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Existence and uniqueness theorem for slant immersions in Kenmotsu space forms

Pradeep Kumar Pandey, Ram Shankar Gupta

Abstract

In this paper we have obtained a general existence as well as uniqueness theorem for slant immersions into a Kenmotsu-space form.

Key Words: Kenmotsu manifold, slant immersion, mean curvature, sectional curvature.

1. Introduction

B. Y. Chen has defined and studied slant immersions by generalizing the concept of holomorphic and totally real immersions [5]. Latter, it was A. Lotta [14], who introduced the concept of slant immersion of a Riemannian manifold into an almost contact metric manifold. B. Y. Chen and Y. Tazawa [8] have obtained examples of n -dimensional proper slant submanifolds in the complex Euclidean n -space C^n . On the other hand, Chen and Vrancken [6] have established the existence of n -dimensional proper slant submanifolds into a non-flat complex space form $\bar{M}^n(4c)$ and in contact geometry J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, and M. A. Fernandez [2] have established the existence and uniqueness theorem in Sasakian space form. Later, R. S. Gupta, S. M. K. Haider and A. Sharfuddin [10] have obtained the existence and uniqueness theorem into a non-flat cosymplectic space form.

The purpose of the present paper is to establish a general existence and uniqueness theorem for slant immersions in Kenmotsu-space forms.

In section 2, we review some basic formulae and results for our subsequent use.

2. Preliminaries

Let \bar{M} be a $(2m+1)$ -dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, g) , where ϕ is a $(1,1)$ tensor field, ξ a vector field, η a 1-form and g is the Riemannian metric on \bar{M} . These tensors satisfy [1]

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$$\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, \eta(\xi) = 1, \eta(\phi X) = 0$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi)$$

(2.1)

for any $X, Y \in T\bar{M}$, where $T\bar{M}$ denotes the Lie algebra of vector fields on \bar{M} .

An almost contact metric manifold \bar{M} is called a Kenmotsu manifold if [12],

$$(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \text{ and } \bar{\nabla}_X \xi = X - \eta(X)\xi$$

(2.2)

where $\bar{\nabla}$ denotes the Levi-Civita connection on \bar{M} .

The curvature tensor \bar{R} of Kenmotsu space form $\bar{M}(c)$ is given by [12],

$$\bar{R}(X, Y)Z = \frac{c-3}{4} \{g(Y, Z)X - g(X, Z)Y\} + \frac{c+1}{4} \left\{ \begin{array}{l} \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi \\ -\eta(X)g(Y, Z)\xi - g(\phi X, Z)\phi Y \\ +g(\phi Y, Z)\phi X + 2g(X, \phi Y)\phi Z \end{array} \right\}$$

(2.3)

for all $X, Y, Z \in T\bar{M}$.

Now, let M be an m -dimensional Riemannian manifold isometrically immersed in a Kenmotsu manifold \bar{M} . Denoting by TM the tangent bundle of M and by $T^\perp M$ the set of all vector fields normal to M , we write,

$$\varphi X = PX + FX \text{ and } \varphi N = tN + fN$$

(2.4)

for any $X \in TM$ and $N \in T^\perp M$, where PX (resp. FX) denotes the tangential (resp. normal) component of φX , and tN (resp. fN) denotes the tangential (resp. normal) component of φN .

From now on, we assume that the structure vector field ξ is tangent to M . We take the orthogonal direct decomposition $TM = D \oplus \{\xi\}$.

A submanifold M is said to be slant if for any non zero X tangent to M at x such that X is not proportional to ξ_x , the angle $\theta(X)$ between φX and $T_x M$ is constant, i.e. $\theta(X)$ is independent of the choice of $x \in M$ and $X \in T_x M - \{\xi_x\}$. Sometime the angle $\theta(X)$ is termed as Wirtinger angle of the slant immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

Let $\bar{\nabla}$ (resp. ∇) denote the Riemannian connection on \bar{M} (resp. M) and ∇^\perp denote the connection in the normal bundle $T^\perp M$ of M . Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

(2.5)

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N$$

(2.6)

for any $X, Y \in TM$ and $N \in T^\perp M$.

The second fundamental forms h and A_N are related by

$$g(A_N X, Y) = g(h(X, Y), N). \quad (2.7)$$

Denote by R the curvature tensor of M and by R^\perp the curvature tensor of the normal connection. Then the equations of Gauss, Ricci and Codazzi are given by

$$\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)) \quad (2.8)$$

$$\bar{R}(X, Y, U, V) = R^\perp(X, Y, U, V) - g([A_U, A_V] X, Y) \quad (2.9)$$

and

$$(\bar{R}(X, Y)Z)^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) \quad (2.10a)$$

for all $X, Y, Z, W \in TM$ and $U, V \in T^\perp M$, where $(\bar{R}(X, Y)Z)^\perp$ denotes the normal component of $\bar{R}(X, Y)Z$, and $(\bar{\nabla}_X h)(Y, Z)$ is given by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla_X^\perp(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (2.10b)$$

Now if P is the endomorphism given by (2.4), then we have

$$g(PX, Y) + g(X, PY) = 0. \quad (2.11)$$

Thus, it is obvious that the operator P^2 , which is denoted by Q , is self adjoint. Also,

$$(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y) \quad (2.12)$$

$$(\nabla_X F)Y = \nabla_X^\perp(FY) - F(\nabla_X Y) \quad (2.13)$$

for any $X, Y \in TM$.

Now, Gauss and Weingarten formulae together with (2.2) and (2.9) imply

$$(\nabla_X P)Y = A_{FY}X + th(X, Y) + g(Y, PX)\xi - \eta(Y)PX \quad (2.14)$$

$$\nabla_X^\perp(FY) - F(\nabla_X Y) = fh(X, Y) - h(X, PY) - \eta(Y)FX \quad (2.15)$$

for any $X, Y \in TM$.

For each $X \in TM$, we denote

$$X^* = \frac{FX}{\sin \theta}. \quad (2.16)$$

Now, one can define a symmetric bilinear TM -valued form δ on M , given by

$$\delta(X, Y) = th(X, Y). \quad (2.17)$$

Moreover, using (2.2), we have

$$\delta(X, \xi) = 0. \quad (2.18)$$

Also, from (2.4), (2.16) and (2.17), we get

$$\varphi\delta(X, Y) = P\delta(X, Y) + \sin\theta \delta^*(X, Y). \quad (2.19)$$

Now using (2.4) and (2.17), we get

$$\varphi h(X, Y) = \delta(X, Y) + \sigma^*(X, Y), \quad (2.20)$$

where σ is a symmetric bilinear D-valued form on M. Applying φ on (2.10) and using (2.19) with (1.4), we find

$$-h(X, Y) = P\delta(X, Y) + \sin \theta \delta^*(X, Y) + t\sigma^*(X, Y) + f\sigma^*(X, Y). \quad (2.21)$$

Equating tangential as well as normal parts in the above equation, we have

$$(a) \quad P\delta(X, Y) = -t\sigma^*(X, Y)$$

and

$$(b) \quad -h(X, Y) = \sin \theta \delta^*(X, Y) + f\sigma^*(X, Y).$$

Moreover, $\varphi^2\sigma(X, Y) = -\sigma(X, Y) = P^2\sigma(X, Y) + FP\sigma(X, Y) + tF\sigma(X, Y) + fF\sigma(X, Y)$

