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HANI KHASHAN

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Some properties of gr-multiplication ideals

Hani A. Khashan

Abstract

In this paper, we study some of the properties of gr-multiplication ideals in a graded ring R . We first characterize finitely generated gr-multiplication ideals and then give a characterization of gr-multiplication ideals by using the gr-localization of R . Finally we determine the set of gr- P -primary ideals of R when P is a gr-multiplication gr-prime ideal of R .

Key Words: Graded Rings, Graded Ideals, Gr-primary Ideals and Gr-multiplication Ideals.

1. Introduction

Let G be a group. A ring (R, G) is called a G -graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for each g and h in G . For simplicity, we will denote the graded ring (R, G) by R . An element of a graded ring R is called homogeneous if it belongs to $\bigcup_{g \in G} R_g$ and this set of homogeneous elements is denoted by $h(R)$. If $x \in R_g$ for some $g \in G$, then we say that x is of degree g . A graded ideal I of a graded ring R is an ideal verifying $I = \bigoplus_{g \in G} (I \cap R_g) = \bigoplus_{g \in G} I_g$. Equivalently, I is graded in R if and only if I has a homogeneous set of generators. If $R = \bigoplus_{g \in G} R_g$ and $R' = \bigoplus_{g \in G} R'_g$ are two graded rings, then a mapping $\eta : R \rightarrow R'$ with $\eta(1_R) = 1_{R'}$ is called a gr-homomorphism if $\eta(R_g) \subseteq R'_g$ for all $g \in G$. A graded ideal P of a graded ring R is called gr-prime if whenever $x, y \in h(R)$ with $xy \in P$, then $x \in P$ or $y \in P$. A graded ideal M of a graded ring R is called gr-maximal if it is maximal in the lattice of graded ideals of R . A graded ring R is called a gr-local ring if it has unique gr-maximal ideal.

Let R be a graded ring and let $S \subseteq h(R)$ be a multiplicatively closed subset of R . Then the ring of fractions $S^{-1}R$ is a graded ring which is called the gr-ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g \in G} (S^{-1}R)_g$ where

$$(S^{-1}R)_g = \left\{ \frac{r}{s} : r \in R, s \in S \text{ and } g = (\deg s)^{-1}(\deg r) \right\}.$$

Consider the ring gr-homomorphism $\eta : R \rightarrow S^{-1}R$ defined by $\eta(r) = \frac{r}{1}$. For any graded ideal I of R , the ideal of $S^{-1}R$ generated by $\eta(I)$ is denoted by $S^{-1}I$. Similar to non graded case, one can prove that

$$S^{-1}I = \left\{ \lambda \in S^{-1}R : \lambda = \frac{r}{s} \text{ for } r \in I \text{ and } s \in S \right\}$$

and that $S^{-1}I \neq S^{-1}R$ if and only if $S \cap I = \Phi$. Moreover, similar to the non graded case, we have the following properties for graded ideal I and J of R :

- (1) $S^{-1}(I \cap J) = S^{-1}I \cap S^{-1}J$,
- (2) $S^{-1}(IJ) = (S^{-1}I)(S^{-1}J)$ and
- (3) $S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$ if J is finitely generated.

If \mathcal{J} is a graded ideal in $S^{-1}R$, then $\mathcal{J} \cap R$ will denote the graded ideal $\eta^{-1}(\mathcal{J})$ of R . Moreover, similar to the non graded case one can prove that $S^{-1}(\mathcal{J} \cap R) = \mathcal{J}$.

Let P be any gr-prime ideal of a graded ring R and consider the multiplicatively closed subset $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of R by R_P^g and we call it the gr-localization of R . This ring is gr-local with the unique gr-maximal $S^{-1}P$ which will be denoted by PR_P^g . Moreover, for graded ideals I and J of R , if $IR_P^g = JR_P^g$ for every gr-prime (gr-maximal) ideal P of R , then $I = J$. For a positive integer n the graded ideal $(PR_P^g)^n \cap R$ of R is denoted by $P^{(n)g}$. For more definitions and theorems about gr-ring of fractions of graded rings, one can see [8].

