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Abstract

In this paper we give definition of m-fold completeness and prove a theorem on completeness of elementary generalized solution of corresponding boundary value problems at which the equation describes the process of corrosive fracture of metals in aggressive media and the principal part of the equation has multiple characteristics.

Key Words: Hilbert space, existence of generalized solution, operator-differential equation.

1. Introduction

Many problems of mechanics and mathematical physics are connected in part to eigen and adjoint vectors of operator pencils. As an example, we can show the following papers.

Study of trace problems for solving some elliptic equations in a semi-cylinder precedes the completeness problems.

Necessary and sufficient conditions are formulated for boundary values providing the belongness of the solution to energetic space.

As is known, stress-strain state of a plate may be separated into internal and external layers [1,4]. Construction of a boundary layer is related with sequential solution of plane problems of elasticity theory in a semi-strip. In Papkovich's paper [5] and in others a boundary value problem of elasticity theory in a semi-strip $x > 0, |y| \leq 1$ is reduced to the definition of Airy biharmonic functions in the form

$$u = \sum_{\text{Im }\sigma_k > 0} C_k \varphi_k(y) e^{i\sigma_k x},$$

where φ_k are Papkovich functions [5,6], σ_k are corresponding values of a self-adjoint boundary value problem, and C_k are unknown coefficients. In this connection, in [6] there is a problem on representation of a pair of functions f_1 and f_2 in the form

$$\sum_{k=1}^{\infty} C_k P_k \varphi_k = f_1, \quad \sum_{k=1}^{\infty} C_k Q_k \varphi_k = f_2, \tag{1}$$

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where P_k, Q_k are differential operators defined by boundary conditions for $x = 0$. In papers [7, 8] some sufficient conditions for the convergence of expansion (1) is given for the cases when coefficients C_k are obviously defined with the help of generalized orthogonality.

I. I. Vorovich [9] indicated the relation of the given problem with the n- fold completeness of M. V. Keldysh [10, 11] and suggested a new approach based on immediate study of initial boundary value problem. In [12] the coefficients C_k are uniquely defined by the boundary values of a biharmonic function and its derivatives. Thus, completeness and base properties of elementary solutions are closely connected with differential properties of a biharmonic function in a corner points domain (trace problem).

In the later investigations of [13, 14], are statements on n- fold completeness in the space L_2 of a part of eigen and adjoint vectors of an operator pencil generated by some boundary value problem for an elliptic equation on a semi-strip. The trace problem for a two-dimensional domain with piecewise smooth boundary was studied in paper [15]. Paper [16] deals with differential properties of solutions of general elliptic equations in domains with canonical and corner points. Some new results for a biharmonic equation are in [17]. Investigations to behaviour of solution of problems of elasticity theory in the vicinity of singular points at the boundary in are papers [18-20]. M. B. Orazov [21], S. S. Mirzoyev [22], Dj. Allahverdiyev and E. E. Gasanov [23] studied the problem when a principal part of the equation is of the form $(-1)^m \frac{d^{2m}}{dt^{2m}} + A^{2m}$, where A is a self-adjoint operator pencil and it has multiple characteristics that differ from above-mentioned papers.

2. Problem statement

Let H be a separable Hilbert space, and A be a positive-definite self-adjoint operator in H with domain of definition $D(A)$. Denote by H_γ a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D(A^\gamma)$, $(\gamma \geq 0)$, $(x, y)_\gamma = = (A^\gamma x, A^\gamma y)\,, x, y \in D(A^\gamma)$. We denote by $L_2\left((a,b); H\right)$ $(-\infty \leq a < b \leq \infty)$ a Hilbert space of vector-functions $f(t)$ determined in (a, b) almost everywhere with values from H measurable, square integrable in the Bochner's sense

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f\|_\gamma^2 \, dt \right)^{1/2}.$$

Further, we define a Hilbert space for natural $m \geq 1$ [24]

$$W_2^m\left((a,b); H\right) = \left\{ u/u^{(m)} \in L_2\left((a,b); H\right), A^m u \in L_2\left((a,b); H_m\right) \right\}$$

with norm

$$\|u\|_{W_2^m((a,b);H)} = \left(\left\| u^{(m)} \right\|^2_{L_2((a,b);H)} + \|A^m u\|^2_{L_2((a,b);H)} \right)^{1/2}.$$

