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LUCYNA REMPULSKA

KAROLINA TOMCZAK

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Approximation by certain linear operators preserving x^2

Lucyna Rempulska, Karolina Tomczak

Abstract

We investigate certain positive linear operators L_n preserving the functions $e_k(x) = x^k$, $k = 0, 1$, and modified operators L_n^* which preserve e_0 and e_2 . We show that the error of approximation of f by $L_n^*(f)$ is smaller than for $L_n(f)$.

Key Words: Positive linear operator, polynomial weighted space, degree of approximation.

1. Introduction

1.1. Let as usual $N = \{1, 2, \dots\}$, $N_0 = N \cup \{0\}$, and let I be the interval $[0, \infty)$ or $(0, \infty)$.

Similar to [5], let $p \in \mathbb{N}_0$,

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1} \quad \text{if } p \geq 1, \quad (1.1)$$

for $x \in I$, and let $B_p \equiv B_p(I)$ be the set of all functions $f : I \rightarrow \mathbb{R}$ for which fw_p is bounded on I and the norm is given by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in I} w_p(x)|f(x)|. \quad (1.2)$$

Moreover, let $C_p \equiv C_p(I)$ with $p \in \mathbb{N}_0$ be the set of all $f \in B_p$ for which fw_p is a uniformly continuous function on I . The norm in C_p is defined by (2).

The spaces B_p and C_p are called polynomial weighted spaces.

We see that if $p, q \in \mathbb{N}_0$ and $p < q$, then $B_p \subset B_q$, $C_p \subset C_q$ and $\|f\|_q \leq \|f\|_p$ for every $f \in B_p$.

1.2. It is known ([1–7, 14, 15]) that several classical positive linear operators, e.g. the Szász-Mirakyan, Baskakov, Post-Widder and Stancu operators, act from the space C_p to C_p for every $p \in \mathbb{N}_0$ and preserve the functions $e_0(x) = 1$ and $e_1(x) = x$ but does not preserve $e_2(x) = x^2$.

Recently O. Duman and M. A. Özarslan in papers [8, 9] have introduced modified Szász-Mirakyan operators S_n^* which preserve the functions e_0 and e_2 . They have shown that the error of approximation of f , with the certain function space, by $S_n^*(f)$ is smaller than for the classical Szász-Mirakyan operators $S_n(f)$.

The similar problems were considered for the Bernstein polynomials and the MKZ type operators in [10] and [11].

1.3. The purpose of this paper is to extend the Duman-Özarslan idea ([8]) to certain sequences of linear positive operators L_n acting from the space C_p to B_p . In Section 2 we shall give definition of operators and their basic properties. The main theorems will be given in Section 3.

2. Definition and lemmas

2.1. Let $x \in I$ be a fixed point and let

$$\varphi_x(t) := |t - x| \quad \text{for } t \in I. \tag{2.3}$$

We consider a sequence $(L_n)_{n_0}^\infty$, $n_0 \in \mathbb{N}$, of positive linear operators satisfying the following conditions:

- (i) $L_n : C_p(I) \rightarrow B_p(I)$ for every $p \in \mathbb{N}_0$, $n \geq n_0$, and $L_n(f; 0) = f(0)$ for every $f \in C_p([0, \infty))$;
- (ii) For every $e_k(x) = x^k$, $k \in \mathbb{N}_0$, and $n \geq n_0$ there exists an algebraic polynomial $P_{n,k}$ of the order k such that $L_n(e_k; x) = P_{n,k}(x)$ and

$$L_n(e_0; x) = 1, \quad L_n(e_1; x) = x \quad \text{for } x \in I, n \geq n_0; \tag{2.4}$$

- (iii) There exist numbers $a, b \geq 0$, $a^2 + b^2 > 0$, and a numerical increasing and unbounded sequence $(\lambda_n)_{n_0}^\infty$ such that $\lambda_{n_0} > 0$ and

$$L_n(e_2; x) = x^2 + \frac{ax^2 + bx}{\lambda_n} \quad \text{for } x \in I, n \geq n_0; \tag{2.5}$$

- (iv) For functions

$$T_{n;p}(x) := L_n(\varphi_x^p(t); x), \quad x \in I, n \geq n_0, p \in \mathbb{N}_0. \tag{2.6}$$

there holds

$$\lim_{n \rightarrow \infty} \|T_{n;p}\|_p = 0. \tag{2.7}$$

Now, using L_n we define for $f \in C_p$, $p \in \mathbb{N}_0$, the following operators:

$$L_n^*(f; x) := L_n(f; u_n(x)), \quad x \in I, n \geq n_0, \tag{2.8}$$

with

$$u_n(x) := \frac{-b + \sqrt{b^2 + 4\lambda_n(a + \lambda_n)x^2}}{2(a + \lambda_n)}. \tag{2.9}$$

From (8), (9) and properties (i)–(iv) of L_n we deduce that L_n^* , $n \geq n_0$, is a positive linear operator well defined on every space C_p , $p \in \mathbb{N}_0$.