Let I be a graded ideal in a graded ring R . The graded radical of I (denoted by $g-rad(I)$) is defined in [9] as the set of all $x \in R$ such that for each $g \in G$, there exists $n_g \geq 0$ such that $x_g^{n_g} \in I$.

A graded ideal Q of a graded ring R is called gr-primary if $Q \neq R$ and whenever $a, b \in h(R)$ with $ab \in Q$, then $a \in Q$ or $b \in g-rad(Q)$. If Q is gr-primary ideal of R , then $g-rad(Q) = P$ is a gr-prime ideal of R and we say that Q is gr- P -primary. If I is a graded ideal of R with $g-rad(I) = M$, a gr-maximal ideal of R , then I is gr- M -primary; see [9].

Recall that a graded ring R is called gr- PIR if every graded ideal of R is gr-principal, where a gr-principal ideal of a graded ring R is generated by some homogeneous element in R . Also, recall that a graded ring R is called gr- $SPIR$ if R has unique gr-prime ideal P and every graded ideal of R is a power of P . Similar to the non graded case, one can prove that if R is a gr- $SPIR$, then R is a gr- PIR and the unique gr-prime ideal of R is nilpotent.

An ideal I of a ring R is called multiplication if whenever J is an ideal of R with $J \subseteq I$, then there is an ideal K of R such that $J = IK$. If every ideal in a ring R is multiplication, then R is called a multiplication ring. Multiplication ideals and rings have been studied in detail in [1], [2] and [7]. A generalization of multiplication graded ideals and rings to gr-multiplication ideals and rings have been studied in [3], [4] and [5].

In this paper, we study more properties of gr-multiplication ideals in a graded ring R and give a characterization for finitely generated gr-multiplication ideals. For an ideal I of a graded ring R , we define the graded ideal $\theta^g(I)$ and use it together with the gr-localization of R to give a general characterization for gr-multiplication ideals. Finally, we determine the set of gr- P -primary ideals of a graded ring R when P is both gr-prime and gr-multiplication in R .

2. Properties for gr-multiplication ideals

Definition 2.1 Let R be a graded ring graded by the group G . A graded ideal I of R is called a *gr-multiplication ideal* of R if whenever J is a graded ideal of R with $J \subseteq I$, then there is a graded ideal K of R such that $J = KI$. If every graded ideal in a graded ring R is gr-multiplication, then R is called a *gr-multiplication ring*.

Clearly, any graded ideal which is multiplication is a gr-multiplication ideal. A graded ideal I of a graded ring R is called a *gr-invertible ideal* if there exists a graded ideal J of R such that $IJ = R$. Also, one can easily see that every gr-invertible ideal is gr-multiplication. In particular, the gr-principal ideals are gr-multiplication.

The class of gr-multiplication domains has been characterized in [5] as the class of gr-Dedekind domains which is the class of graded domains in which every graded ideal is gr-invertible. In [10], we can see an example of a gr-multiplication ring which is not multiplication. Indeed, the group ring $R[\mathbb{Z}]$, where R is a Dedekind domain is gr-Dedekind domain and so it is gr-multiplication domain. On the other hand, if R is not a field, then $R[\mathbb{Z}]$ is not a Dedekind domain and so it is not a multiplication domain, see [6].

If I and J are two graded ideals in a graded ring R , then the ideal $(J : I) = \{x \in R : xI \subseteq J\}$ is a graded ideal, see [4]. In the following theorem, we can see another equivalent definition of gr-multiplication ideals.

Theorem 2.2 Let I be a graded ideal in a graded ring R . Then I is gr-multiplication iff $I \cap J = I(J : I)$ for every graded ideal J of R .