Here and in conat follows the derivatives are understood in the sense of distributions theory [24]. We assume

$$L_2\left((0,\infty); H\right) \equiv L_2(R_+; H),\ L_2\left((-\infty,\infty); H\right) \equiv L_2(R; H),$$

$$W_2^m\left((0,\infty); H\right) \equiv W_2^m\left(R_+; H\right),\ W_2^m\left((-\infty,+\infty); H\right) \equiv W_2^m\left(R; H\right).$$

Then we determine the spaces

$$W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)=\left\{u|u\in W_2^m\left(R_+;H\right),u^{(\nu)}(0)=0,\ \nu=\overline{0,m-1}\right\}.$$

Obviously, by the trace theorem in [24] the space $W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$ is a closed subspace of the Hilbert space $W_2^m\left(R_+;H\right)$.

Let's define a space of $D\left([a,b];H_\gamma\right)$-times infinitely differentiable functions for $a\le t\le b$ with values in H_γ having a compact support in $[a,b]$. As is known, a linear set $D\left([a,b];H_\gamma\right)$ is everywhere dense in the space $W_2^m\left((a,b);H\right)$ ([24]).

It follows from the trace theorem that the space

$$D\left(R_+;H_m;\{\nu\}_{\nu=0}^{m-1}\right)=\left\{u|u\in D\left(R_+;H_m\right),u^{(\nu)}(0)=0,\ \nu=\overline{0,m-1}\right\}$$

and is also everywhere dense in the space.

Let's consider a polynomial operator pencil

$$P(\lambda)=\left(-\lambda^2E+A^2\right)^m+\sum_{j=1}^{m}A_j\lambda^{m-j}. \tag{2}$$

Bind the polynomial pencil (2) with the boundary value problem

$$\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t)+\sum_{j=1}^{m}A_j u^{(m-j)}(t)=0,\quad t\in R_+=(0,+\infty), \tag{3}$$

$$u^{(\nu)}(0)=\varphi_\nu,\quad \nu=\overline{0,m-1},\ \varphi_\nu\in H_{m-\nu-1/2}. \tag{4}$$

Here we assume that the following conditions are fulfilled

1) A is a positive-definite self-adjoint operator with completely continuous inverse $C=A^{-1}\in\sigma_\infty$.

2) The operators

$$B_j=A^{-j/2}A_jA^{-j/2}\ \left(j=2k,\ k=\overline{1,m}\right)$$

and

$$B_j=A^{-(j-1)/2}A_jA^{-(j-1)/2}\ \left(j=2k-1,\ k=\overline{1,m-1}\right)$$

are linear in H.

3) Operators $(B+E_m)$ are bounded in H; E_m are unit operators.

Equation (3) describes a process of corrosion fracture in aggressive media that were studied in the paper [25].

3. Some definition and auxiliary facts

Denote

$$P_0\left(\frac{d}{dt}\right)u(t)\equiv\left(-\frac{d^2}{dt^2}+A^2\right)^m u(t),\quad u(t)\in D\left(R_+;H_m\right), \tag{5}$$

$$P_1\left(\frac{d}{dt}\right)u(t) \equiv \sum_{j=1}^{m-1} A_j u^{(m-j)}(t), \quad u(t) \in D\left(R_+; H_m\right). \tag{6}$$

Lemma 1. *Let A be a positive-definite self-adjoint operator, the operators $B_j = A^{-j/2} A_j A^{-j/2}$ $\left(j = 2k, k = \overline{1,m}\right)$ and $B_j = A^{-(j-1)/2} A_j A^{-(j-1)/2}$ $\left(j = 2k-1, k = \overline{1,m-1}\right)$ be bounded in H. Then a bilinear functional*

$$P_1(u,\psi) \equiv \left(P_1(d/dt)u, \psi\right)_{L_2(R_+;H)}$$

determined for all vector-functions $u \in D\left(R_+; H_m\right)$ and $\psi \in D\left(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}\right)$ continues on the space $W_2^m\left(R_+; H\right) \oplus W_2^m\left(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}\right)$ that acts in the following way:

$$P_1(u,\psi) = \sum_{(j=2k)} (-1)^{m-j/2}\left(A_j u^{(m-j/2)}, \psi^{(m-j/2)}\right)_{L_2} +$$

$$+ \sum_{j=(2k-1)} (-1)^{m-(j+1)/2}\left(A_j u^{(m-(j-1)/2)}, \psi^{(m-(j-1)/2)}\right)_{L_2}. \tag{7}$$

In the first term, the summation is taken over even j, in the second term over odd j.