From (8), (9) and (3)–(5) we deduce that

$$L_n^*(e_0; x) = 1, \quad L_n^*(e_1; x) = u_n(x), \quad L_n^*(e_2; x) = x^2, \tag{2.10}$$

and

$$L_n(\varphi_x^2(t); x) = \frac{ax^2 + bx}{\lambda_n}, \quad L_n^*(\varphi_x^2(t); x) = 2x(x - u_n(x)), \tag{2.11}$$

for every $x \in I$ and $n \geq n_0$.

In this paper we shall denote by $M_k(p)$, $k \in \mathbb{N}$, suitable positive constants depending only on indicated parameter p .

2.2. Here we shall give some auxiliary results.

Lemma 1 *Let u_n and w_p be functions defined by (9) and (1). Then we have*

$$0 \leq u_n(x) \leq x, \quad 0 \leq 2x(x - u_n(x)) \leq (ax^2 + bx) / \lambda_n, \tag{2.12}$$

$$\sqrt{\frac{ax^2 + bx}{\lambda_n}} - \sqrt{2x(x - u_n(x))} \geq \sqrt{\frac{ax^2 + bx}{\lambda_n} \frac{2ax + b}{4[2ax + b + 2\lambda_n x]}}, \tag{2.13}$$

$$0 < \frac{w_p(x)}{w_p(u_n(x))} \leq 1, \tag{2.14}$$

$$(w_p(x))^2 \leq w_{2p}(x), \quad (w_p(x))^{-2} \leq 2/w_{2p}(x), \tag{2.15}$$

for all $x \in I$, $n \geq n_0$ and $p \in \mathbb{N}_0$.

Proof. From (9) we deduce that $u_n(0) = 0$ for $n \geq n_0$,

$$\begin{aligned} 0 < u_n(x) &= \frac{2\lambda_n x^2}{b + \sqrt{b^2 + 4\lambda_n(a + \lambda_n)x^2}} < \frac{2\lambda_n x^2}{\sqrt{4\lambda_n^2 x^2}} = x, \\ 0 < x - u_n(x) &= \frac{1}{2(a + \lambda_n)} \frac{[2(a + \lambda_n)x + b]^2 - b^2 - 4\lambda_n(a + \lambda_n)x^2}{2(a + \lambda_n)x + b + \sqrt{b^2 + 4\lambda_n(a + \lambda_n)x^2}} \\ &= \frac{2(ax^2 + bx)}{2(a + \lambda_n)x + b + \sqrt{b^2 + 4\lambda_n(a + \lambda_n)x^2}} < \frac{ax + b}{2\lambda_n}, \end{aligned}$$

and

$$\begin{aligned} \sqrt{(ax^2 + bx)/\lambda_n} - \sqrt{2x(x - u_n(x))} &= \frac{(ax^2 + bx)/\lambda_n - 2x(x - u_n(x))}{\sqrt{(ax^2 + bx)/\lambda_n} + \sqrt{2x(x - u_n(x))}} \\ &\geq \frac{1}{2\sqrt{(ax^2 + bx)/\lambda_n}} \left(\frac{ax^2 + bx}{\lambda_n} - \frac{4x(ax^2 + bx)}{2(a + \lambda_n)x + b + \sqrt{b^2 + 4\lambda_n(a + \lambda_n)x^2}} \right) \\ &= \sqrt{\frac{ax^2 + bx}{\lambda_n}} \frac{2ax + b + \sqrt{b^2 + 4\lambda_n(a + \lambda_n)x^2} - 2\lambda_n x}{2(2ax + b + 2\lambda_n x + \sqrt{b^2 + 4\lambda_n(a + \lambda_n)x^2})} \\ &> \sqrt{\frac{ax^2 + bx}{\lambda_n} \frac{2ax + b}{4(2ax + b + 2\lambda_n x)}}, \end{aligned}$$

for every $x > 0$ and $n \geq n_0$.