Proof. Suppose that $J \subseteq I$ for a graded ideal J of R . Then $J = I \cap J = I(J : I) = IJ$

Conversely, suppose that I is a gr-multiplication ideal in R . Let J be any graded ideal of R . Then $I \cap J \subseteq I$ and so there is a graded ideal K of R with $I \cap J = IK$. Therefore, $K \subseteq (I \cap J : I) \subseteq (J : I)$ and then $I \cap J = IK \subseteq I(J : I)$. On the other hand, clearly, $I(J : I) \subseteq I \cap J$ and therefore, $I(J : I) = I \cap J$. \square

The following theorem is a characterization of gr-multiplication ideals in gr-local rings; see [3].

Theorem 2.3 Let R be a gr-local ring with the unique gr-maximal ideal M . A graded ideal I of R is gr-multiplication iff I is gr-principal.

Proof. If $I = \langle x \rangle$ for some $x \in h(R)$, then clearly I is a gr-multiplication ideal of R .

Conversely, suppose that I is gr-multiplication in R . Since I is graded, then it is generated by a set of homogeneous elements, say, $\{a_\alpha : \alpha \in \Lambda\}$. Now, for each $\alpha \in \Lambda$, $\langle a_\alpha \rangle \subseteq I$ and so there is a graded ideal B_α of R such that $\langle a_\alpha \rangle = IB_\alpha$. Therefore, $I = \sum_{\alpha \in \Lambda} \langle a_\alpha \rangle = \sum_{\alpha \in \Lambda} IB_\alpha = I \sum_{\alpha \in \Lambda} B_\alpha$. If $\sum_{\alpha \in \Lambda} B_\alpha = R$, then

$B_{\alpha_0} = R$ for some $\alpha_0 \in \Lambda$, since otherwise if $B_\alpha \subset R$ for each $\alpha \in \Lambda$, then $B_\alpha \subseteq M$ for each $\alpha \in \Lambda$ and so $R = \sum_{\alpha \in \Lambda} B_\alpha \subseteq M$, a contradiction. Therefore, $\langle a_{\alpha_0} \rangle = IB_{\alpha_0} = I$ and I is gr-principal. If $\sum_{\alpha \in \Lambda} B_\alpha \neq R$, then

$\sum_{\alpha \in \Lambda} B_\alpha \subseteq M$ and then $I = I \sum_{\alpha \in \Lambda} B_\alpha \subseteq IM \subseteq I$. Therefore, $I = IM$ and then $I = 0$ by proposition 2.4 in [4].

It follows that I is gr-principal. \square

Theorem 2.4 *If I is a gr-multiplication ideal of a graded ring R and $S \subseteq h(R)$ is a multiplicatively closed subset of R , then $S^{-1}I$ is a gr-multiplication ideal of $S^{-1}R$.*

Proof. Let \mathcal{J} be a graded ideal of $S^{-1}R$ such that $\mathcal{J} \subseteq S^{-1}I$. Then $\mathcal{J} = S^{-1}J$ for some graded ideal J of R . Now, $I \cap J \subseteq I$ and therefore, there is a graded ideal K of R such that $I \cap J = IK$. Thus

$$\mathcal{J} = S^{-1}I \cap S^{-1}J = S^{-1}(I \cap J) = S^{-1}(IK) = (S^{-1}I)(S^{-1}K).$$

Therefore, $S^{-1}I$ is a gr-multiplication ideal in $S^{-1}R$. □

Definition 2.5 *A graded ideal I of a graded ring R is called locally gr-principal if IR_P^g is gr-principal for any gr-prime ideal P of R .*

As a corollary of theorem 2.3, we have the following.

Corollary 2.6 *Any gr-multiplication ideal in a graded ring R is locally gr-principal.*

In [3], it has been proved that if I is a finitely generated graded ideal of R , then I is gr-multiplication if and only if I is locally gr-principal. In the following theorem, we can see another characterization of finitely generated gr-multiplication ideals. First, we have the following technical lemma.

