S', P', h')$  compatible with  $(H, \Gamma_{\partial H}, \xi|_H)$ . The page  $S'$  is a thrice



**Figure 9.** The surfaces  $S \times \{-1\}$  and  $P \times \{0\}$ , curves  $\alpha$  and  $\beta$ , the intersections  $x_{ijk}$ , and the regions  $R_i$  for  $r = 3$  in Example 4.



**Figure 10.** The partial open book decomposition  $(S, P, h)$  in Example 5.

punctured sphere that can be obtained by attaching a 1-handle  $P'$  to the annulus  $R'_+ = S$ . The monodromy map  $h'$  is the restriction onto  $P'$  of a right-handed Dehn twist along the curve  $c$  which is shown in Figure 13. There is a single generator  $\mathbf{x}$  as shown in Figure 14 in the chain complex  $CF(\Sigma, \beta, \alpha)$  and thus  $EH(M, \Gamma, \xi)$  is a generator of  $SFH(-H, -\Gamma_{\partial H}) \cong \mathbb{Z}$ .

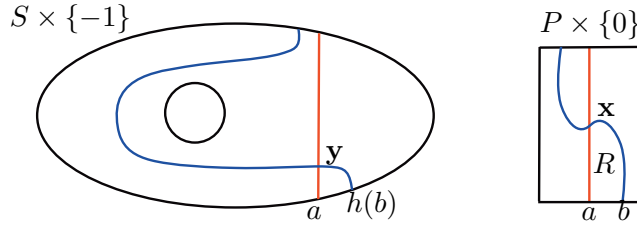
Similarly we know that

$$N = (P \times [0, 1]) / \sim$$

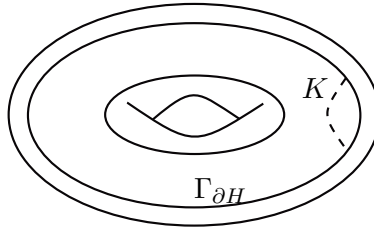
where  $(x, t) \sim (x, t')$  for  $x \in A$  and  $t, t' \in [0, 1]$ . There is a unique (up to isotopy) compatible tight contact structure on  $N$  whose dividing set  $\Gamma_{\partial N}$  on  $\partial N$  is  $\partial P \times \{1/2\}$  (cf. Proposition 2.2). We observe that  $(N, \Gamma_{\partial N}, \xi|_N)$  is the standard contact 3-ball. It follows that  $EH(N, \Gamma_{\partial N}, \xi|_N)$  is a generator of the sutured Floer homology group  $SFH(-N, -\Gamma_{\partial N}) \cong \mathbb{Z}$  (see Example 3).

**Example 6.** Let us denote the neighborhood of an overtwisted disk, in Example 5 as  $(B^3, \xi)$ . It is clear that when we glue two copies of this overtwisted 3-ball along their common boundary we obtain an overtwisted 3-sphere  $(S^3, \xi_{ot})$ . In this example, we explicitly construct the corresponding open book decomposition of  $(S^3, \xi_{ot})$  using the partial open book decomposition of  $(B^3, \xi)$ , and determine  $\xi_{ot}$  up to isotopy.

In the following, we digress to explain briefly how to extend a partial open book decomposition of a compact submanifold  $M \subset (Y, \xi)$  into an open book decomposition of the closed contact 3-manifold  $(Y, \xi)$



**Figure 11.** The surfaces  $S \times \{-1\}$  and  $P \times \{0\}$ , curves  $\alpha$  and  $\beta$ , the intersections  $\mathbf{x}$  and  $\mathbf{y}$ , and the region  $R$  in Example 5.