Comparison of tangential and normal parts yields

$$(c) \quad -\sin^2 \theta \sigma(X, Y) = tF\sigma(X, Y)$$

and

$$(d) \quad FP\sigma(X, Y) = -fF\sigma(X, Y).$$

Now from (a) and (c), we get

$$\sigma(X, Y) = \csc \theta P\delta(X, Y) \quad (2.22)$$

Also, (b) and (d) after making use of (2.22), give

$$h(X, Y) = -\csc \theta \delta^*(X, Y) \quad (2.23)$$

Using (2.19), we get

$$h(X, Y) = \csc^2 \theta (P\delta(X, Y) - \varphi\delta(X, Y)). \quad (2.24)$$

Now, from (2.14)

$$g((\nabla_X P)Y, Z) = -g(\delta(X, Z), Y) + g(\delta(X, Y), Z) + \eta(Z)g(PX, Y) + \eta(Y)g(X, PZ). \quad (2.25)$$

From (2.3), we have

$$\begin{aligned} \bar{R}(X, Y, Z, W) = & \frac{c-3}{4} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\ & + \frac{c+1}{4} \left\{ \begin{aligned} & \eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ & + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z) - g(\varphi X, Z)g(\varphi Y, W) \\ & + g(\varphi Y, Z)g(\varphi X, W) + 2g(X, \varphi Y)g(\varphi Z, W) \end{aligned} \right\} \end{aligned} \quad (2.26)$$

for $X, Y, Z, W \in TM$.

Using (2.1), (2.4) and (2.8) in (2.26), we find

$$R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)) = \frac{c-3}{4}\{g(Y, Z)g(X, W) - g(X, Z)(Y, W)\} + \frac{c+1}{4} \left\{ \begin{array}{l} \eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ +\eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z) - g(PX, Z)g(PY, W) \\ +g(PY, Z)g(PX, W) + 2g(X, PY)g(PZ, W) \end{array} \right\}, \quad (2.27)$$

which in the view of (2.23) and the relation $g(FX, FY) = \sin^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\}$, gives

$$R(X, Y, Z, W) = \csc^2 \theta \{g(\delta(X, W), \delta(Y, Z)) - g(\delta(X, Z), \delta(Y, W))\} + \frac{c-3}{4} \{g(Y, Z)g(X, W) - g(X, Z)(Y, W)\} + \frac{c+1}{4} \left\{ \begin{array}{l} \eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\ +\eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z) \\ -g(PX, Z)g(PY, W) + g(PY, Z)g(PX, W) + 2g(X, PY)g(PZ, W) \end{array} \right\}. \quad (2.28)$$

Now taking normal part of equation (2.3), we get

$$[\bar{R}(X, Y)Z]^\perp = \frac{c+1}{4} \{-g(PX, Z)FY + g(PY, Z)FX + 2g(X, PY)FZ\} \quad (2.29)$$

We have,

$$\begin{aligned} \nabla_X^\perp(h(Y, Z)) &= \nabla_X^\perp(-\csc \theta \delta^*(Y, Z)) \\ &= -\csc^2 \theta \nabla_X^\perp(F\delta(Y, Z)) \\ &= -\csc^2 \theta \{(\nabla_X F)\delta(Y, Z) + F(\nabla_X \delta(Y, Z))\} \end{aligned}$$

Using (2.15), we get

$$\nabla_X^\perp(h(Y, Z)) = -\csc^2 \theta \left\{ \begin{array}{l} fh(X, \delta(Y, Z)) - h(X, P\delta(Y, Z)) + F((\nabla_X \delta)(Y, Z)) \\ +\delta(\nabla_X Y, Z) + \delta(Y, \nabla_X Z) \end{array} \right\}.$$

From (2.23), we obtain

$$h(\nabla_X Y, Z) = -\csc \theta \delta^*(\nabla_X Y, Z) = -\csc^2 \theta F\delta(\nabla_X Y, Z)$$

Also, $h(Y, \nabla_X Z) = -\csc^2 \theta F\delta(Y, \nabla_X Z)$.

Hence using (2.10) (b), we get

$$(\bar{\nabla}_X h)(Y, Z) = -\csc^2 \theta \{fh(X, \delta(Y, Z)) - h(X, P\delta(Y, Z)) + F((\nabla_X \delta)(Y, Z))\}.$$

Since, $fh(X, Y) = \csc^2 \theta FP\delta(X, Y)$, we have

$$\begin{aligned} (\bar{\nabla}_X h)(Y, Z) &= -\csc^2 \theta [\csc^2 \theta FP\delta(X, \delta(Y, Z)) + \\ &+ \csc^2 \theta F\delta(X, P\delta(Y, Z)) + F((\nabla_X \delta)(Y, Z))]. \end{aligned} \tag{2.30}$$

Now using (2.29) and (2.30) in Codazzi equation, we obtain

$$\begin{aligned} &(\nabla_X \delta)(Y, Z) + \csc^2 \theta \{P\delta(X, \delta(Y, Z)) + \delta(X, P\delta(Y, Z))\} + \\ &+ \frac{c+1}{4} \sin^2 \theta \{g(X, PZ)(Y - \eta(Y)\xi) + g(X, PY)(Z - \eta(Z)\xi)\} \\ &= (\nabla_Y \delta)(X, Z) + \csc^2 \theta \{P\delta(Y, \delta(X, Z)) + \delta(Y, P\delta(X, Z))\} + \\ &+ \frac{c+1}{4} \sin^2 \theta \{g(Y, PZ)(X - \eta(X)\xi) + g(Y, PX)(Z - \eta(Z)\xi)\}. \end{aligned} \tag{2.31}$$

3. Existence theorem for slant immersions into Kenmotsu space form

In this section we shall obtain a general existence theorem for slant immersions into Kenmotsu space form. In order to prove the existence theorem, we need the following result.

Theorem A ([9]). *Let us take a manifold S with complete connection \bar{D} having parallel torsion and curvature tensors. Let M be a simply connected manifold and E be a vector bundle with connection \bar{D} over M having the algebraic structure (\bar{R}, \bar{T}) of S . Let $F : TM \rightarrow E$ be a vector bundle homomorphism satisfying the equations*

$$\begin{aligned} \bar{D}_V F(W) - \bar{D}_W F(V) - F([V, W]) &= \bar{T}(F(V), F(W)) \\ \bar{D}_V \bar{D}_W U - \bar{D}_W \bar{D}_V U - \bar{D}_{[V, W]} U &= \bar{R}(F(V), F(W))U \end{aligned}$$

for any sections V, W of TM and U of E . Then there exists a smooth map $f : M \rightarrow S$ and a parallel bundle isomorphism $\bar{\Phi} : E \rightarrow f^* TS$ preserving \bar{T} and \bar{R} such that $df = \bar{\Phi} \circ F$.