Proof. Let $u \in D(R_+; H_m), \psi \in D\ \left(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}\right)$. Then integrating by parts we get

$$P_1(u,\psi)_{L_2} \equiv \left(P_1(d/dt)u, \psi\right)_{L_2} = \sum_{j=0}^{m}\left(A_j u^{(m-j)}, \psi\right)_{L_2} = \sum_{(j=2k)} (-1)^{m-j/2}\left(A_j u^{(m-j/2)}, \psi^{(m-j/2)}\right)_{L_2} +$$

$$+ \sum_{j=(2k-1)} (-1)^{m-(j+1)/2}\left(A_j u^{(m-(j-1)/2)}, \psi^{(m-(j-1)/2)}\right)_{L_2}.$$

Since

$$P_1(u,\varphi) = \sum_{(j=2k)} (-1)^{m-j/2}\left(B_j A^{j/2} u^{(m-j/2)}, A^{j/2}\psi^{(m-j/2)}\right)_{L_2} +$$

$$+ \sum_{j=(2k-1)} (-1)^{m-(j+1)/2}\left(B_j A_j u^{(m-(j-1)/2)}, A^{(j-1)/2}\psi^{(m-(j-1)/2)}\right)_{L_2},$$

from belongless of $u \in D\left(R_+; H_m\right)$ and $\psi \in D\left(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}\right)$ by intermediate derivatives theorem [24], it follows that

$$|P_1(u,\varphi)| \le \sum_{(j=2k)} \|B_j\| \left\|A^{j/2} u^{(m-j/2)}\right\|_{L_2} \left\|A^{j/2}\psi^{(m-j/2)}\right\|_{L_2} +$$

$$+ \sum_{j=(2k-1)} \|B_j\| \left\|A^{(j+1)/2} u^{(m-(j-1)/2)}\right\|_{L_2} \left\|A^{(j-1)/2}\psi^{(m-(j-1)/2)}\right\|_{L_2} \le const\, \|u\|_{W_2^m(R_+;H)} \|\psi\|_{W_2^m(R_+;H)},$$

i.e. $P_1(u,\varphi)$ is continuous in the space $D(R_+; H_m) \oplus D\left(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}\right)$ therefore it continues by continuity on the space $W_2^m(R_+; H) \oplus W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$. The lemma is proved. □

Definition 1. *The vector-function $u(t) \in W_2^m(R_+;H)$ is said to be a generalized solution of (3), (4), if*

$$\lim_{t\to 0}\left\|u^{(\nu)}(t)-\varphi_\nu\right\|_{H_{m-\nu-1/2}}=0,\quad \nu=\overline{0,m-1},$$

and for any $\psi(t) \in W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$ it is fulfilled the identity

$$\langle u,\psi\rangle=(u,\psi)_{W_2^m(R_+;H)}+\sum_{p=1}^{m-1}C_m^p\left(A^p u^{(m-p)},A^p\psi^{(m-p)}\right)_{L_2(R_+;H)}+P_1(u,\psi)=0,$$

where

$$C_m^p=\frac{m(m-1)...(m-p+1)}{p!}=\left(\begin{array}{c}m\\p\end{array}\right).$$

Definition 2. *If a non-zero vector $\varphi_0\neq 0$ is a solution of the equation $P(\lambda_0)\varphi_0=0$, then λ_0 is said to be an eigen-value of the pencil $P(\lambda)$ and φ_0 is an eigenvector corresponding to the number λ_0.*

Definition 3. *The system $\{\varphi_1,\varphi_2,...,\varphi_m\}\in H_m$ is said to be a chain of eigen and adjoint vectors φ_0 if it satisfies the equations*

$$\sum_{i=0}^{q}\frac{1}{i}\frac{d^i}{d\lambda^i}P(\lambda)|_{\lambda=\lambda_0}\cdot\varphi_{q-i}=0,\quad q=\overline{1,m}.$$

Definition 4. *Let $\{\varphi_0,\varphi_1,...,\varphi_m\}$ be a chain of eigen and adjoint vectors corresponding to eigenvalues λ_0, then vector-functions*

$$\varphi_h(t)=e^{\lambda_0 t}\left(\frac{t^h}{h!}\varphi_0+\frac{t^{h-1}}{(h-1)!}\varphi_1+...+\varphi_n\right),\quad h=\overline{0,m}$$

satisfy equation (3) and are said to be its elementary solutions corresponding to the eigen-value λ_0.