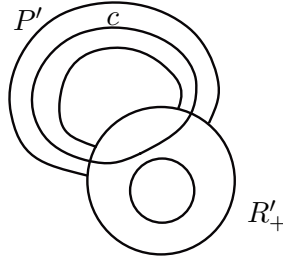


**Figure 12.** A Legendrian arc  $K$  in  $H$  connecting two points on  $\Gamma_{\partial H}$ .

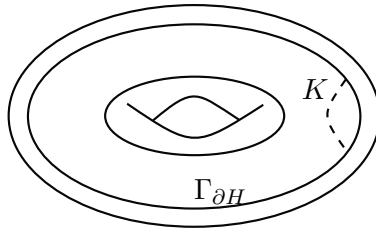
(see the proof of Theorem 4.5 in [12]). The contact 3-manifold  $(M, \xi)$  has a compatible partial open book decomposition  $(S, P, h)$  which is given by a Legendrian graph  $K$  in  $(M, \xi|_M)$ . Let  $M' = \overline{Y \setminus M}$ . Since  $\partial M'$  is convex,  $M'$  has a compatible partial open book decomposition  $(S', P', h')$  given by a Legendrian graph  $K'$  in  $(M', \xi|_{M'})$ . Here we assume that  $K$  and  $K'$  intersect the dividing set on the boundary at distinct points. Then one can glue the partial open book decompositions  $(S, P, h)$  and  $(S', P', h')$  to obtain a compatible open book decomposition for  $(Y, \xi)$ . Notice that the handlebodies  $(M \setminus N(K)) \cup N(K')$  and  $(M' \setminus N(K')) \cup N(K)$  give a Heegaard decomposition of  $Y$ . In fact, we can think of each of these handlebodies as the one half of an open book decomposition on  $Y$ . Moreover this open book decomposition is compatible with  $\xi$ , since the partial open book decompositions on  $M$  and  $M'$  are compatible with  $\xi|_M$  and  $\xi|_{M'}$ , respectively. The page of this open book decomposition is  $T = S \cup P' = S' \cup P$  and the monodromy  $g : T \rightarrow T$  is a diffeomorphism which extends  $h : P \rightarrow S$ .

Now in order to illustrate a special case of the above construction of Honda, Kazez and Matić, we take the double of the partial open book decomposition  $(S, P, h)$  of the neighborhood  $(B^3, \xi)$  of an overtwisted disk which we described in Example 5. It follows that the page of the open book decomposition of the overtwisted 3-sphere  $(S^3, \xi_{ot})$  obtained by doubling  $(B^3, \xi)$ , is a twice punctured disk  $T = S \cup P'$  as shown in Figure 15. Moreover the embeddings of  $P$  and  $P'$  are depicted (on the left) in Figure 16. The monodromy map  $g : T \rightarrow T$  which extends these embeddings is the composition of a right-handed Dehn twist along one of the punctures and a left-handed Dehn twist along the boundary of the disk as illustrated (on the right) in Figure 16.

Next, we would like to determine the overtwisted contact structure  $\xi_{ot}$  in  $S^3$  up to isotopy. Recall that Eliashberg [1] classified overtwisted contact structures: Two overtwisted contact structures are isotopic if and only if their underlying oriented plane fields are homotopic. Moreover, the homotopy class of an oriented plane field on  $S^3$  is distinguished by its  $d_3$ -invariant (see [8] or [13] for a detailed discussion). A simple computation



**Figure 13.** The surface  $S' = R'_+ \cup P'$ , and the curve  $c$  in  $S'$ .



**Figure 14.** The surfaces  $S' \times \{-1\}$  and  $P' \times \{0\}$ , curves  $\alpha$  and  $\beta$ , the intersection  $\mathbf{x}$  in Example 5.

[6] shows that  $d_3(\xi_{ot}) = 1/2$ , which pins down precisely the homotopy (and hence the isotopy) class of  $\xi_{ot}$  in  $S^3$ .

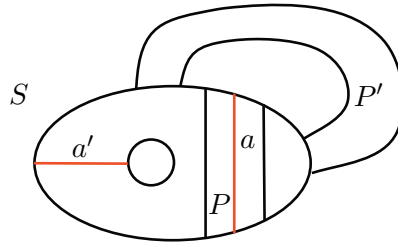
In Theorem 4.5 in [12], Honda-Kazez and Matić, actually relates the EH-class of a submanifold  $M \subset (Y, \xi)$  to the EH-class of the closed contact 3-manifold  $(Y, \xi)$ . In order to establish such a relation they make use of the extension of a partial open book decomposition of  $(M, \xi|_M)$  to an open book decomposition of  $(Y, \xi)$ . Finally we would like to illustrate their construction in our example. A basis  $\{a, a'\}$  to compute  $EH(S^3, \xi_{ot})$  is shown in Figure 15. In Figure 17 we depicted a Heegaard diagram  $(\Sigma, \{\beta, \beta'\}, \{\alpha, \alpha'\})$  for the open book decomposition of  $(S^3, \xi_{ot})$ . There are seven generators of  $CF(\Sigma, \beta, \alpha)$  and the boundary relation  $\partial(y, x') = (x, x')$  implies that  $EH(S^3, \xi_{ot}) = [(x, x')] = 0$ , as expected. The crucial point here is that when we erase  $\alpha', \beta'$  and the handle  $P'$  in Figure 17 we get a copy of Figure 3.