Theorem 3.1 (Existence). *Let c and θ be two constants with $0 < \theta \leq \pi/2$ and M be a simply-connected $(m + 1)$ -dimensional Riemannian manifold with metric tensor g . Suppose that there exists a unit global vector field ξ on M , an endomorphism P of the tangent bundle TM and a symmetric bilinear TM -valued form δ on M such that*

$$P(\xi) = 0, \quad g(\delta(X, Y), \xi) = 0, \quad \nabla_X \xi = X - \eta(X)\xi \tag{3.1}$$

$$P^2 X = -\cos^2 \theta (X - \eta(X)\xi) \tag{3.2}$$

$$g(PX, Y) + g(X, PY) = 0 \tag{3.3}$$

$$\delta(X, \xi) = 0 \tag{3.4}$$

$$g((\nabla_X P)Y, Z) = g(\delta(X, Y), Z) - g(\delta(X, Z), Y) + g(PX, Y)\eta(Z) + g(X, PZ)\eta(Y) \tag{3.5}$$

$$\begin{aligned}
 R(X, Y, Z, W) &= \csc^2 \theta \{g(\delta(X, W), \delta(Y, Z)) - g(\delta(X, Z), \delta(Y, W))\} \\
 &+ \frac{c-3}{4} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
 &+ \frac{c+1}{4} \left\{ \begin{aligned} &\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) + \eta(Y)\eta(W)g(X, Z) \\ &-\eta(X)\eta(W)g(Y, Z) - g(PX, Z)g(PY, W) + g(PY, Z)g(PX, W) \\ &+ 2g(X, PY)g(PZ, W) \end{aligned} \right\} \tag{3.6}
 \end{aligned}$$

and

$$\begin{aligned}
 &(\nabla_X \delta)(Y, Z) + \csc^2 \theta \{P\delta(X, \delta(Y, Z)) + \delta(X, P\delta(Y, Z))\} \\
 &+ \frac{c+1}{4} \sin^2 \theta \{g(X, PZ)(Y - \eta(Y)\xi) + g(X, PY)(Z - \eta(Z)\xi)\} \\
 &= (\nabla_Y \delta)(X, Z) + \csc^2 \theta \{P\delta(Y, \delta(X, Z)) + \delta(Y, P\delta(X, Z))\} \\
 &+ \frac{c+1}{4} \sin^2 \theta \{g(Y, PZ)(X - \eta(X)\xi) + g(Y, PX)(Z - \eta(Z)\xi)\} \tag{3.7}
 \end{aligned}$$

for all $X, Y, Z \in TM$, where η denotes the dual 1-form of ξ . Then there exists a θ -slant immersion from M into Kenmotsu space form $\bar{M}^{2m+1}(c)$ whose second fundamental form $h(X, Y) = \csc^2 \theta (P\delta(X, Y) - \varphi\delta(X, Y))$ is given by the relation

$$h(X, Y) = \csc^2 \theta (P\delta(X, Y) - \varphi\delta(X, Y)). \tag{3.8}$$

Proof. Let c, θ, M, ξ, P and δ satisfy the conditions stated above. Suppose $TM \oplus D$ be the Whitney sum. We identify $(X, 0)$ with X for each $X \in TM$, and $(0, Z)$ by Z^* for each $Z \in D$. In particular, we identify $(\xi, 0)$ with $\hat{\xi}$ for ξ . We denote the product metric on $TM \oplus D$ by \hat{g} . Hence, if we denote the dual 1-form of $\hat{\xi}$ by $\hat{\eta}$ then, $\hat{\eta}(X, Z) = \eta(X)$, for any $X \in TM$ and $Z \in D$.

The endomorphism $\hat{\phi}$ on $TM \oplus D$ can be defined as

$$\hat{\phi}(X, 0) = (PX, \sin \theta (X - \eta(X)\xi)), \quad \hat{\phi}(0, Z) = (-\sin \theta Z, -PZ) \tag{3.9}$$

for any $X \in TM$ and $Z \in D$.

It is easy to see that $\hat{\phi}^2(X, 0) = -(X, 0) + \hat{\eta}(X, 0)\hat{\xi}$, and $\hat{\phi}^2(0, Z) = -(0, Z)$. Thus, $\hat{\phi}^2(X, Z) = -(X, Z) + \hat{\eta}(X, Z)\hat{\xi}$, for any $X \in TM$ and $Z \in D$.

Now, using (3.2), (3.3) and (3.9) it can readily be seen that $(\hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ is an almost contact metric structure on $TM \oplus D$.

Now we can take A, h and ∇^\perp as following:

$$A_{Z^*}X = \csc \theta \{(\nabla_X P)Z - \delta(X, Z) - g(Z, PX)\xi\} \tag{3.10}$$

$$h(X, Y) = -\csc \theta \delta^*(X, Y) \tag{3.11}$$

$$\nabla_X^\perp Z^* = (\nabla_X Z - \eta(\nabla_X Z)\xi)^* + \csc^2 \theta \{(P\delta(X, Z))^* + \delta^*(X, PZ)\} \tag{3.12}$$

for any $X, Y \in TM$ and $Z \in D$. It is easy to check that A is an endomorphism on TM , h is a $(D)^*$ -valued symmetric bilinear form on TM and ∇^\perp is a metric connection of the vector bundle $(D)^*$ over M .

Let $\hat{\nabla}$ denote the connection on $TM \oplus D$ induced from equations (3.10)-(3.12). Then from (3.1), (3.2), (3.4) and (3.9), we have

$$\left(\hat{\nabla}_{(X,0)} \hat{\varphi} \right)(Y, 0) = g(\hat{\varphi}(X, 0), (Y, 0)) \hat{\xi} - \hat{\eta}(Y, 0) \hat{\varphi}(X, 0), \quad \left(\hat{\nabla}_{(X,0)} \hat{\varphi} \right)(0, Z) = 0$$

for any $X, Y \in TM$ and $Z \in D$.

Let R^\perp denote the curvature tensor associated with the connection ∇^\perp on $(D)^*$, that is $R^\perp(X, Y)Z^* = \nabla_X^\perp \nabla_Y^\perp Z^* - \nabla_Y^\perp \nabla_X^\perp Z^* - \nabla_{[X, Y]}^\perp Z^*$, for any $X, Y \in TM$ and $Z \in D$. Then by virtue of (3.1), (3.2), (3.3), (3.4), (3.7) and (3.12), a straightforward computation yields

$$\begin{aligned} R^\perp(X, Y)Z^* &= \{R(X, Y)Z - \eta(R(X, Y)Z)\xi\}^* \\ &+ \frac{c+1}{4} \left[\begin{aligned} &P \{g(Y, PZ)X + 2g(Y, PX)Z - g(X, PZ)Y\} \\ &+ \{g(Y, P^2Z)(X - \eta(X)\xi) + 2g(Y, PX)PZ - g(X, P^2Z)(Y - \eta(Y)\xi)\} \end{aligned} \right]^* \\ &+ \csc^2 \theta \left\{ \begin{aligned} &(\nabla_X P)\delta(Y, Z) - (\nabla_Y P)\delta(X, Z) - \eta(\nabla_X(P\delta(Y, Z)))\xi + \eta(\nabla_Y(P\delta(X, Z)))\xi \\ &+ \delta(Y, (\nabla_X P)Z) - \delta(X, (\nabla_Y P)Z) - \eta(\nabla_X(\delta(Y, PZ)))\xi + \eta(\nabla_Y(\delta(X, PZ)))\xi \end{aligned} \right\}^* \\ &+ \{\eta(X)\eta(\nabla_Y Z)\xi - \eta(Y)\eta(\nabla_X Z)\xi - \eta(\nabla_Y Z)X + \eta(\nabla_X Z)Y\}^*. \end{aligned} \tag{3.13}$$

