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Oscillation of higher-order nonlinear delay differential equations with oscillatory coefficients

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Abstract

A criterion is established on the bounded solutions of type higher-order nonlinear neutral differential equations of type oscillatory or tending to zero at infinity

$$\left[a(t)\left[x(t)+r(t)x(\kappa(t))\right]^{(n-1)}\right]'+p(t)F(x(\tau(t)))+q(t)G(x(\sigma(t)))=\phi(t),$$

where $t \geq t_0$, $n \geq 2$, a, p are positive, r, q, ϕ are allowed to alternate in sign infinitely many times, F, G are continuous functions, and κ, τ, σ are strictly increasing unbounded continuous delay functions.

Key Words: Neutral differential equations, positive and oscillating coefficients

1. Introduction

In this paper, we are concerned with the oscillation and asymptotic behavior of bounded solutions of higher-order nonlinear neutral differential equations of the form:

$$\left[a(t)\left[x(t)+r(t)x(\kappa(t))\right]^{(n-1)}\right]'+p(t)F(x(\tau(t)))+q(t)G(x(\sigma(t)))=\phi(t), \tag{1}$$

where $t \geq t_0$ and $n \geq 2$, under the following primary conditions:

(A1) $\kappa, \tau, \sigma \in C([t_0, \infty), \mathbb{R})$ are strictly increasing unbounded functions satisfying $\sigma^{-1} \circ \tau \in C^1([t_0, \infty), \mathbb{R})$ and $\kappa(t), \tau(t), \sigma(t) \leq t$ for all sufficiently large t;

(A2) $a \in C([t_0, \infty), \mathbb{R}^+)$ is nondecreasing, $p \in C([t_0, \infty), \mathbb{R}^+)$ and $r, q \in C([t_0, \infty), \mathbb{R})$ are allowed to oscillate;

(A3) $F, G \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing, $|F(u)| \geq |G(u)|$ and $uF(u), uG(u) > 0$ for all $u \in \mathbb{R}\backslash\{0\}$.

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Up to now, asymptotic and oscillatory behavior of solutions of neutral delay differential equations have been a wide subject of interest. Most of the results hold in the cases when $r \equiv c \in \mathbb{R}$ and/or $r > 0$ (or < 0), $q \equiv 0$. Results on those type of equations can be found in [1, 2, 3, 5, 6, 7, 8, 9] and references cited therein. The case where r is allowed to oscillate is harder to deal with and thus there are very few papers focused on this problem, the readers may find some results in [4, 8, 10]. Our paper is constructed to improve/extend their results. Therefore, application of our technique to (1) reveals different oscillatory behaviour of bounded solutions compared to that in the two papers mentioned above.

For the fundamental theory of delay differential equations, readers are referred to [2, 3, 5, 6].

Set $\delta(t) := \min\{\kappa(t), \tau(t), \sigma(t)\}$ for $t \geq t_0$ and $t_{-1} := \delta(t_0)$. By a *solution* of (1), we mean a function $x \in C([t_{-1}, \infty), \mathbb{R})$ such that $x(t) + r(t)x(\kappa(t))$ is $n-1$ times differentiable and $a(t)\big[x(t) + r(t)x(\kappa(t))\big]^{(n-1)}$ is differentiable for all $t \geq t_0$, and satisfies (1) for all $t \geq t_0$.

As is usual, a solution of (1) is called *nonoscillatory* if it has eventually constant sign; otherwise, the solution is called *oscillatory*.

2. Main result

For the sake of convenience, we define the function P by

$$P(t) := p(t) + [\sigma^{-1}(\tau(t))]' q(\sigma^{-1}(\tau(t)))$$

for $t \geq \delta^{-1}(t_0)$.

We give our results under the following conditions:

(H1) $P(t) \geq 0 (\not\equiv 0)$ holds for all sufficiently large t;

(H2) there exist two nonnegative constants λ, Λ with $\lambda + \Lambda < 1$ such that $-\lambda \leq r(t) \leq \Lambda$ holds for all sufficiently large t;

(H3) $\big|\int^{\infty} v^{n-2}/a(v) \int_{\sigma^{-1}(\tau(v))}^{v} q(u)dudv\big| < \infty$ holds;

(H4) $\int^{\infty} v^{n-1} P(v)/a(v)dv = \infty$ holds;

(H5) there exists a function $\Phi \in C^{(n-1)}([t_0, \infty), \mathbb{R})$ such that $a\Phi^{(n-1)} \in C^{(1)}([t_0, \infty), \mathbb{R})$, $\big[a\Phi^{(n-1)}\big]' = \phi$ and $\lim_{t\to\infty} \Phi(t)$ exists and is finite.

Theorem 1 *Assume that (A1)–(A3), (H1)–(H5) hold. Then, every bounded solution of* (1) *oscillates or tends to zero as* $t \to \infty$.