Inequalities (14) and (15) are obvious for each $p \in \mathbb{N}_0$ by (1) and (12). \square

Lemma 2 For each $p \in \mathbb{N}_0$, there exists $M_1(p) = \text{const.} > 0$ such that for the above operators L_n and L_n^* there holds

$$\|L_n^*(1/w_p)\|_p \leq \|L_n(1/w_p)\|_p \leq M_1(p), \quad n \geq n_0, \quad (2.16)$$

and

$$\|L_n^*(f)\|_p \leq \|L_n(f)\|_p \leq M_1(p)\|f\|_p, \quad (2.17)$$

for every $f \in C_p$ and $n \geq n_0$.

Formulas (8) and (9) and inequality (17) show that L_n^* , $n \geq n_0$, is linear positive operator acting from the space C_p to B_p for every $p \in \mathbb{N}_0$.

Proof. The inequalities (16) and (17) with $M_1(p) = 1$ are obvious for $p = 0$ by (1), (2), (4) and (12).

If $p \in \mathbb{N}$, then by (1)–(3) and properties of L_n , we have

$$L_n(1/w_p(t); x) = 1 + L_n(e_p; x) \leq 1 + 2^p x^p + 2^p L_n(\varphi_x^p(t); x) \quad \text{for } x \in I, n \geq n_0. \quad (2.18)$$

Next from (7) it results that there exists $M_2(p) = \text{const.} > 0$ such that $\|T_{n;p}\|_p \leq M_2(p)$ for $n \geq n_0$, which by (1), (2), (6) and (18) implies that

$$\|L_n(1/w_p)\|_p \leq 2^p (1 + M_2(p)) \quad \text{for } n \geq n_0.$$

Thus the inequality (16) for L_n is proved.

From (8), (14) and (16) for L_n we deduce that

$$\begin{aligned} w_p(x)L_n^*(1/w_p(t); x) &= \frac{w_p(x)}{w_p(u_n(x))} w_p(u_n(x)) L_n(1/w_p(t); u_n(x)) \\ &\leq \|L_n(1/w_p)\|_p \leq M_1(p) \quad \text{for } x \in I, n \geq n_0. \end{aligned}$$

Analogously for $f \in C_p$, $p \in \mathbb{N}_0$, we get

$$w_p(x) |L_n^*(f(t); x)| \leq \|L_n(f)\|_p \quad \text{for } x \in I, n \geq n_0. \quad (2.19)$$

Moreover, for $f \in C_p$, $p \in \mathbb{N}_0$, we have

$$|L_n(f; x)| \leq L_n(|f|; x), \quad x \in I, n \geq n_0,$$

which by (2) implies that

$$\|L_n(f)\|_p \leq \|f\|_p \|L_n(1/w_p)\|_p, \quad n \geq n_0. \quad (2.20)$$

Now from (19), (20) and (16) immediately follows (17). \square

For L_n and L_n^* there holds the following:

Lemma 3 *Let $f, g \in C_p$ with a fixed $p \in \mathbb{N}_0$. Then*

$$|L_n(f(t)g(t); x)| \leq (L_n(f^2(t); x))^{1/2} (L_n(g^2(t); x))^{1/2} \text{ for } x \in I, n \geq n_0,$$

and

$$\|L_n(fg)\|_{2p} \leq \|L_n(f^2)\|_{2p}^{1/2} \|L_n(g^2)\|_{2p}^{1/2} \text{ for } n \geq n_0.$$

Identical inequalities there hold for operators L_n^* . □

Applying (4) and (10)–(16), we can derive the following

Corollary 1 *Let φ_x and w_p be defined by (3) and (1). Then*

$$\begin{aligned} L_n(\varphi_x(t); x) &\leq (L_n(\varphi_x^2(t); x))^{1/2} = ((ax^2 + bx)/\lambda_n)^{1/2}, \\ L_n^*(\varphi_x(t); x) &\leq (L_n^*(\varphi_x^2(t); x))^{1/2} = (2x(x - u_n(x)))^{1/2} \leq ((ax^2 + bx)/\lambda_n)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} w_p(x)L_n(\varphi_x(t)/w_p(t); x) &\leq \sqrt{2}\|L_n(1/w_{2p})\|_{2p}^{1/2} (L_n(\varphi_x^2(t); x))^{1/2}, \\ w_p(x)L_n^*(\varphi_x(t)/w_p(t); x) &\leq \sqrt{2}\|L_n(1/w_{2p})\|_{2p}^{1/2} (L_n^*(\varphi_x^2(t); x))^{1/2}, \end{aligned}$$

for every $x \in I$ and $n \geq n_0$.

3. Theorems

Here we shall prove two approximation theorems for the above operators. Let C_p^1 , $p \in \mathbb{N}_0$, be the set of all functions $f \in C_p$ having first derivative $f' \in C_p$.