Also, from (3.1), (3.5), (3.10), and (3.11), we have

$$\begin{aligned} \sin^2 \theta g([A_{Z^*}, A_{W^*}]X, Y) &= g((\nabla_Y P)Z, (\nabla_X P)W) - g((\nabla_Y P)Z, \delta(X, W)) \\ &- g(W, PX)\eta((\nabla_Y P)Z) - g(\delta(Y, Z), (\nabla_X P)W) + g(\delta(Y, Z), \delta(X, W)) \\ &- g(Z, PY)\eta((\nabla_X P)W) + g(Z, PY)g(W, PX) - g((\nabla_Y P)W, (\nabla_X P)Z) \\ &+ g((\nabla_Y P)W, \delta(X, Z)) + g(Z, PX)\eta((\nabla_Y P)W) + g(\delta(Y, W), (\nabla_X P)Z) \\ &- g(\delta(Y, W), \delta(X, Z)) + \eta((\nabla_X P)Z)g(W, PY) - g(W, PY)g(Z, PX). \end{aligned} \tag{3.14}$$

From (3.3), we have

$$g(\delta(Y, Z), PW) + g(P\delta(Y, Z), W) = 0. \tag{3.15}$$

Taking covariant derivative of (3.15) with respect to X and using (3.3), we get

$$g(\delta(Y, Z), (\nabla_X P)W) + g((\nabla_X P)\delta(Y, Z), W) = 0. \tag{3.16}$$

Moreover, by virtue of (3.5), we get

$$\begin{aligned} g((\nabla_Y P)W, (\nabla_X P)Z) &= g(\delta(Y, W), (\nabla_X P)Z) - g(\delta(Y, (\nabla_X P)Z), W) + \\ &+ g(PY, W)\eta((\nabla_X P)Z) + g(Y, P((\nabla_X P)Z))\eta(W). \end{aligned} \tag{3.17}$$

Also,

$$\begin{aligned} g((\nabla_Y P)Z, (\nabla_X P)W) &= g((\nabla_X P)W, (\nabla_Y P)Z) \\ &= g(\delta(X, W), (\nabla_Y P)Z) - g(\delta(X, (\nabla_Y P)Z), W) + g(PX, W)\eta((\nabla_Y P)Z). \end{aligned}$$

Using this in equations (2.3), (3.2), (3.3) and (3.18), we get

$$\begin{aligned} g(R^\perp(X, Y)Z^*, W^*) - g([AZ^*, AW^*]X, Y) \\ = \left(\frac{c+1}{4}\right) [\sin^2 \theta \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} + 2g(Y, PX)g(PZ, W)]. \end{aligned} \tag{3.18}$$

Equations (2.3), (3.2), (3.3) and (3.18) imply that (M, A, ∇^\perp) satisfies the Ricci equation for an $(m+1)$ -dimensional θ -slant submanifold in $\bar{M}^{2m+1}(c)$. Moreover, (2.28) and (2.31) imply that (M, h) satisfies the equations of Gauss and Codazzi for a θ -slant submanifold. Thus, we have a vector bundle $(TM \oplus D)$ over M equipped with product metric \hat{g} , the shape operator A , the second fundamental form h and the connections ∇^\perp and $\hat{\nabla}$ satisfying the structure equations of $(m+1)$ -dimensional θ -slant submanifold in $\bar{M}^{2m+1}(c)$. Therefore, from theorem A, there exists a θ -slant isometric immersion of M in $\bar{M}^{2m+1}(c)$ with h as its second fundamental form, A as its shape operator and ∇^\perp as its normal connection. \square

4. Uniqueness theorem for slant immersions into Kenmotsu space form

Theorem 4.1 (*Uniqueness*). *Let $x^1, x^2 : M \rightarrow \bar{M}(c)$ be two slant immersions with slant angle θ ($0 < \theta \leq \pi/2$), of a connected Riemannian manifold M^{m+1} into a Kenmotsu space-form $\bar{M}^{2m+1}(c)$. Let h^1, h^2 denote the second fundamental forms of x^1 and x^2 , respectively. Suppose that there is a vector field $\hat{\xi}$ on M such that $x_{*p}^i(\hat{\xi}_p) = \xi_{x^i(p)}$ for any $i = 1, 2$ and $p \in M$; and*

$$g(h^1(X, Y), \varphi x_*^1 Z) = g(h^2(X, Y), \varphi x_*^2 Z) \tag{4.1}$$

for all vector fields X, Y, Z tangent to M . Moreover, we also assume that one of the following conditions hold:

- (i) $\theta = \pi/2$
- (ii) There exists a point p of M such that $P_1 = P_2$.
- (iii) $c \neq -1$.

Then, there exists an isometry ϕ of $\bar{M}^{2m+1}(c)$ such that $x^1 = \phi \circ x^2$.

Proof. Let us take any point p of M . We may assume that $x^1(p) = x^2(p)$ and $x_*^1(p) = x_*^2(p)$. Take a geodesic γ through $p = \gamma(0)$. Now, we define $\gamma_1 = x^1(\gamma)$ and $\gamma_2 = x^2(\gamma)$. To prove the theorem it is sufficient to show that γ_1 and γ_2 coincide. It is known that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1'(0) = \gamma_2'(0)$. Let $E_1, E_2, \dots, E_m, \hat{\xi}$ be any orthonormal frame along γ . We can define a frame along γ_1 and γ_2 as follows:

$$a_i = x_*^1(E_i), B_i = x_*^2(E_i), A_{n+i} = (x_*^1(E_i))^*, B_{n+i} = (x_*^2(E_i))^*, \text{ where, } X^* = \frac{FX}{\sin \theta}$$

for any $X \in D$.

From (3.11), $h(X, Y) = -\csc \theta \delta^*(X, Y)$, and therefore $h^i = -\csc \theta (\delta^i)^*$ for any $i = 1, 2$.

From (4.1), we have

$$\begin{aligned} g(\csc \theta (\delta^1)^*(X, Y), \varphi x_*^1 Z) &= g(\csc \theta (\delta^2)^*(X, Y), \varphi x_*^2 Z) \\ g((\delta^1)^*(X, Y), F x_*^1 Z) &= g((\delta^2)^*(X, Y), F x_*^2 Z) \\ g(\delta^1(X, Y), x_*^1 Z) &= g(\delta^2(X, Y), x_*^2 Z) \end{aligned}$$

Since, $x_*^1(p) = x_*^2(p)$ and Z is arbitrary, we conclude that $\delta^1 = \delta^2$.

Now, we have to show that $P_1 = P_2$.

If (i) is satisfied then we see that $P_1 = P_2 = 0$.