Obviously, elementary solutions $\varphi_h(t)$ have traces in the zero

$$\varphi_h^{(\nu)}=\frac{d^\nu}{dt^\nu}\varphi_h|_{t=0},\quad \nu=\overline{0,m-1}.$$

By means of $\varphi_h^{(\nu)}$ we define the vectors

$$\left\{\widetilde{\varphi}_h=\left(\varphi_h^{(0)},\varphi_h^{(1)}\right),\ h=\overline{0,m}\right\}\subset H^m=\underbrace{H\times...\times H}_{m\ \text{times}}.$$

Later by $K(\Pi_-)$ we denote all possible vectors $\widetilde{\varphi}_h$ corresponding to all eigen-values from the left half-plane $(\Pi_-=\{\lambda/\mathrm{Re}\,\lambda<0\})$.

Definition 5. *The system $K(\Pi_-)$ is said to be m-fold complete in the trace space, if the system $K(\Pi_-)$ is complete in the space $\overset{m}{\underset{i=0}{\oplus}} H_{m-i-1/2}$.*

It holds

Lemma 2. *Let conditions 1)-2) be fulfilled and*

$$\alpha = \sum_{j=1}^{m} C_j \left\| B_{m-j} \right\| < 1, \tag{8}$$

where

$$C_j = \begin{cases} d_{m,j/2}^{m/2}, & j = 2k, \ k = \overline{0, m} \\ \left(d_{m,(j-1)/2} d_{m,(j+1)/2}\right)^{m/2}, & j = 2k-1, k = \overline{1, m-1} \end{cases}$$

and

$$d_{m,j} = \begin{cases} \left(\frac{j}{m}\right)^{\frac{j}{m}} \left(\frac{m-j}{m}\right)^{\frac{m-j}{m}}, & j = \overline{1, m-1} \\ 1, & j = 0, m. \end{cases}$$

Then, for any $\psi \in W_2^m \left(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}\right)$, holds the inequality

$$Re\ P(\psi, \psi) \geq (1-\alpha) P_0(\psi, \psi),$$

where

$$P_0(\psi, \psi) = \left(\left(-\frac{d}{dt} + A\right)^m \psi, \left(-\frac{d}{dt} + A\right)^m \psi\right)_{L_2}$$

and

$$P(u, \psi) = P_0(u, \psi) + P_1(u, \psi).$$

$P_1(u, \psi)$ is determined from lemma 1.

Proof. Let $\psi \in D\left(R_+; H_m; \{\nu\}_{\nu=0}^{m-1}\right)$. Then for any ψ

$$\text{Re } P(\psi, \psi) = \text{ Re } P_0(\psi, \psi) + \text{ Re } P_1(\psi, \psi) = \left(\left(-\frac{d}{dt} + A\right)^m \psi, \left(-\frac{d}{dt} + A\right)^m \psi\right)_{L_2} +$$

$$+ \text{ Re } P_1(\psi, \psi) \geq \left\| \left(-\frac{d}{dt} + A\right)^m \psi \right\|_{L_2}^2 - \left| P_1(\psi, \psi) \right|.$$

Since

$$\left\| A^k \psi^{(m-k)} \right\|_{L_2} \leq d_{m,m-k}^{m/2} \left\| u \right\|_{W_2^m}$$

then

$$\left| P_1(\psi, \psi) \right| \leq$$

$$\leq \left(\sum_{(j=2k)} \left\| B_j \right\| d_{m,m-j/2}^{m/2} + \sum_{(j=2k-1)} \left\| B_j \right\| d_{m,m-(j-1)/2}^{m/2} d_{m,m-(j+1)/2}^{m/2} \right) \left\| \psi \right\|_{W_2^m}.$$

Here, $d_{0,0} = d_{m,m} = 1$, thus

$$|P_1(\psi,\psi)| \le \sum_{j=0}^{m} \|B_{m-j}\| \, C_j,$$

where

$$C_j = \begin{cases} d_{m,j/2}^{m/2}, \ j = 2k, \ \ k = \overline{0,m} \\ \left(d_{m,(j+1)/2} d_{m,(j-1)/2}\right)^{m/2}, \ j = 2k-1, \ \ k = \overline{1,m-1}. \end{cases}$$

Thus

$$|P_1(\psi,\psi)_{L_2}| \le \alpha \, \|\psi\|^2_{W_2^m(R_+;H)}.$$

Thus

$$P(\psi,\psi)_{L_2(R_+;H)} \ge (1-\alpha) P_0(\psi,\psi)_{L_2(R_+;H)}.$$

The lemma is proved. □

Remark. From the proof we can show that for $m = 2$, $c_1 = c_3 = 1/2$, $c_2 = 1/4$, $c_4 = 1$.