#### 4. Gluing Simple Pieces

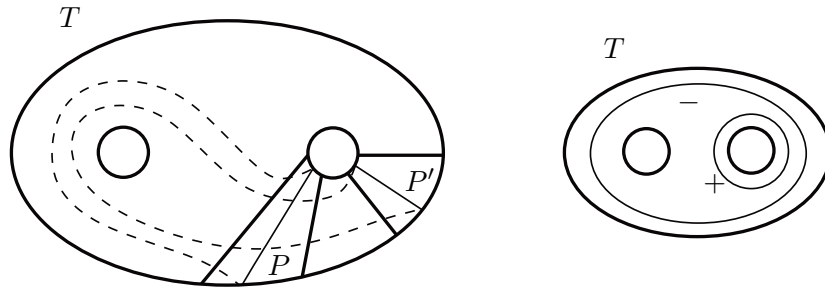
In this section we generalize the discussion at the end of Example 5 to an arbitrary contact (connected) 3-manifold  $(M, \Gamma, \xi)$  with the dividing set  $\Gamma$  on its convex boundary. It is clear that  $(M, \Gamma, \xi)$  can be obtained by gluing two tight (see Proposition 2.2) contact handlebodies  $(H, \Gamma_{\partial H}, \xi|_H)$  and  $(N, \Gamma_{\partial N}, \xi|_N)$  along parts of their boundaries.

**Proposition 4.1** *The contact class  $EH(H, \Gamma_{\partial H}, \xi|_H)$  is a generator of the sutured Floer homology group  $SFH(-H, -\Gamma_{\partial H}) \cong \mathbb{Z}$ .*

**Proof.** Let  $S$  be a connected genus  $g$  surface with  $n$  boundary components. Then there is a unique (up to isotopy) compatible tight contact structure on  $H = (S \times [-1, 0]) / \sim$ , where  $(x, t) \sim (x, t')$  for  $x \in \partial S$



**Figure 15.** The page  $T = S \cup P'$ , and a basis  $\{a, a'\}$  of  $T$ .



**Figure 16.** On the left: the embeddings of  $P$  and  $P'$  are represented by the dashed lines, which are the images of their cores, respectively.

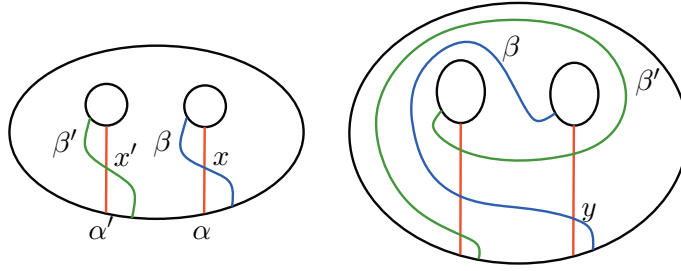
and  $t, t' \in [-1, 0]$ , whose dividing set  $\Gamma_{\partial H}$  on  $\partial H$  is  $\partial S \times \{0\}$  (cf. Proposition 2.2). In the following we will describe a partial open book decomposition  $(S', P', h')$  for  $(H, \Gamma_{\partial H}, \xi|_H)$ . We observe that when we cut the solid handlebody  $H$  along compressing disks we get a standard contact 3-ball  $B^3$ . Note that  $\Gamma_{\partial B^3}$  is obtained by gluing  $\Gamma_{\partial H}$  and the dividing sets on the compressing disks. Let  $K$  be a properly embedded Legendrian arc in  $B^3$  connecting two points on  $\Gamma_{\partial H}$ . Then  $K$  can be viewed as a Legendrian arc in  $H$  connecting two distinct points of  $\Gamma_{\partial H}$  and it satisfies the conditions in Theorem 2.4 by our construction (see Example 3). Then  $S'$  will be obtained from  $S$  by attaching a 1-handle  $P'$ . Let  $c$  be the indicated curve in Figure 18. Then  $h'$  is the restriction to  $P'$  of a right-handed Dehn twist along  $c$ . It follows (just as in Example 5, Figure 14) that there is a unique intersection point of the  $\alpha$  and  $\beta$  curves and consequently,  $EH(H, \Gamma_{\partial H}, \xi|_H)$  is a generator of  $SFH(-H, -\Gamma_{\partial H}) \cong \mathbb{Z}$ .  $\square$

**Remark 4.2** In [9], Juhász proves that the sutured Floer homology group of a product sutured 3-manifold is isomorphic to  $\mathbb{Z}$ . In fact, the proof of Prop 4.1 generalizes to give the following result.