Lemma 2.7 *Let R be a gr-local ring with gr-maximal ideal M and I be a gr-principal ideal in R . If $I = \langle a_1, a_2, \dots, a_n \rangle$, then $I = \langle a_j \rangle$ for some $j \in \{1, 2, \dots, n\}$.*

Proof. Suppose that $I = \langle a \rangle$ for some $a \in h(R)$ and suppose that $I = \langle a_1, a_2, \dots, a_n \rangle$. Then $a = \sum_{i=1}^n a_i r_i$ where $r_i \in R$ for all i . Also for all i , $a_i = ax_i$ for some $x_i \in R$. Thus, $a(1 - \sum_{i=1}^n x_i r_i) = 0$. If $1 - \sum_{i=1}^n x_i r_i$ is a unit in R , then $a = 0$ and so $I = \langle 0 \rangle = \langle a_i \rangle$ for all $i = 1, 2, \dots, n$ since $a_i = 0$ for all i . If $1 - \sum_{i=1}^n x_i r_i$ is not a unit, then $\sum_{i=1}^n x_i r_i \notin M$ and so $\sum_{i=1}^n x_i r_i$ is a unit. Therefore, there is some $j \in \{1, 2, \dots, n\}$ such that $x_j r_j$ is a unit and then x_j is also a unit. Hence, $a = ax_j x_j^{-1} = a_j x_j^{-1}$ and $I = \langle a \rangle \subseteq \langle a_j \rangle$. Hence, $I = \langle a_j \rangle$ for some $j \in \{1, 2, \dots, n\}$. □

Theorem 2.8 *Let $I = \langle a_1, a_2, \dots, a_n \rangle$ be a finitely generated graded ideal of a graded ring R . Then the following are equivalent.*

- (1) I is gr-multiplication.
- (2) I is locally gr-principal.
- (3) $\sum_{i=1}^n (\langle a_i \rangle : I_i) = R$, where $I_i = \langle a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle$.

Proof. (1) \Leftrightarrow 2): see [3].

(2) \Rightarrow 3): Suppose that I is locally gr-principal. Then for each gr-prime ideal P of R , we have $IR_P^g = \langle \frac{a_1}{1}, \frac{a_2}{1}, \dots, \frac{a_n}{1} \rangle = \langle \frac{a_j}{1} \rangle = \langle a_j \rangle R_P^g$ for some $j \in \{1, 2, \dots, n\}$ by lemma 2.7. Hence, for any gr-prime ideal P of R , $(\langle a_j \rangle R_P^g : I_j R_P^g) = R_P^g$, and then

$$\left(\sum_{i=1}^n (\langle a_i \rangle : I_i) \right) R_P^g = \sum_{i=1}^n (\langle a_i \rangle R_P^g : I_i R_P^g) = R_P^g,$$

since I_i is finitely generated for each i . Therefore, $\sum_{i=1}^n (\langle a_i \rangle : I_i) = R$.

(3) \Rightarrow 2): Suppose that $\sum_{i=1}^n (\langle a_i \rangle : I_i) = R$. Then for any gr-prime ideal P of R , we have

$$\sum_{i=1}^n (\langle a_i \rangle R_P^g : IR_P^g) = \left(\sum_{i=1}^n (\langle a_i \rangle : I_i) \right) R_P^g = \left(\sum_{i=1}^n (\langle a_i \rangle : I_i) \right) R_P^g = R_P^g.$$

Therefore, there is $j \in \{1, 2, \dots, n\}$ such that $(\langle a_j \rangle R_P^g : IR_P^g) = R_P^g$ and then $IR_P^g \subseteq \langle a_j \rangle R_P^g = \langle \frac{a_j}{1} \rangle$. It follows that $IR_P^g = \langle \frac{a_j}{1} \rangle$ for each gr-prime ideal P of R and I is locally gr-principal. \square

If I is a graded ideal in a graded ring R , then we define the subset $\theta^g(I)$ of R as $\theta^g(I) = \sum_{x \in I \cap h(R)} (\langle x \rangle : I)$.