Lemma 3. *Let the conditions of lemma 2 be fulfilled. Then, for any $x \in H_m$ and $\xi \in R$, holds the inequality*

$$(P(i\xi)X, X)_H > (1-\alpha)\left(P_0(i\xi)X, X\right)_H.$$

Proof. It follows from the conditions that, for all $\psi(t) \in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$, holds the inequality

$$\left(P(\psi,\psi)\right)_{L_2(R_+;H)} \ge (1-\alpha) P_0(\psi,\psi)_{L_2(R_+;H)}. \tag{9}$$

Let $\psi(t) = g(t) \cdot X, X \in H_m$ and a scalar function $g(t) \in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$. Then from (7)we get

$$\left(P(i\xi) g(t) \cdot X, g(t) \cdot X\right)_{L_2(R_+;H)} \ge (1-\alpha)\left(P_0(i\xi) \cdot X, X\right) \|g(t)\|^2_{L_2(R_+;H)},$$

then

$$\left(P(i\xi)X, X\right) \|g(t)\|^2_{L_2(R_+;H)} \ge (1-\alpha)\left(P_0(i\xi)X, X\right) \|g(t)\|^2_{L_2(R_+;H)},$$

i.e.

$$\left(P(i\xi)X, X\right) \ge (1-\alpha)\left(P_0(i\xi)X, X\right)$$

The lemma is proved. □

4. On the existence of generalized solution of boundary value problems

First of all we consider the problem

$$P_0\left(\frac{d}{dt}\right) u(t) = \left(-\frac{d^2}{dt^2} + A^2\right)^m u(t) = 0, \ \ t \in R_+ = (0,+\infty), \tag{10}$$

$$u^{(\nu)}(0) = \varphi_\nu, \quad \nu = \overline{0, m-1}. \tag{11}$$

The folowing theorem holds.

Theorem 1. *For any collection* $\varphi_\nu \in H_{m-\nu-1/2}$ $\left(\nu = \overline{0, m-1}\right)$*, problems (10), (11) has a unique generalized solution.*

Proof. Let $c_0, c_1, ..., c_{m-1} \in H_{m-\nu-1/2}$ $\left(\nu = \overline{0, m-1}\right)$, e^{-At} be a holomorphic semi-group of bounded operators generated by the operator $(-A)$. Then the vector-function

$$u_0(t) = e^{-tA}\left(c_0 + \frac{t}{1!}Ac_1 + ... + \frac{t^{m-1}}{(m-1)!}A^{m-1}c_{m-1}\right)$$

belongs to the space $W_2^m(R_+; H)$. Really, using spectral expansion of the operator A we see that each term

$$\frac{t^{m-\nu}}{(m-\nu)!}A^{m-\nu}e^{-tA} \in W_2^m(R_+; H) \quad \text{for } c_\nu \in H_{m-1/2} \quad \left(\nu = \overline{0, m-1}\right).$$

Then it is easily verified that $u_0(t)$ is a generalized solution of equation (10), i.e. it satisfies the relation

$$(u_0, \varphi)_{W_2^m} + \sum_{p=1}^{m-1} C_m^p \left(A^{m-p}u_0^{(p)}, A^{m-p}\varphi^{(p)}\right) = 0$$

for any $\varphi \in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$.

Show that $u^{(\nu)}(0) = \varphi_\nu$, $\nu = \overline{0, m-1}$.