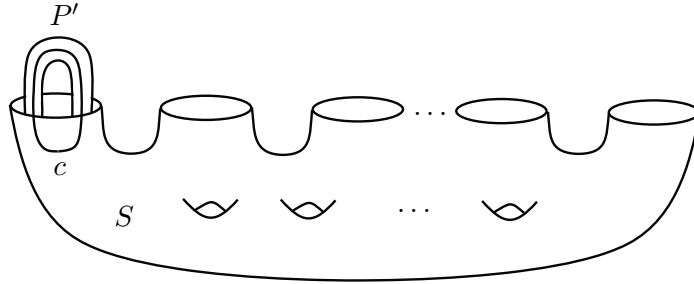
**Theorem 4.3** Let  $(W, \Gamma)$  be a product sutured 3-manifold and let  $\xi$  be the unique compatible tight contact structure on  $W$ . The contact class  $EH(W, \Gamma, \xi)$  is a generator of the sutured Floer homology group  $SFH(-W, -\Gamma) \cong \mathbb{Z}$ .

Since  $(N, \Gamma_{\partial N})$  is a product sutured manifold, we have the following result as a particular case of Theorem 4.3.





**Figure 17.** A Heegaard diagram  $(\Sigma, \{\beta, \beta'\}, \{\alpha, \alpha'\})$  for the open book decomposition of  $(S^3, \xi_{ot})$ .



**Figure 18.** The surface  $S' = S \cup P'$ , and the curve  $c$  in  $S'$ .

**Proposition 4.4** *Suppose that  $N_1, N_2, \dots, N_n$  are the connected components of the handlebody  $N$ . Then  $EH(N, \Gamma_{\partial N}, \xi|_N)$  is a generator of*

$$SFH(N, \Gamma_{\partial N}, \xi|_N) = \bigotimes_{i=1}^n SFH(-N_i, -\Gamma_{\partial N_i}) \cong \mathbb{Z}.$$

**Corollary 4.5** *Every contact 3-manifold can be obtained by gluing tight contact handlebodies whose  $EH$ -classes are nontrivial.*

**Proof.** Suppose that  $(Y, \xi)$  is a closed connected contact 3-manifold. We know that  $(Y, \xi)$  has a compatible open book decomposition with page  $S$  and monodromy  $h : S \rightarrow S$  (cf. [7]). This implies that

$$Y \cong (S \times [-2, 0]) / \sim,$$

where  $(h(x), 0) \sim (x, -2)$  for  $x \in S$  and  $(x, t) \sim (x, t')$  for  $x \in \partial S$  and  $t, t' \in [-2, 0]$ . Thus  $(Y, \xi)$  is obtained by gluing the contact handlebodies  $(H_1, \Gamma_{\partial H_1}, \xi|_{H_1})$  and  $(H_2, \Gamma_{\partial H_2}, \xi|_{H_2})$  along their convex boundaries using the monodromy map. Here

$$H_1 \cong (S \times [-1, 0]) / \sim,$$

where  $(x, 0) \sim (x, -1)$  for  $x \in S$  and  $(x, t) \sim (x, t')$  for  $x \in \partial S$  and  $t, t' \in [-1, 0]$  and

$$H_2 \cong (S \times [-2, -1]) / \sim$$

where  $(x, -1) \sim (x, -2)$  for  $x \in S$  and  $(x, t) \sim (x, t')$  for  $x \in \partial S$  and  $t, t' \in [-2, -1]$ . Then Proposition 4.1 implies that  $EH(H_i, \Gamma_{\partial H_i}, \xi|_{H_i}) \neq 0$  for  $i = 1, 2$ .

Now suppose that  $(Y, \xi)$  is a connected contact 3-manifold with nonempty convex boundary. Then  $Y$  admits a partial open book decomposition  $Y \cong H \cup N$  and the result is immediate by Propositions 4.1 and 4.4. If  $Y$  has more than one connected components then we can apply the above argument to each of its components to obtain the desired result.  $\square$

**Remark 4.6** *It is well-known that an overtwisted closed contact 3-manifold has trivial Ozsváth-Szabó invariant [15]. Every such manifold is obtained by gluing two tight contact handlebodies whose  $EH$ -classes are nontrivial by Corollary 4.5.*

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