And if (ii) is satisfied, it follows from (3.5) that,

$$g(\nabla_X (P_1 - P_2)Y, Z) = g((P_1 - P_2)X, Y) \eta(Z) + g(X, (P_1 - P_2)Z) \eta(Y).$$

Since it is true for any X, Y, Z and we have $P_1 = P_2$ at any point $p \in M$, therefore we have $P_1 = P_2$ everywhere.

Now suppose that (iii) is satisfied and assume that $P_1 \neq P_2$ and (i) and (ii) are not satisfied. First we want to show that $P_1 = -P_2$.

From (3.6), we find that

$$\begin{aligned} g(P_1 X, W) g(P_1 Y, Z) - g(P_1 X, Z) g(P_1 Y, W) + 2g(P_1 Z, W) g(P_1 Y, X) \\ = g(P_2 X, W) g(P_2 Y, Z) - g(P_2 X, Z) g(P_2 Y, W) + 2g(P_2 Z, W) g(P_2 Y, X). \end{aligned} \quad (4.2)$$

Putting $X = W, Y = Z$, and using skew symmetric property of P_1 and P_2 , equation (4.2) reduces to

$$g(P_1 Y, X)^2 = g(P_2 Y, X)^2. \quad (4.3)$$

Now putting $e_1 = X$ and $e_2 = P_1 X$, and letting that $P_2 e_1$ has a component in the direction of vector e_3 which is orthogonal to both e_1 and e_2 , a contradiction follows from (4.3) which states that

$$g(P_2 e_1, e_3)^2 = g(P_1 e_1, e_3)^2 = g(e_2, e_3)^2 = 0.$$

Now using (3.2) and (3.3), we have $P_1 \nu = \pm P_2 \nu$ for any tangent vector ν .

We choose a basis $\{e_1, \dots, e_m, e_{m+1}\}$ of the tangent space at a point $p \in M$. Then there exists a number $\varepsilon_i \in \{-1, 1\}$ such that $P_1 e_i = \varepsilon_i P_2 e_i$. Hence we have

$$\pm P_1(e_i + e_j) = P_2(e_i + e_j) = \varepsilon_i P_1 e_i + \varepsilon_j P_1 e_j.$$

Thus the above formula shows that all ε_i have to be same, and so either $P_1 \nu = P_2 \nu$ or $P_1 \nu = -P_2 \nu$ for all $\nu \in T_p M$.

Since M is connected, we have either $P_1 = P_2$ or $P_1 = -P_2$ in case (iii).

Now, assume that we have two immersions with $P_1 = -P_2$. From (3.5) it follows that

$$g((\nabla_X P_1)Y, Z) = g(\delta^1(X, Y), Z) - g(\delta^1(X, Z), Y) + g(P_1 X, Y) \eta(Z) + g(X, P_1 Z) \eta(Y)$$

and

$$\begin{aligned} g((\nabla_X P_2)Y, Z) &= -g((\nabla_X P_1)Y, Z) \\ &= g(\delta^2(X, Y), Z) - g(\delta^2(X, Z), Y) + g(P_2 X, Y) \eta(Z) + g(X, P_2 Z) \eta(Y). \end{aligned}$$

Since $\delta^1 = \delta^2 = \delta$, we get

$$g(\delta(X, Y), Z) = g(\delta(X, Z), Y). \tag{4.4}$$

Writing equation (3.7) for both the immersions, we get

$$\begin{aligned} \{(\nabla_X \delta^1)(Y, Z) - (\nabla_Y \delta^1)(X, Z)\} &= \csc^2 \theta \{P_1 \delta^1(Y, \delta^1(X, Z)) + \delta^1(Y, P_1 \delta^1(X, Z)) \\ &\quad - P_1 \delta^1(X, \delta^1(Y, Z)) - \delta^1(X, P_1 \delta^1(Y, Z))\} + \frac{c+1}{4} \sin^2 \theta \{g(Y, P_1 Z)(X - \eta(X)\xi) \\ &\quad - g(X, P_1 Z)(Y - \eta(Y)\xi) - 2g(X, P_1 Y)(Z - \eta(Z)\xi)\} \end{aligned}$$

and

$$\begin{aligned} \{\nabla_X \delta^2)(Y, Z) - (\nabla_Y \delta^2)(X, Z)\} &= \csc^2 \theta \{P_2 \delta^2(Y, \delta^2(X, Z)) + \delta^2(Y, P_2 \delta^2(X, Z)) \\ &\quad - P_2 \delta^2(X, \delta^2(Y, Z)) - \delta^2(X, P_2 \delta^2(Y, Z))\} + \frac{c+1}{4} \sin^2 \theta \{g(Y, P_2 Z)(X - \eta(X)\xi) \\ &\quad - g(X, P_2 Z)(Y - \eta(Y)\xi) - 2g(X, P_2 Y)(Z - \eta(Z)\xi)\}. \end{aligned}$$

Now using $P_1 = -P_2 = P$ in the above equations, and subtracting the two, we get

$$\begin{aligned} 0 &= 2 \csc^2 \theta \{P \delta(Y, \delta(X, Z)) + \delta(Y, P \delta(X, Z)) - P \delta(X, \delta(Y, Z)) - \delta(X, P \delta(Y, Z))\} \\ &\quad + 2 \frac{c+1}{4} \sin^2 \theta \{g(Y, P Z)(X - \eta(X)\xi) - g(X, P Z)(Y - \eta(Y)\xi) - 2g(X, P Y)(Z - \eta(Z)\xi)\}, \end{aligned}$$

or

$$\begin{aligned} \{P \delta(Y, \delta(X, Z)) + \delta(Y, P \delta(X, Z)) - P \delta(X, \delta(Y, Z)) - \delta(X, P \delta(Y, Z))\} \\ + \frac{c+1}{4} \sin^4 \theta \{g(Y, P Z)(X - \eta(X)\xi) - g(X, P Z)(Y - \eta(Y)\xi) - 2g(X, P Y)(Z - \eta(Z)\xi)\} = 0, \end{aligned}$$

or

$$\begin{aligned} P \delta(X, \delta(Y, Z)) + \delta(X, P \delta(Y, Z)) - P \delta(Y, \delta(X, Z)) - \delta(Y, P \delta(X, Z)) \\ + \frac{c+1}{4} \sin^4 \theta \{g(X, P Z)(Y - \eta(Y)\xi) - g(Y, P Z)(X - \eta(X)\xi) + \\ + 2g(X, P Y)(Z - \eta(Z)\xi)\} = 0. \end{aligned} \tag{4.5}$$

Taking inner product of equation (4.5) with a vector W , we get

$$\begin{aligned} g(P \delta(X, \delta(Y, Z)), W) + g(\delta(X, P \delta(Y, Z)), W) - g(P \delta(Y, \delta(X, Z)), W) - g(\delta(Y, P \delta(X, Z)), W) \\ + \frac{c+1}{4} \sin^4 \theta \{g(X, P Z) g(Y, W) - g(X, P Z) \eta(Y) \eta(W) - g(Y, P Z) g(X, W) + g(Y, P Z) \eta(X) \eta(W) \\ + 2g(X, P Y) g(Z, W) - 2g(X, P Y) \eta(Z) \eta(W)\} = 0, \end{aligned}$$

or

$$\begin{aligned}
 & -g(\delta(X, PW), \delta(Y, Z)) + g(\delta(Y, PW), \delta(X, Z)) + g(\delta(X, W), P\delta(Y, Z)) - \\
 & -g(\delta(Y, W), P\delta(X, Z)) + \frac{c+1}{4} \sin^4 \theta \{g(X, PZ)g(Y, W) - g(X, PZ)\eta(Y)\eta(W) - \\
 & -g(Y, PZ)g(X, W) + g(Y, PZ)\eta(X)\eta(W) + \\
 & + 2g(X, PY)g(Z, W) - 2g(X, PY)\eta(Z)\eta(W)\} = 0.
 \end{aligned} \tag{4.6}$$

If δ vanishes identically at a point, then a contradiction follows from (4.6) since $c \neq -1$.