Clearly, $\theta^g(I)$ is a graded ideal of R .

Lemma 2.9 *Let I be a gr-multiplication ideal of a graded ring R . Then*

- (1) $I = I\theta^g(I)$;
- (2) $J = J\theta^g(I)$ for any graded ideal $J \subseteq I$.

Proof. (1) For $x \in I \cap h(R)$, $\langle x \rangle \subseteq I$ and so $\langle x \rangle = I(\langle x \rangle : I)$. Therefore

$$I = \sum_{x \in I \cap h(R)} \langle x \rangle = \sum_{x \in I \cap h(R)} I(\langle x \rangle : I) = I \sum_{x \in I \cap h(R)} (\langle x \rangle : I) = I\theta^g(I).$$

(2) Suppose that J is a graded ideal with $J \subseteq I$. Then $J = IK$ for some graded ideal K of R . Hence,

$$J = IK = I\theta^g(I)K = IK\theta^g(I) = J\theta^g(I).$$

\square

Lemma 2.10 *Let I and J be graded ideals in a graded ring R and let $S \subseteq h(R)$ be a multiplicatively closed subset of R . Then*

- (1) $\theta^g(I)\theta^g(J) \subseteq \theta^g(IJ)$;
- (2) $S^{-1}(\theta^g(I)) \subseteq \theta^g(S^{-1}I)$.

Proof. (1) Let $a \in I \cap h(R)$ and let $b \in J \cap h(R)$. It is enough to prove that $(\langle a \rangle : I)(\langle b \rangle : J) \subseteq (\langle ab \rangle : IJ)$. Let $\sum_{i=1}^n x_i y_i \in (\langle a \rangle : I)(\langle b \rangle : J)$ where $x_i \in (\langle a \rangle : I)$ and $y_i \in (\langle b \rangle : J)$ for $i = 1, 2, \dots, n$. Then $x_i I \subseteq \langle a \rangle$ and $y_i J \subseteq \langle b \rangle$ for $i = 1, 2, \dots, n$. Hence, $x_i y_i IJ \subseteq \langle ab \rangle$ and then $x_i y_i \in (\langle ab \rangle : IJ)$. Therefore, $\sum_{i=1}^n x_i y_i \in (\langle ab \rangle : IJ)$.

$$\begin{aligned} (2) \quad S^{-1}(\theta^g(I)) &= S^{-1}\left(\sum_{x \in I \cap h(R)} (\langle x \rangle : I)\right) = \sum_{x \in I \cap h(R)} S^{-1}(\langle x \rangle : I) \\ &\subseteq \sum_{x \in I \cap h(R)} \left\langle \frac{x}{1} \right\rangle : S^{-1}I \subseteq \theta^g(S^{-1}I). \end{aligned}$$

□

Recall that a graded ideal I in a graded ring R is called gr-finitely generated if I is generated by a finite set of homogeneous elements in R .

Theorem 2.11 *Let I be a graded ideal in a graded ring R . Then I is gr-finitely generated and locally gr-principal iff $\theta^g(I) = R$.*

Proof. Let M be a gr-maximal ideal in R . Then $IR_M^g = \langle x \rangle R_M^g$ for some $x \in I \cap h(R)$. Hence, $R_M^g = (\langle x \rangle R_M^g : IR_M^g) = (\langle x \rangle : I) R_M^g$ since I is gr-finitely generated. Therefore, $R_M^g = \theta^g(I) R_M^g$ and then $\theta^g(I) = R$.