Proof. Assume that (1) has a bounded nonoscillatory solution x. Further, assume that x is not tending to zero as $t \to \infty$; i.e., α is a positive constant, where $\alpha := \limsup_{t\to\infty} |x(t)|$. Without loss of generality, we may assume that x is eventually positive; the proof is so similar when x is eventually negative, and thus this case is omitted. So, there exists a $t_1 \geq t_0$ such that $x(\delta(t)) > 0$ holds for all $t \geq t_1$. We set

$$y_x(t) := x(t) + r(t)x(\kappa(t)) \tag{2}$$

and

$$z_x(t) := y_x(t) - \int_t^{\infty} \frac{(t-v)^{n-2}}{(n-2)!} \frac{1}{a(v)} \int_{\sigma^{-1}(\tau(v))}^{v} q(u)G(x(\sigma(u)))dudv - \Phi(t), \tag{3}$$

for $t \geq t_1$. It is easy to see that y_x and z_x are bounded because of (H2), (H3), (H5) and boundedness of x. Then from (1), (2), (3) and (H5), we get

$$\begin{aligned}\left[a(t)z_x^{(n-1)}(t)\right]' &= \left[a(t)y_x^{(n-1)}(t)\right]' + q(t)G(x(\sigma(t)))\\ &\quad - [\sigma^{-1}(\tau(t))]'q(\sigma^{-1}(\tau(t)))G(x(\tau(t))) - \phi(t)\\ &\leq -P(t)F(x(\tau(t))) \leq 0,\end{aligned}$$

for all $t \geq t_2$ for some sufficiently large $t_2 \geq t_1$. It follows that $az_x^{(n-1)}$ is either eventually negative or eventually positive, and thus, $z_x^{(n-1)}$ is either eventually negative or eventually positive since a is nonnegative. This indicates that $z_x^{(k)}$ is strictly monotonic and of constant sign eventually for $k = 0, 1, \ldots, n-2$. Hence, $\beta := \lim_{t\to\infty} z_x(t)$ exists and is a finite constant. By (H3), (H5), (3) and boundedness of x, we have $\lim_{t\to\infty} y_x(t) = \beta + \gamma$, where $\gamma := \lim_{t\to\infty} \Phi(t)$. One can easily show that $\liminf_{t\to\infty} x(t) = 0$ is true by following the steps in the proof of [9, Theorem 1], in which the nondecreasing nature of a and the condition (H4) are needed. Then, for any increasing divergent $\{\zeta_n\}_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} x(\zeta_n) = 0$, we have

$$\beta + \gamma = \lim_{n\to\infty} y(\zeta_n) \leq \lim_{n\to\infty} x(\zeta_n) + \Lambda \lim_{n\to\infty} x(\kappa(\zeta_n)) \leq \Lambda\alpha. \tag{4}$$

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be an increasing divergent sequence satisfying $\lim_{n\to\infty} x(\xi_n) = \alpha$. Since x is bounded, we may assume existence of the limit $\lim_{n\to\infty} x(\kappa(\xi_n))$. Therefore, $\lim_{n\to\infty} x(\kappa(\xi_n)) \leq \alpha$ is true. Then, we deduce

$$\beta + \gamma = \lim_{n\to\infty} y(\xi_n) \geq \lim_{n\to\infty} x(\xi_n) - \lambda \lim_{n\to\infty} x(\kappa(\xi_n)) \geq (1-\lambda)\alpha,$$

which together with (H2), (4) and $\alpha > 0$ indicates that $0 \geq [1-(\lambda+\Lambda)]\alpha > 0$ is true. This is a contradiction, and therefore the proof is complete. □

Now, we give an example.

Example 1 *Consider the following neutral delay equation*

$$\left[\frac{t}{t+1}\left[x(t) + \left(\frac{1}{3}\sin(t) + \frac{2}{9}\right)x(t/2)\right]^{(3)}\right]' + \frac{1}{t^4}\left[x(t/3)\right]^3 - \frac{\sin(t)}{\mathrm{e}^t}\left[x(t/2)\right]^3 = \frac{\sin(t)}{t^5} \tag{5}$$

for $t \geq 5$. Clearly, $n = 4$, $a(t) = t/(t+1)$, $r(t) = \sin(t)/3 + 2/9$, $\kappa(t) = t/2$, $p(t) = 1/t^4$, $\tau(t) = t/3$, $G(u) = u^3$, $q(t) = -\sin(t)/\mathrm{e}^t$, $\sigma(t) = t/2$, $H(u) = u^3$ and $\phi(t) = \sin(t)/t^5$ for $t \geq 5$ and $u \in \mathbb{R}$. In this case, we have $\lambda = 1/9$, $\Lambda = 5/9$, $P(t) = 1/t^4 - 2\sin(2t/3)/(3\mathrm{e}^{2t/3})$ for $t \geq 5$, and

$$\Phi(t) = \int_t^{\infty} \frac{(v-t)^2(v+1)}{2v} \int_v^{\infty} \frac{\sin(u)}{u^5} dudv \to 0$$

as $t \to \infty$. *By calculation, we obtain*

$$\int_{5}^{\infty} v^2 \frac{v+1}{v} \int_{2v/3}^{v} \frac{-\sin(u)}{\mathrm{e}^u} du dv = -\frac{459}{16}\frac{\cos(10/3)}{\mathrm{e}^{10/3}} + \frac{63}{8}\frac{\sin(10/3)}{\mathrm{e}^{10/3}} + \frac{71}{4}\frac{\cos(5)}{\mathrm{e}^5} - \frac{13}{4}\frac{\sin(5)}{\mathrm{e}^5}$$

and

$$\int_{5}^{\infty} v^3 \frac{v+1}{v} \left(\frac{1}{v^4} - \frac{2\sin(2t/3)}{3\mathrm{e}^{2t/3}} \right) dv = \infty.$$

Therefore, (5) *satisfies all the conditions of Theorem 1, and hence every bounded solution oscillates or tends to zero at infinity.*

The example given above illustrates the significance of our result because none of the results in [1, 4, 7, 8, 9, 10] can be applied to (5).

Remark It is not hard to extend Theorem 1 to equations involving several coefficients.

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