For this purpose we must determine the vectors c_ν $\left(\nu = \overline{0, m-1}\right)$ from condition (11). Obviously, in order to determine the vectors c_ν $\left(\nu = \overline{0, m-1}\right)$ from condition (11) we get a system of equations with respect to the vectors

$$\begin{pmatrix} E & 0 & \cdots & 0 \\ -E & E & \cdots & 0 \\ E & -E & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{m-1}\begin{pmatrix} 1 \\ m-1 \end{pmatrix}E & (-1)^{m-2}\begin{pmatrix} 2 \\ m-2 \end{pmatrix}E & \cdots & E \end{pmatrix} \times$$

$$\times \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{m-1} \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ A^{-1}\varphi_1 \\ A^{-2}\varphi_2 \\ \vdots \\ A^{-(m-1)}\varphi_{m-1} \end{pmatrix}, \tag{12}$$

where E is a unique operator in H and $\begin{pmatrix} p \\ m-s \end{pmatrix} = C_{m-s}^p$. Since the principal operator determinant is invertible, we can uniquely determine c_ν $\left(\nu = \overline{0, m-1}\right)$. Obviously, for any ν the vector $A^{-(m-\nu)}\varphi_\nu \in$

$H_{m-1/2}$, since $\varphi_\nu \in H_{m-\nu-1/2}$. As the vector at the right hand side of the equation (12) belongs to the space

$$\underbrace{H_{m-1/2} \oplus ... \oplus H_{m-1/2}}_{m \text{ times}} = \left(H_{m-1/2}\right)^m,$$

then taking into account the fact that the principal operator matrix $\widetilde{E}$ as a product of the invertible scalar matrix by matrix where $\widetilde{E}$ is a unique matrix in $\left(H_{m-1/2}\right)^m$, then it is unique. Therefore, each vector c_ν $\left(\nu = \overline{0, m-1}\right)$ is a linear combination of elements $A^{-(m-\nu)}\varphi_\nu \in H_{m-1/2}$, that is why the vector c_ν $\left(\nu = \overline{0, m-1}\right)$ is determined uniquely and belongs to the space $H_{m-1/2}$. The theorem is proved. □

In the space $W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$ we define a new norm

$$|||u|||_{W_2^m(R_+;H)} = \left(\|u\|^2_{W_2^m(R_+;H)} + \sum_{p=1}^{m-1} C_m^p \left\|A^{m-p}u^{(p)}\right\|^2_{L_2(R_+;H)}\right)^{1/2}.$$

By the intermediate derivatives theorem [24] the norms $|||u|||_{W_2^m(R_+;H)}$ and $\|u\|_{W_2^m(R_+;H)}$ are equivalent in the space $W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$. Therefore, the numbers

$$N_j\left(R_+; \{\nu\}_{\nu=0}^{m-1}\right) = \sup_{0 \neq u \in W_2^m\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)} \left\|A^{m-j}u^{(j)}\right\|_{L_2(R_+;H)} |||u|||^{-1}_{W_2^m(R_+;H)}, \quad j = \overline{0, m}.$$

are finite.

The next lemma enables one to find exact values of these numbers.

Lemma 4. *The numbers* $N_j\left(R_+; \{\nu\}_{\nu=0}^{m-1}\right)$ *are determined as follows:*

$$N_j\left(R_+; \{\nu\}_{\nu=0}^{m-1}\right) = d_{m,j}^{m/2},$$

where

$$d_{m,j} = \begin{cases} \left(\frac{j}{m}\right)^{\frac{j}{m}} \left(\frac{m-j}{m}\right)^{\frac{m-j}{m}}, & j = \overline{1, m-1} \\ 1, & j = 0, m. \end{cases}$$

Using the method of papers [22, 26] the lemma is easily proved.

Now, let's prove a theorem on the existence of generalized solutions of problem (4), (5).

Theorem 2. *Let* A *be a positive-definite self-adjoint operator, the operators* $B_j = A^{-j/2}A_jA^{-j/2}$ $\left(j = 2k, k = \overline{0, m}\right)$ *and* $B_j = A^{-(j-1)/2}A_jA^{-(j-1)/2}$ $\left(j = 2k-1, k = \overline{1, m-1}\right)$ *be bounded in* H *and let hold the inequality*

$$\alpha = \sum_{j=1}^{m} C_j \|B_{m-j}\| < 1,$$

where C_j *are determined from Lemma 2. Then for any* $\varphi_\nu \in D\left(A^{m-n-1/2}\right)$, $\left(\nu=\overline{0,m-1}\right)$ *problem (4), (5) has a unique generalized solution, and thus holds the inequality*