Now we take a fixed point p of M and consider a function f defined on the set of all unit tangent vectors UM_p by

$$f(\nu) = g(\delta(\nu, \nu), \nu), \text{ for all } \nu \in UM_p.$$

Since UM_p is compact there exists a vector u such that f attains an absolute maximum at u . Let w be a unit vector orthogonal to u . Then the function $f(t) = f(g(t))$, where the relation $g(t) = (\cos t)u + (\sin t)w$ satisfies the conditions $f'(0) = 0$ and $f''(0) \leq 0$. The first condition implies that $g(\delta(u, u), w) = 0$, whereas the second condition implies $g(\delta(u, w), w) \leq \frac{1}{2}g(\delta(u, u), u)$.

Using the total symmetry of δ , it follows that we can choose an orthonormal basis $e_1 = u, e_2, e_3, \dots, e_m, e_{m+1}$ such that

$$\delta(e_1, e_1) = \lambda_1 e_1, \quad \delta(e_1, e_i) = \lambda_i e_i \tag{4.7}$$

with $i > 1$ and $\lambda_i \leq \frac{1}{2}\lambda_1$. Since δ is not identically 0, it follows from total symmetry of (4.4) that $\lambda_1 > 0$.

Using (4.4) and (4.7) in (4.6), with $X = Z = W = e_1$ and $Y = e_i$, we find

$$\begin{aligned}
 & -g(\delta(e_1, Pe_1), \delta(e_i, e_1)) + g(\delta(e_i, Pe_1), \delta(e_1, e_1)) + g(\delta(e_1, e_1), P\delta(e_i, e_1)) \\
 & -g(\delta(e_i, e_1), P\delta(e_1, e_1)) + \frac{c+1}{4} \sin^4 \theta \{g(e_1, Pe_1)g(e_i, e_1) - g(e_1, Pe_1)\eta(e_i)\eta(e_1) \\
 & -g(e_i, Pe_1)g(e_1, e_1) + g(e_i, Pe_1)\eta(e_1)\eta(e_1) + 2g(e_1, Pe_1)g(e_1, e_1) - 2g(e_1, Pe_1)\eta(e_1)\eta(e_1)\} = 0,
 \end{aligned}$$

or

$$\begin{aligned}
 & -g(\delta(e_1, Pe_1), \lambda_i e_i) + g(\delta(e_i, Pe_1), \lambda_1 e_1) + g(\lambda_1 e_1, P\lambda_i e_i) - g(\lambda_i e_i, P\delta(e_1, e_1)) \\
 & \frac{c+1}{4} \sin^4 \theta \{-g(e_i, pe_1)g(e_1, e_1) + 2g(e_1, Pe_i)g(e_1, e_1)\} = 0,
 \end{aligned}$$

or

$$(\lambda_i^2 + \lambda_i \lambda_1 + 3\frac{c+1}{4} \sin^4 \theta)g(Pe_1, e_i) = 0. \tag{4.8}$$

Now, we show that Pe_1 is an eigen vector of $\delta(e_1, \cdot)$. In order to do so, we put $X = Z = W = e_1, W = e_j$ and $Y = e_i$ for $i, j > 1$. Then, we get

$$\begin{aligned}
 & -g(\delta(e_1, Pe_j), \delta(e_i, e_1)) + g(\delta(e_i, Pe_j), \delta(e_1, e_1)) + g(\delta(e_1, e_j), P\delta(e_i, e_1)) \\
 & -g(\delta(e_i, e_j), P\delta(e_1, e_1)) + \frac{c+1}{4} \sin^4 \theta \{g(e_1, Pe_1)g(e_i, e_j) - g(e_1, Pe_1)\eta(e_i)\eta(e_j) \\
 & -g(e_i, Pe_1)g(e_1, e_j) + g(e_i, Pe_1)\eta(e_1)\eta(e_j) + 2g(e_1, Pe_i)g(e_1, e_j) - 2g(e_1, Pe_i)\eta(e_1)\eta(e_j)\} = 0,
 \end{aligned}$$

or

$$(\lambda_i^2 - \lambda_i \lambda_1 + \lambda_i \lambda_j) g(Pe_j, e_i) + \lambda_1 g(\delta(e_i, e_j), Pe_1) = 0. \quad (4.9)$$

Interchanging the indices i and j in (4.9), we get

$$(\lambda_j^2 - \lambda_j \lambda_1 + \lambda_i \lambda_j) g(Pe_i, e_j) + \lambda_1 g(\delta(e_i, e_j), Pe_1) = 0. \quad (4.10)$$

Combining (4.9) and (4.10), we obtain

$$(\lambda_i^2 - \lambda_i \lambda_1 + \lambda_j^2 - \lambda_j \lambda_1 + 2\lambda_i \lambda_j) g(Pe_j, e_i) = 0,$$

or

$$(\lambda_i + \lambda_j)(\lambda_1 - \lambda_i - \lambda_j) g(Pe_j, e_i) = 0. \quad (4.11)$$

Since, $\lambda_1 \geq 2\lambda_i$, we get that $\lambda_1 - \lambda_i - \lambda_j = 0$ only if $\lambda_i = \lambda_j = \frac{1}{2}\lambda_1$.

Now if we put $X = W = e_1$, $Z = e_j$ and $Y = e_i$ for $i, j > 1$ in (4.6), we find that

$$\begin{aligned} & -g(\delta(e_1, Pe_1), \delta(e_i, e_j)) + \lambda_j g(\delta(e_i, e_j), Pe_1) - \lambda_i \lambda_j g(e_i, Pe_j) - \\ & -\lambda_1 g(\delta(e_i, e_j), Pe_1) - \frac{c+1}{4} \sin^4 \theta \{g(e_i, Pe_j)\} = 0, \end{aligned}$$

or

$$\begin{aligned} & g(\delta(e_1, Pe_1), \delta(e_i, e_j)) - \lambda_j g(\delta(e_i, e_j), Pe_1) + \lambda_i \lambda_j g(e_i, Pe_j) \\ & + \lambda_1 g(\delta(e_i, e_j), Pe_1) + \frac{c+1}{4} \sin^4 \theta \{g(e_i, Pe_j)\} = 0. \end{aligned} \quad (4.12)$$