Theorem 1 *For each $p \in \mathbb{N}_0$, there exists $M_3(p) = \text{const.} > 0$ such that for every $f \in C_p^1$ there holds*

$$w_p(x) |L_n(f; x) - f(x)| \leq M_3(p) \|f'\|_p \sqrt{(ax^2 + bx)/\lambda_n} \tag{3.21}$$

and

$$w_p(x) |L_n^*(f; x) - f(x)| \leq M_3(p) \|f'\|_p \sqrt{2x(x - u_n(x))}, \tag{3.22}$$

for $x \in I$ and $n \geq n_0$.

Proof. We shall prove only (22) because the proof of (21) is identical by the above lemmas.

For a fixed $f \in C_p^1$ and $x \in I$ we can write

$$f(t) - f(x) = \int_x^t f'(u) du, \quad t \in I,$$

and

$$\left| \int_x^t f'(u) du \right| \leq \|f'\|_p \left| \int_x^t \frac{du}{w_p(u)} \right| \leq \|f'\|_p (1/w_p(t) + 1/w_p(x)) |t - x|.$$

Using now L_n^* and (10), (3) and Corollary 1, we get

$$\begin{aligned} w_p(x) |L_n^*(f(t); x) - f(x)| &\leq w_p(x) L_n^* \left(\left| \int_x^t f'(u) du \right|; x \right) \\ &\leq \|f'\|_p (w_p(x) L_n^*(\varphi_x(t)/w_p(t); x) + L_n^*(\varphi_x(t); x)) \\ &\leq \|f'\|_p \left(\sqrt{2} \|L_n(1/w_{2p})\|_{2p}^{1/2} + 1 \right) \sqrt{2x(x - u_n(x))}, \end{aligned}$$

which by (16) implies (22). \square

Theorem 2 For each $p \in \mathbb{N}_0$, there exists $M_4(p) = \text{const.} > 0$ such that for every $f \in C_p$, $x \in I$ and $n \geq n_0$, we have

$$w_p(x) |L_n(f; x) - f(x)| \leq M_4(p) \omega \left(f; \sqrt{(ax^2 + bx)/\lambda_n} \right)_p \quad (3.23)$$

and

$$w_p(x) |L_n^*(f; x) - f(x)| \leq M_4(p) \omega \left(f; \sqrt{2x(x - u_n(x))} \right)_p \leq M_4(p) \omega \left(f; \sqrt{(ax^2 + bx)/\lambda_n} \right)_p \quad (3.24)$$

where $\omega(f; \cdot)_p$ is the modulus of continuity of $f \in C_p$, i.e.

$$\omega(f; t)_p := \sup_{0 \leq h \leq t} \|\Delta_h f(\cdot)\|_p \text{ for } t \geq 0, \quad (3.25)$$

and $\Delta_h f(x) = f(x + h) - f(x)$.

Proof. Similar to [5], we use the Steklov function f_h of $f \in C_p$, i.e.

$$f_h(x) := \frac{1}{h} \int_0^h f(x + t) dt, \quad x \in I, h > 0. \quad (3.26)$$

We have $f_h \in C_p^1$ and by (26) and (25) we get

$$\|f_h - f\|_p \leq \omega(f; h)_p \quad (3.27)$$

and

$$\|f_h'\|_p \leq h^{-1} \omega(f; h)_p, \text{ for } h > 0. \quad (3.28)$$

By the above properties of f_h and (10) we can write for $f \in C_p$ and L_n^* :

$$|L_n^*(f; x) - f(x)| \leq |L_n^*(f(t) - f_h(t); x)| + |L_n^*(f_h(t); x) - f_h(x)| + |f_h(x) - f(x)|, \quad (3.29)$$

for $x \in I$, $n \geq n_0$ and $h > 0$. Next, by (17) and (27) we have

$$\|L_n^*(f - f_h)\|_p \leq M_1(p) \|f - f_h\|_p \leq M_1(p) \omega(f; h)_p. \quad (3.30)$$

Theorem 1 for f_h and (28) imply that

$$w_p(x) |L_n^*(f_h(t); x) - f_h(x)| \leq M_3(p) \|f_h'\|_p \sqrt{2x(x - u_n(x))} \leq M_3(p) h^{-1} \sqrt{2x(x - u_n(x))} \omega(f; h)_p. \quad (3.31)$$

Combining (29)-(31) and (27), we get

$$w_p(x) |L_n^*(f; x) - f(x)| \leq \omega(f; h)_p \left(1 + M_1(p) + M_3(p) h^{-1} \sqrt{2x(x - u_n(x))}\right), \quad (3.32)$$

for each $x \in I$, $n \geq n_0$ and $h > 0$. Putting $h = \sqrt{2x(x - u_n(x))}$ with $x > 0$ to (32), we obtain the desired estimation for $x > 0$.