Conversely, suppose $\theta^g(I) = R$. Then there exist $x_1, x_2, \dots, x_n \in I \cap h(R)$ such that $R = \theta^g(I) = (\langle x_1 \rangle : I) + (\langle x_2 \rangle : I) + \dots + (\langle x_n \rangle : I)$. Thus,

$$\begin{aligned} I &= I\theta^g(I) = I(\langle x_1 \rangle : I) + I(\langle x_2 \rangle : I) + \dots + I(\langle x_n \rangle : I) \\ &\subseteq \langle x_1 \rangle + \langle x_2 \rangle + \dots + \langle x_n \rangle \subseteq I. \end{aligned}$$

So, $I = \langle x_1, x_2, \dots, x_n \rangle$ is gr-finitely generated. Now, let M be a gr-maximal ideal of R . Since $\theta^g(I) = R$, there is $x \in I \cap h(R)$ with $(\langle x \rangle : I) \not\subseteq M$. Therefore, there exists $r \in R - M$ with $rI \subseteq \langle x \rangle$ and then $rIR_M^g = \langle r \rangle R_M^g$. $IR_M^g = IR_M^g \subseteq \langle x \rangle R_M^g$. Hence, $IR_M^g = \langle x \rangle R_M^g$ for any gr-maximal ideal M of R and so I is locally gr-principal. □

Definition 2.12 *A graded ideal I of a graded ring R is called meet-gr-principal if $JI \cap K = (J \cap (K : I))I$ for all graded ideals J and K of R .*

We are ready now for the following characterization of gr-multiplication ideals similar to that in the non graded case; see [1].

Theorem 2.13 *Let I be a graded ideal in a graded ring R . Then the following are equivalent:*

- (1) I is meet-gr-principal.
- (2) I is gr-multiplication.
- (3) $IR_M^g = \langle 0 \rangle R_M^g$ for any gr-maximal ideal M of R with $M \supseteq \theta^g(I)$.

Proof. (1) \Rightarrow 2): Let J be a graded ideal of R with $J \subseteq I$. Then $J = RI \cap J = (R \cap (J : I))I = (J : I)I$ and then I is a gr-multiplication ideal by theorem 2.2.

(2) \Rightarrow 3): Suppose that I is a gr-multiplication ideal. Let M be any gr-maximal ideal of R such that $\theta^g(I) \subseteq M$. Let $x \in I \cap h(R)$. Then $\langle x \rangle$ is a graded ideal and $\langle x \rangle = \langle x \rangle \theta^g(I)$ by lemma 2.9 and so, $\langle x \rangle R_M^g = \langle x \rangle R_M^g \theta^g(I) R_M^g$. By proposition 2.4 in [4], we see that $\langle x \rangle R_M^g = \langle 0 \rangle R_M^g$ and so $IR_M^g = \langle 0 \rangle R_M^g$.

(3) \Rightarrow 1): Let J and K be graded ideals of R . We prove that $JI \cap K = (J \cap (K : I))I$. Clearly, $JI \cap K \supseteq (J \cap (K : I))I$ is always true. We prove the other containment locally. Let M be a gr-maximal ideal of R . If $\theta^g(I) \subseteq M$, then $IR_M^g = \langle 0 \rangle R_M^g$ by assumption and so $((J \cap (K : I))I) R_M^g = \langle 0 \rangle R_M^g = (JI \cap K) R_M^g$. Suppose that $\theta^g(I) \not\subseteq M$. Then $(\langle x \rangle : I) \not\subseteq M$ for some $x \in I \cap h(R)$ and so there is $r \in R$ such that $rI \subseteq \langle x \rangle$ and $r \notin M$. Let $b = y_1 z_1 + y_2 z_2 + \dots + y_n z_n \in JI \cap K$ where $y_k \in J$ and $z_k \in I$ for $k = 1, 2, \dots, n$. Then there exist $r_1, r_2, \dots, r_n \in R$ such that

$$\begin{aligned} rb &= r(y_1 z_1 + y_2 z_2 + \dots + y_n z_n) = y_1(rz_1) + y_2(rz_2) + \dots + y_n(rz_n) \\ &= y_1(r_1 x) + y_2(r_2 x) + \dots + y_n(r_n x) = (y_1 r_1 + y_2 r_2 + \dots + y_n r_n)x. \end{aligned}$$

where the third equality holds since $rI \subseteq \langle x \rangle$. Now,

$$(y_1 r_1 + y_2 r_2 + \dots + y_n r_n)rI \subseteq (y_1 r_1 + y_2 r_2 + \dots + y_n r_n)\langle x \rangle = \langle rb \rangle \subseteq K.$$