Interchanging the indices i and j in (4.12), we get

$$\begin{aligned} & g(\delta(e_1, Pe_1), \delta(e_i, e_j)) - \lambda_i g(\delta(e_i, e_j), Pe_1) + \lambda_i \lambda_j g(e_j, Pe_i) + \\ & + \lambda_1 g(\delta(e_i, e_j), Pe_1) + \frac{c+1}{4} \sin^4 \theta \{g(e_j, Pe_i)\} = 0. \end{aligned} \quad (4.13)$$

Combining (4.12) and (4.13), we get

$$(\lambda_i - \lambda_j) g(\delta(e_i, e_j), Pe_1) + 2\lambda_i \lambda_j g(e_i, Pe_j) + \frac{c+1}{2} \sin^4 \theta g(e_i, Pe_j) = 0. \quad (4.14)$$

Now, we summarize the previous equations in the following manner. First, by taking $i = j$ in (4.9), we get

$$g(\delta(e_i, e_i), Pe_1) = 0. \quad (4.15)$$

Hence, we have $g(\delta(\nu, \nu), Pe_1) = 0$ if ν is an eigenvector of $\delta(e_1, \cdot)$. Moreover, the symmetry of δ then implies that $g(\delta(e_i, e_j), Pe_1) = 0$, whenever $\lambda_i = \lambda_j$.

We now consider the following four different cases.

(1) $\lambda_i + \lambda_j \neq 0$, but not $\lambda_i = \lambda_j = \frac{1}{2}\lambda_1$. In this case (4.11) implies $g(Pe_i, e_j) = 0$.

(2) $\lambda_i + \lambda_j = 0$, and $\lambda_i \neq 0$. In this case, (4.9) implies $g(\delta(e_i, e_j), Pe_1) = \lambda_i g(Pe_j, e_i)$.

Using the consequence of case (2) in (4.14), we get $g(Pe_j, e_i) = 0$.

(3) $\lambda_i = \lambda_j = 0$. In this case it follows from (4.14) that $g(e_i, Pe_j) = 0$;

(4) $\lambda_i = \lambda_j = \frac{1}{2}\lambda_1$.

If $e_{i_1}, e_{i_2}, \dots, e_{i_k}$ are eigenvectors belonging to an eigenvalue different from $\frac{1}{2}\lambda_1$, then each Pe_{i_l} , $l = 1, \dots, k$, can only have a component in the direction of e_1 , say $Pe_{i_l} = \mu_l e_1$. Thus $\mu_l Pe_1 = -\cos^2 \theta e_{i_l}$. Consequently, either $k = 1$ or there does not exist an eigenvector with eigenvalue different from $\frac{1}{2}\lambda_1$. If $k = 1$, then obviously Pe_1 is an eigenvector. In the latter case $\delta(e_1, \cdot)$, restricted to the space e_1^\perp , only has one eigenvalue, namely $\frac{1}{2}\lambda_1$. Since, Pe_1 is always orthogonal to e_1 , Pe_1 is also an eigenvector in this case. Hence Pe_1 is always an eigenvector of $\delta(e_1, \cdot)$.

Without any loss of generality, we may assume that e_2 is in the direction of Pe_1 . Then it follows immediately that $\delta(e_1, Pe_1) = \lambda_2 Pe_1$, where λ_2 satisfies the equation

$$\lambda_2^2 + \lambda_2 \lambda_1 + \frac{3(c+1)}{4} \sin^4 \theta = 0 \tag{4.16}$$

by virtue of (4.8).

If we choose $X = Z = e_1$, $W = Pe_1$ and $Y = e_i$ for $i > 2$ in (4.6), then

$$g(\delta(e_1, Pe_1), \lambda_i Pe_i) - \lambda_1 g(\delta(e_i, Pe_1), Pe_1) = 0$$

or, $g(\lambda_2 Pe_1, \lambda_i Pe_i) - \lambda_1 g(\delta(e_i, Pe_1), Pe_1) = 0$

or, $\lambda_i \lambda_2 \cos^2 \theta g(e_1, e_i) - \lambda_1 g(\delta(e_i, Pe_1), Pe_1) = 0$

or, $\lambda_1 g(\delta(e_i, Pe_1), Pe_1) = \lambda_1 g(\delta(Pe_1, Pe_1), e_i) = 0$.

Thus $\delta(Pe_1, Pe_1) = \lambda_2 \cos^2 \theta e_1$.

Putting $X = Z = W = Pe_1$ and $Y = e_1$ in (4.6), we get

$$-\lambda_2^2 - \lambda_2 \lambda_1 + \frac{3(c+1)}{4} \sin^4 \theta = 0. \tag{4.17}$$

Now from (4.16) and (4.17) we get $\frac{3(c+1)}{4} \sin^4 \theta = 0$, which is a contradiction since $c \neq -1$. Therefore $P_1 = P_2$.

It can be easily seen from relations (3.10)-(3.12) that

$$g(\gamma'_1, A_k) = g(\gamma'_2, B_k) \text{ and } g(\hat{\nabla}_\gamma A_k, A_l) = g(\hat{\nabla}_\gamma B_k, B_l) \text{ for } k, l = 1, \dots, 2m$$

such that, by [16, Proposition 3], $\gamma_1 = \gamma_2$. □

5. Applications and examples

Let $\psi = \psi(x), \psi_i = \psi_i(x), i = 1, 2, 3$ be four functions defined on an open interval containing 0. Let c and θ be two constants with $0 < \theta \leq \pi/2$ and M be a simply connected open neighbourhood of the origin $(0, 0, 0) \in R^3$. Now we suppose that

$$f(x) = \exp \int \phi_3(x) dx \tag{5.1}$$

$$\eta = dz. \tag{5.2}$$

Let the warped metric on M be defined by

$$g = \eta \otimes \eta + e^{2z} (dx \otimes dx + f^2(x) dy \otimes dy). \tag{5.3}$$

Now, we consider the vectors

$$e_1 = \frac{1}{e^z} \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{f e^z} \frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial z}.$$

Then it can be readily seen that $\{e_1, e_2, e_3\}$ is a local orthonormal frame of TM and that η is a dual 1-form of ξ . Moreover, we have

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_3 & \nabla_{e_1} e_2 &= 0 & \nabla_{e_1} e_3 &= e_1 \\ \nabla_{e_2} e_1 &= \frac{\psi_3}{e^z} e_2 & \nabla_{e_2} e_2 &= -\frac{\psi_3}{e^z} e_1 - e_3 & \nabla_{e_2} e_3 &= e_2 \\ \nabla_{e_3} e_1 &= 0 & \nabla_{e_3} e_2 &= 0 & \nabla_{e_3} e_3 &= 0. \end{aligned}$$

Let us define a tensor field φ such that

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi \xi = 0,$$

and a symmetric bilinear TM -valued form δ on M given by

$$\delta(e_1, e_1) = \psi e_1 + \psi_1 e_2, \quad \delta(e_1, e_2) = \psi_1 e_1 + \psi_2 e_2, \quad \delta(e_2, e_2) = \psi_2 e_1 - \psi_1 e_2 \tag{5.4}$$

$$\delta(e_1, \xi) = 0, \quad \delta(e_2, \xi) = 0, \quad \delta(\xi, \xi) = 0. \tag{5.5}$$