$$\|u\|_{W_2^m(R_+;H)} \leq const \sum_{\nu=0}^{m-1} \|\varphi\|_{m-\nu-1/2}.$$

Proof. Let $\psi \in D\left(R_+;H;\{\nu\}_{\nu=0}^{m-1}\right)$. Then for any ψ

$$\text{Re } P(\psi,\psi) = \text{ Re } P_0(\psi,\psi) + \text{ Re } P_1(\psi,\psi) = \left(\left(-\frac{d}{dt}+A\right)^m \psi, \left(-\frac{d}{dt}+A\right)^m \psi\right) +$$

$$+ \text{ Re } P_1(\psi,\psi) \geq \left\|\left(-\frac{d}{dt}+A\right)^m \psi\right\|^2 - |\text{ Re } P_1(\psi,\psi)| \geq \left\|\left(-\frac{d}{dt}+A\right)^m \psi\right\|^2_{L_2(R_+;H)} - |P_1(\psi,\psi)|_{L_2(R_+;H)}.$$

Since by lemma 2

$$\left\|A^k \psi^{(m-k)}\right\|_{L_2(R_+;H)} \leq d_{m,m-k}^{m/2} \|u\|_{W_2^m(R_+;H)},$$

then

$$|P_1(\psi,\psi)| \leq$$

$$\leq \left(\sum_{(j=2k)} \|B_{m-j}\| \, d_{m,m-k}^{m/2} + \sum_{(j=2k-1)} \|B_{m-j}\| \, d_{m,m-k+1}^{m/2} d_{m,m-k-1}^{m/2}\right) \|\psi\|^2_{W_2^m}.$$

Here $d_{0,0} = d_{m,m} = 1$ and

$$d_{m,k} = \left(\frac{k}{m}\right)^{\frac{k}{m}} \left(\frac{m-k}{m}\right)^{\frac{m-k}{m}}, \quad \left(k=\overline{1,m-1}\right),$$

thus

$$|P_1(\psi,\psi)| \leq \sum_{j=1}^{m} C_j \|B_{m-j}\|,$$

where

$$C_j = \begin{cases} d_{m,j/2}^{m/2}, \quad j=2k, k=\overline{0,m} \\ \left(d_{m,(j+1)/2}^{m/2} d_{m,(j-1)/2}^{m/2}\right)^{m/2}, \ j=2k-1, k=\overline{1,m-1}. \end{cases}$$

Consequently

$$|P_1(\psi,\psi)| \leq \alpha \|\psi\|^2_{W_2^m(R_+;H)}.$$

Then

$$\text{Re } P(\psi,\psi)_{L_2(R_+;H)} \geq (1-\alpha) P_0(\psi,\psi)_{L_2(R_+;H)}. \tag{13}$$

Now we look for a generalized solution of problem (4), (5) in the form

$$u(t) = u_0(t) + \theta(t),$$

where $u_0(t)$ is a generalized solution of problem (10), (11) and $\theta(t) \in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$. To define $\theta(t)$ we get relation

$$\langle\theta; \psi\rangle = (\theta, \psi)_{W_2^m(R_+;H)} + \sum_{p=1}^{m-1} C_m^p \left(A^{m-p}\theta, A^{m-p}\psi\right) + P_1(\theta, \psi) = P_1(u_0, \psi). \tag{14}$$

Since the right hand side of the equality is a continuous functional in $W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$, and the left hand side $\langle\theta; \psi\rangle$ is a bilinear functional in the space $W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right) \oplus W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$, then by inequality (13) it satisfies conditions of Lax-Milgram theorem [25]. Consequently, there exists a unique vector-function $\theta(t) \in W_2^m\left(R_+; H; \{\nu\}_{\nu=0}^{m-1}\right)$ that satisfies equality (14) and $u(t) = u_0(t) + \theta(t)$ is a generalized solution of problem (4), (5).

Further, by $J(R_+; H)$ we denote a set of generalized solutions of problem (4), (5) and define the operator $\Gamma : J(R_+; H) \to \widetilde{H} = \overset{m-1}{\underset{k=0}{\oplus}} H_{m-k-1/2}$ acting in the following way $\Gamma u = \left(u^{(k)}(0)\right)_{k=0}^{m-1}$. Obviously $J(R_+; H)$ is a closed set and by the trace theorem $\|\Gamma u\|_{\widetilde{H}} \leq C \|u\|_{W_2^m(R_+;H)}$. Then by the Banach theorem on the inverse operator there exists the inverse operator $\Gamma^{-1} : \widetilde{H} \to J(R_+; H)$. Consequently

$$\||u|\|_{W_2^m(R_+;H)} \leq const \sum_{k=0}^{m-1} \|\varphi\|_{m-k-1/2}.$$

The theorem is proved. □

Lemma 5. *Let conditions 1)-3) and solvability conditions be fulfilled, then estimation* $\left\|A^m p^{-1}(i\xi) A^m\right\| \leq const$ *is true.*

The proof of this lemma is easily obtained from Keldysh lemma [12] and lemma 2.