Hence,

$$(y_1 r_1 + y_2 r_2 + \dots + y_n r_n)r \in J \cap (K : I).$$

and then

$$\frac{y_1 r_1 + y_2 r_2 + \dots + y_n r_n}{1} \in (J \cap (K : I)) R_M^g.$$

Now,

$$\frac{b}{1} = \left(\frac{r}{1}\right)^{-1} \left(\frac{rb}{1}\right) = \left(\frac{r}{1}\right)^{-1} \left(\frac{y_1 r_1 + y_2 r_2 + \dots + y_n r_n}{1}\right) \left(\frac{x}{1}\right) \in (J \cap (K : I)) R_M^g IR_M^g.$$

Therefore, $(JI \cap K) R_M^g = (J \cap (K : I)) R_M^g IR_M^g$ for any gr-maximal ideal M of R and so $JI \cap K = (J \cap (K : I))I$. \square

We have the following as a corollary of the previous theorem and lemma 2.10.

Corollary 2.14 *If I and J are gr-multiplication ideals of a graded ring R , then IJ is gr-multiplication.*

Proof. Let M be a gr-maximal ideal of R such that $\theta^g(IJ) \subseteq M$. Then $\theta^g(I)\theta^g(J) \subseteq \theta^g(IJ) \subseteq M$ and so either $\theta^g(I) \subseteq M$ or $\theta^g(J) \subseteq M$. Hence, by theorem 2.13, either $IR_M^g = 0R_M^g$ or $JR_M^g = 0R_M^g$. In both cases, $(IJ)R_M^g = 0R_M^g$ and then IJ is a gr-multiplication ideal of R again by theorem 2.13. \square

3. Gr-primary ideals with gr-multiplication gr-radicals

Definition 3.1 *Let P be a gr-prime ideal in a graded ring R . Then we define the graded rank of P (denoted by $gr\text{-rank}(P)$) as the supremum of the lengths of all chains of distinct proper gr-prime ideals of R having P*

as last term. The *gr-dimension* of a graded ring R is defined as the supremum of the lengths of all chains of distinct *gr-prime* ideals of R and is denoted by $gr-dim(R)$.

Now, any *gr-prime* ideal in the graded ring R_P^g is of the form $P'R_P^g$ where P' is a *gr-prime* ideal of R with $P' \subseteq P$. Therefore we conclude that $gr-dim(R_P^g) = gr-rank(P)$. Recall that a *gr-prime* ideal P of R is called *minimal gr-prime* over a graded ideal I if there is no *gr-prime* ideal Q of R such that $I \subseteq Q \subset P$.

Definition 3.2 Let I be a graded ideal in a graded ring R . Then the *graded rank* of I (denoted by $gr-rank(I)$) is defined as the infimum of the values of $gr-rank(P)$ as P runs over all of the *minimal gr-prime* ideals of I .

Theorem 3.3 Let I be a *gr-multiplication* ideal in a graded ring R . If $gr-rank(I) \gneq 0$, then I is *gr-finitely generated*.

Proof. Suppose that I is not *gr-finitely generated*, then by theorem 2.11, $\theta^g(I) \neq R$ and, therefore, $\theta^g(I) \subseteq M$ for some *gr-maximal* ideal M of R . Hence, $IR_M^g = 0R_M^g$ by theorem 2.13 and so $gr-rank(I) \leq gr-rank(IR_M^g) = 0$, a contradiction. \square

Now, in the following main theorem, we determine the set of all *gr-P*-primary ideals of a graded ring R where P is any *gr-prime* ideal of R that is *gr-multiplication*. First, we have the following lemma.