It is easy to show that $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold with $(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$ for any $X, Y \in TM$. By putting $P = \cos \theta \varphi$, we can see that (M, g, ξ, P, δ) satisfy equations (3.1), (3.2), (3.3), (3.4) and (3.5). In addition, we can prove that M satisfy conditions (3.6) and (3.7) if we have the following:

$$\psi'_3 = -\psi_3^2 - e^{2z} \csc^2 \theta \{\psi \psi_2 - 2\psi_1^2 - \psi_2^2\} - e^{2z} \frac{(c+1)}{4} (1 + 3 \cos^2 \theta); \tag{5.6}$$

$$\psi'_2 = (-2\psi_2 + \psi)\psi_3 - e^z \csc \theta \cot \theta (\psi_2 + \psi)\psi_1; \tag{5.7}$$

$$\psi'_1 = -3\psi_1\psi_3 + e^z \csc \theta \cot \theta (\psi_2 + \psi)\psi_2 + 3e^z \frac{(c+1)}{4} \sin^2 \theta \cos \theta; \tag{5.8}$$

$$\psi'_1 = -3\psi_1\psi_3 + e^z \csc \theta \cot \theta (\psi_2 + \psi)\psi_2 - 3e^z \frac{(c+1)}{4} \sin^2 \theta \cos \theta. \tag{5.9}$$

But we see that (5.8) and (5.9) hold simultaneously if and only if

$e^z \frac{(c+1)}{4} \sin^2 \theta \cos \theta = 0$. Since, $0 < \theta \leq \frac{\pi}{2}$, we know that $\sin^2 \theta \neq 0$, and $e^z \neq 0$ for any $z \in R$. Hence, it must be either $c = -1$ or $\theta = \frac{\pi}{2}$.

By applying theorem (4.1), we have the following result.

Theorem 5.1 Let $\psi = \psi(x)$ be a function defined on an open interval containing 0 and c_1, c_2, c_3, c and θ be the five constants with $0 < \theta \leq \frac{\pi}{2}$. Consider the following set of first order differential equations in $y_i = y_i(x)$, for $i = 1, 2, 3$

$$\begin{aligned} y_1' &= -3y_1y_3 + e^z \csc \theta \cot \theta (y_2 + \psi)y_2 \\ y_2' &= (-2y_2 + \psi)y_3 - e^z \csc \theta \cot \theta (y_2 + \psi)y_1 \\ y_3' &= -y_3^2 - e^{2z} \csc^2 \theta (\psi y_2 - 2y_1^2 - y_2^2) - e^{2z} \frac{(c+1)}{4} (1 + 3 \cos^2 \theta) \end{aligned}$$

with the initial conditions: $y_1(0) = c_1, y_2(0) = c_2, y_3(0) = c_3$. Let ψ_1, ψ_2 and ψ_3 be the components of the unique solution of this differentiable system on some open interval containing 0. Let M be a simply connected open neighborhood of the origin $(0, 0, 0) \in R^3$, endowed with the metric given by (5.1)-(5.3). Let δ be the TM-valued form defined by (5.4) and (5.5). Then, we have

1. If $c = -1$, there exists a θ -slant isometric immersion from M into $\bar{M}^5(-1)$, whose second fundamental form is given by

$$h(X, Y) = \csc^2 \theta (P\delta(X, Y) - \varphi \delta(X, Y)).$$

1. If $\theta = \frac{\pi}{2}$, then there exists an anti-invariant immersion from M into $\bar{M}^5(c)$, whose second fundamental form is given by

$$h(X, Y) = -\varphi \delta(X, Y).$$

From theorem 5.1, we have the following existence result for three dimensional submanifolds with prescribed scalar curvature parantez or mean curvature.

Corollary 5.2 For a given constant θ with $0 < \theta < \pi/2$ and a given function $F_1 = F_1(x)$ (resp. $F_2 = F_2(x)$), there exist infinitely many three dimensional θ slant submanifolds in Kenmotsu space form $\bar{M}^5(c)$ with F_1 (resp. F_2) as the prescribed scalar curvature (resp., mean curvature) function for $c = -1$.

References

- [1] Blair, D.E.: Contact manifolds in Riemannian geometry, Lect. Notes in Math. Springer Verlag, Berlin- New York, 509, (1976).
- [2] Cabrerizo, J.L., Carriazo, A., Fernandez, L.M., and Fernandez, M.: Existence and uniqueness theorem for slant immersion in Sasakian manifolds, Publicaciones Mathematicae Debrecen 58, 559-574 (2001).
- [3] Cabrerizo, J.L., Carriazo, A., Fernandez, L.M., and Fernandez, M.: Slant submanifolds in Sasakian manifolds, Glasgow Math. J. 42, 125-138 (2000).
- [4] Cabrerizo, J.L., Carriazo, A., Fernandez, L.M. , and Fernandez, M.: Structure on a slant submanifold of a contact manifold, Indian J. Pure Appl. Math. 31 (7), 857-864 (2000).
- [5] Chen, B.Y.: Geometry of slant submanifolds, Katholieke Universiteit Leuven, (1990).
- [6] Chen, B.Y. and Vrancken, L.: Existence and uniqueness theorem for slant immersions and its applications, Result. Math. 31, 28-39 (1997).

- [7] Chen, B.Y. and Vrancken, L.: (Addendum to) Existence and uniqueness theorem for slant immersions and its applications, *Result. Math.* 39, 18-22 (2001).
- [8] Chen, B.Y. and Tazawa, Y.: Slant submanifolds in complex Euclidean spaces, *Tokyo J. Math.* 14, 101-120 (1991).
- [9] Eschenburg, J. H., and Tribuzy, R.: Existence and uniqueness of maps into affine homogeneous spaces, *Rend. Sem. Mat. Univ. Padova* 89, 11-18 (1993).
- [10] Gupta, R.S., Haider, S.M.K., and Sharfuddin, A.: Existence and uniqueness theorem for slant immersion in cosymplectic space forms, *Publ. Math. Debrecen* 67/1-2, 169-188 (2005).
- [11] Gupta, R.S. and Pandey, P.K.: Structure on slant submanifold of a Kenmotsu manifold, *Diff. Geometry- Dynamical Systems* 10, 139-147 (2008).
- [12] Kenmotsu, K.: A class of almost contact Riemannian manifolds, *Tohoku Math J.* 21, 93-103 (1972).
- [13] Kobayashi, M.: Semi invariant submanifolds of a certain class of almost contact manifolds, *Tensor (N.S.)*, 43, 28-36 (1986).
- [14] Lotta, A.: Slant submanifolds in contact geometry, *Bull. Math. Soc. Roumanie* 39, 183-198 (1996).
- [15] Lotta, A.: Three dimensional slant submanifolds of K-contact manifolds, *Balkan J. Geom. Appl.* 3 (1), 37-51 (1998).
- [16] Reckziegel, H.: On the problem whether the image of a given differentiable map into a Riemannian manifold is contained in a submanifold with parallel second fundamental form, *J. Reine. Angew. Math.* 325, 87-104 (1981).

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