If $x = 0$, then (24) follows by the property (i) of L_n and (8) and (9).

The proof of (23) is analogous. □

From (23), (24) and $\lim_{n \rightarrow \infty} \lambda_n = +\infty$ we can derive the following two corollaries.

Corollary 2 *If $f \in C_p$, $p \in \mathbb{N}_0$, then*

$$\lim_{n \rightarrow \infty} L_n(f; x) = f(x) = \lim_{n \rightarrow \infty} L_n^*(f; x)$$

at every $x \in I$. This convergence is uniform on every interval $[x_1, x_2] \subset I$.

Corollary 3 *The error of approximation of a function $f \in C_p$, $p \in \mathbb{N}_0$, ($f(x) \neq e_k(x)$ for $k = 0, 1$) by $L_n^*(f)$, $n \geq n_0$, is smaller than by $L_n(f)$.*

4. Applications

We present four examples of well-known positive linear operators satisfying the conditions (i)–(iv) given in Section 2 for operators L_n . For these operators we can consider modified operators of the type L_n^* defined by (8) and (9).

4.1. The Szász-Mirakyan operators ([5], [15])

$$S_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \geq 0, n \in \mathbb{N},$$

satisfy conditions (i)–(iv) and $S_n(e_2; x) = x^2 + \frac{x}{n}$, i.e. the condition (5) there holds with $a = 0$, $b = 1$ and $\lambda_n = n$ for $n \in \mathbb{N}$. From (8) and (9) there results that the modified operators S_n^* preserving e_0 and e_2 are defined by the formula $S_n^*(f; x) := S_n(f; u_n(x))$ with

$$u_n(x) = \frac{-1 + \sqrt{1 + 4n^2x^2}}{2n},$$

for $f \in C_p$, $x \geq 0$ and $n \in \mathbb{N}$.

4.2. The Baskakow operators ([5])

$$V_n(f; x) := (1+x)^{-n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{n}\right), \quad x \geq 0, n \in \mathbb{N},$$

satisfy conditions (i)–(iv) and the formula (5) there holds with $a = b = 1$ and $\lambda_n = n$ for $n \in \mathbb{N}$, because $V_n(e_2; x) = x^2 + x(1+x)/n$. Now the modified operators $V_n^*(f; x) := V_n(f; u_n(x))$ are connected with

$$u_n(x) = \frac{-1 + \sqrt{1 + 4n(n+1)x^2}}{2(n+1)}.$$

4.3. The Post-Wildder operators ([7, 12])

$$P_n(f; x) := \int_0^{\infty} f(t)p_n(x, t) dt, \quad x > 0, n \in \mathbb{N}, p_n(x, t) = \frac{(n/x)^n t^{n-1}}{(n-1)!} \exp(-nt/x),$$

have properties (i)–(iv) and $a = 1, b = 0$ and $\lambda_n = n$ in the formula (5), because $P_n(e_2; x) = x^2 + x^2/n$. Hence the operators $P_n^*(f; x) := P_n(f; u_n(x))$ are modified by

$$u_n(x) = \sqrt{\frac{n}{n+1}}x.$$

4.4. The Stancu operators ([14, 13])

$$L_n(f; x) := \int_0^{\infty} f(t)s_n(x, t) dt, \quad x > 0, n \geq 2, s_n(x, t) = \frac{t^{nx-1}}{B(nx, n+1)(1+t)^{nx+n-1}},$$

with the Euler beta function B , satisfy conditions (i)–(iv) and $L_n(e_2; x) = x^2 + \frac{x(x+1)}{n-1}$, for $n \geq 2$, i.e. the formula (5) there holds with $a = b = 1$ and $\lambda_n = n - 1$ for $n \geq 2$. Now modified Stancu operators $L_n^*(f; x) := L_n(f; u_n(x))$ are connected with

$$u_n(x) = \frac{-1 + \sqrt{1 + 4n(n-1)x^2}}{2n}.$$

Applying Theorem 1 and Theorem 2, we can estimate the error of approximation of functions $f \in C_p$ by classical operators S_n, V_n, P_n, L_n and modified operators S_n^*, V_n^*, P_n^* and L_n^* .

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L. REMPULSKA, K. TOMCZAK
 Institute of Mathematics
 Poznan University of Technology
 Piotrowo 3a,
 60-965 Poznan-POLAND
 e-mail: lrempuls@math.put.poznan.pl

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