Lemma 3.4 If I is a graded ideal of a graded ring R such that $g-rad(I)$ is *gr-finitely generated*, then there exists a positive integer t such that $(g-rad(I))^t \subseteq I$.

Proof. Suppose that $g-rad(I) = \langle a_1, a_2, \dots, a_n \rangle$ for $a_1, a_2, \dots, a_n \in h(R)$. Then there exist $t_1, t_2, \dots, t_n \in \mathbb{N}$ such that $a_i^{t_i} \in I$. Let $t = 1 + \sum_{i=1}^n (t_i - 1)$. Then $(g-rad(I))^t$ is a graded ideal generated by

$$L = \left\{ a_1^{k_1} a_2^{k_2} \dots a_n^{k_n} : k_1, k_2, \dots, k_n \in \mathbb{N}, \sum_{i=1}^n k_i = t \right\}.$$

If $k_i \gneq t_i$ for all $i = 1, 2, \dots, n$, then $\sum_{i=1}^n k_i \leq \sum_{i=1}^n (t_i - 1) \gneq t$, a contradiction. Therefore, there exists j , $1 \leq j \leq n$ such that $k_j \geq t_j$ and then $a_1^{k_1} \dots a_j^{k_j} \dots a_n^{k_n} \in I$. Hence, $L \subseteq I$ and then $(g-rad(I))^t \subseteq I$. \square

Theorem 3.5 Let P be a *gr-prime* ideal of a graded ring R that is *gr-multiplication*. If $gr-rank(P) \gneq 0$, then $\{P^n\}_{n=1}^\infty$ is the set of *gr-P*-primary ideals of R . If $gr-rank(P) = 0$, then there is a least positive integer m with $(PR_P^g)^m = 0R_P^g$ and in this case $\{P^n\}_{n=1}^m$ is the set of *gr-P*-primary ideals of R .

Proof. Suppose that $gr-rank(P) \gneq 0$, then P is *gr-finitely generated* by theorem 3.3. Let Q be a *gr-P*-primary ideal in R . Then $P^t \subseteq Q$ for some positive integer t by lemma 3.4. By passing to the graded ring R/P^t , we have, Q/P^t is *gr-P/P^t*-primary ideal and since clearly, $gr-rank(P/P^t) = 0$, it is enough to consider the case where $gr-rank(P) = 0$. Since P is *gr-multiplication*, then PR_P^g is *gr-principal* by theorem 2.3 and since $gr-rank(P) = 0$, then PR_P^g is the only *gr-prime* ideal of R_P^g and each graded ideal of R_P^g is a power

of PR_P^g . Hence, R_P^g is a gr-*SPIR* and so there is a least positive integer m such that $(PR_P^g)^m = \langle 0 \rangle R_P^g$ and the only graded ideals (which are gr- PR_P^g -primary) of R_P^g are $PR_P^g, (PR_P^g)^2, \dots, (PR_P^g)^m$. Therefore, the only gr- P -primary ideals of R are $P^{(1)g}, P^{(2)g}, \dots, P^{(m)g}$. Now, for a fixed i , $1 \leq i \leq m$, we have $P^i \subseteq P^{(i)g}$. Suppose k is the largest integer with $P^{(i)g} \subseteq P^k$. Since by corollary 2.14, P^k is gr-multiplication, there is a graded ideal A of R such that $P^{(i)g} = AP^k$ where $A \not\subseteq P$. Since $P^{(i)g}$ is gr- P -primary, $P^k \subseteq P^{(i)g}$ and so, $P^k = P^{(i)g}$. Now, $(P^{(k)g})R_P^g = P^k R_P^g = P^{(i)g} R_P^g = P^i R_P^g$ and therefore, $i = k$ and $P^{(i)g} = P^i$. It follows that P, P^2, \dots, P^m are the only gr- P -primary ideals of R . \square

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Hani A. KHASHAN
 Department of Mathematics,
 Al al-Bayt University,
 Po Box Number 130095
 Mafraq-JORDAN
 e-mail: hakhashan@aabu.edu.jo

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