5. The basic result

Now, let's prove the principal theorems. The following theorem holds.

Theorem 3. *Let conditions 1)-2) be fulfilled, solvability conditions and one of the following conditions hold:*

a) $A^{-1} \in \sigma_p \ (0 \leq p < 1)$;

b) $A^{-1} \in \sigma_p \ (0 \leq p < \infty)$ *and* $B_j \in \sigma_\infty$.

Then the system of eigen and adjoint vectors from $K(\Pi_-)$ *is complete in the trace space.*

Proof. Denote

$$L(\lambda) = A^{-m} p(\lambda) A^m,$$

where

$$L(\lambda) = \left(-\lambda^2 C^2 + E\right)^m + \sum_{j=1}^{m} \lambda^{m-j} T^j,$$

and

$$T_j = \begin{cases} C^{m-1/2} B_j C^{m-1/2} & \text{for } \ j = 2k, k = \overline{1, m} \\ C^{m-(j-1)/2} B_j C^{m-(j-1)/2} & \text{for } \ j = 2k-1, k = \overline{1, m-1}. \end{cases}$$

Obviously $T_j \in \sigma_{p/m-j}$. Then $L^{-1}(\lambda)$ is represented in the form of relation of two entire functions of order p and minimal order p. Then

$$A^{m-1/2} p^{-1}(\lambda) A^{m-1/2} = A^{-1/2} \left(A^m p^{-1}(\lambda) A^m \right) A^{-1/2}$$

is also represented in the relation of two entire functions of order p and of minimal type for order p. The proof of m-fold completeness of the system $K(\Pi_-)$ is equivalent to the proof of the fact that for any $\varphi_0, \varphi_1, ..., \varphi_{m-1}$ from holomorphic property of the vector-function

$$F(\lambda) = \left(L^*(\overline{\lambda}) \right)^{-1} \left(f(\lambda), f(\overline{\lambda}) \right),$$

where

$$f(\lambda) = \sum_{j=0}^{m-1} \lambda^j C^{j+1/2} \varphi_j.$$

For $\Pi_- = \{\lambda / \ \mathrm{Re}\ \lambda < 0\}$ it follows that $\varphi_j = 0$.

The theorem is proved. □

Now we use theorem 2 and theorem 3 and prove the completeness of elementary solutions of problem (4), (5).

Theorem 4. *Let the conditions of theorem 2 be fulfilled. Then elementary solutions of problem (4), (5) is complete in the space of generalized solutions.*

Proof. It is easy to see that if there exists a generalized solution, then

$$\|u\|_{W_2^m(R_+;H)} \leq const \sum_{j=0}^{m-1} \|\varphi_j\|_{m-j-1/2}.$$

Then it follows from the trace theorem [24] and these inequalities that

$$C_k \sum_{\nu=0}^{m-1} \|\varphi_\nu\|_{m-\nu-1/2} \leq \|u\|_{W_2^m(R_+;H)} = C_k \sum_{\nu=0}^{m-1} \|\varphi_\nu\|_{m-\nu-1/2}. \tag{15}$$

Further, from the theorem on the completeness of the system $K(\Pi_-)$ it follows that for any collection $\{\varphi_\nu\}_{\nu=0}^{m-1}$ and $\varphi_\nu \in H_{m-\nu-1/2}$ there is such a number N and $C_k(\varepsilon, N)$ that

$$\left\| \varphi_\nu - \sum_{k=1}^{N} C_k \varphi_{i,j,h}^{(\nu)} \right\| < \varepsilon / m, \quad \nu = \overline{0, m-1}.$$

Then it follows from (15) that

$$\left\| u(t) - \sum_{k=1}^{N} C_k \varphi_{i,j,h}^{(\nu)} \right\| \leq \varepsilon.$$

The theorem is proved. □

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