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## Some properties of the first eigenvalue of the $p(x)$ -laplacian on riemannian manifolds

*R. A. Mashiyev, G. Alisoy and S. Ogras*

### Abstract

The main result of the present paper establishes a stability property of the first eigenvalue of the associated problem which deals with the  $p(x)$ -Laplacian on Riemannian manifolds with Dirichlet boundary condition.

**Key Words:** Variable exponent Lebesgue and Sobolev spaces; first eigenvalue; Riemannian manifolds;  $p(x)$ -Laplacian.

### 1. Introduction

Over the last decades the variable exponent Lebesgue spaces  $L^{p(x)}$  and the corresponding Sobolev space  $W^{1,p(x)}$  have been a subject of active research stimulated by development of the studies of problems in elasticity, fluid dynamics, calculus of variations, and differential equations with  $p(x)$ -growth (see [2], [3], [12]). We refer the reader to [5], [7], [8] for fundamental properties of these spaces.

The  $p(x)$ -Laplacian equations related to eigenvalue problems have been studied in [6], [9], [10], [11].

Let  $G \subset \mathbb{R}^N$  ( $N \geq 2$ ) is a bounded domain with a smooth boundary. For measurable function  $p(x)$  we denote the variable exponent Lebesgue space by

$$L^{p(x)}(G) = \left\{ u \text{ measurable real functions} : \int_G |u(x)|^{p(x)} dx < \infty \right\},$$

which is equipped with the norm, the so-called Luxemburg norm (see [5], [7])

$$|u|_{p(x)} := |u|_{L^{p(x)}(G)} = \inf \left\{ \delta > 0 : \int_G \left| \frac{u(x)}{\delta} \right|^{p(x)} dx \leq 1 \right\},$$

where

$$1 < \text{ess inf}_{x \in G} p(x) := p^- \leq p(x) \leq \text{ess sup}_{x \in G} p(x) := p^+ < \infty,$$

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where  $(L^{p(x)}(G), |\cdot|_{L^{p(x)}})$  becomes a Banach space, which is known as a variable exponent Lebesgue space.

Define the variable exponent Sobolev space  $W^{1,p(x)}(G)$  by

$$W^{1,p(x)}(G) = \{u \in L^{p(x)}(G); |\nabla u| \in L^{p(x)}(G)\},$$

and equip with the norm

$$\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}(G)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(G).$$

The space  $W_0^{1,p(x)}(G)$  is denoted by the closure of  $C_0^\infty(G)$  in  $W^{1,p(x)}(G)$  which is equipped with the norm for  $u \in W_0^{1,p(x)}(G)$

$$\|u\|_{1,p(x)} = |\nabla u|_{p(x)}, \forall u \in W_0^{1,p(x)}(G).$$

If  $p^- > 1$ , then the spaces  $L^{p(x)}(G)$ ,  $W^{1,p(x)}(G)$  and  $W_0^{1,p(x)}(G)$  are separable and reflexive Banach spaces (see [5], [7]).

**Proposition 1.** ([5], [7]). *Denote*

$$\varrho_{p(x)}(u) = \int_G |u(x)|^{p(x)} dx,$$

and

$$\varrho_{1,p(x)}(u) := \varrho_{p(x)}(\nabla u) = \int_G |\nabla u(x)|^{p(x)} dx, \forall u, \nabla u \in L^{p(x)}(G),$$

then we have

$$\begin{aligned} \min \left\{ |u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+} \right\} &\leq \varrho_{p(x)}(u) \leq \max \left\{ |u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+} \right\}, \\ \min \left\{ |\nabla u|_{p(x)}^{p^-}, |\nabla u|_{p(x)}^{p^+} \right\} &\leq \varrho_{1,p(x)}(u) \leq \max \left\{ |\nabla u|_{p(x)}^{p^-}, |\nabla u|_{p(x)}^{p^+} \right\}. \end{aligned}$$

Let  $M$  be a compact Riemannian manifold with  $\dim M = m$ , and  $\Delta_{p(x)}$  is nonhomogenous  $p(x)$ -Laplacian acting on functions on  $M$ , where  $\Delta_{p(x)}u = \operatorname{div} \left( |\nabla u|^{p(x)-2} \nabla u \right)$ , and  $1 < p(x) < \infty$ . Let  $M^*$  be a compact submanifold of  $\Omega$ , and  $B_\varepsilon$  the tubular neighborhood of  $M^*$  of radius  $\varepsilon > 0$ ; that is,

$$B_\varepsilon = \{x \in M : d(x, M^*) < \varepsilon\},$$

where  $d(\cdot, \cdot)$  is the distance function on  $M$  induced by the Riemannian metric. Denote by  $\Delta_{p(x),\varepsilon}$  the restriction of  $\Delta_{p(x)}$  to those functions on  $M$  vanishing identically in  $B_\varepsilon$ . Set

$$\Omega_\varepsilon = M \setminus B_\varepsilon \text{ and } \partial\Omega_\varepsilon = \partial B_\varepsilon.$$

We consider the following Dirichlet problem

$$\begin{cases} \Delta_{p(x),\varepsilon} u + \lambda(\varepsilon) |u|^{p(x)-2} u = 0 & x \in B_\varepsilon \\ u = 0 & x \in \partial B_\varepsilon. \end{cases}$$

The first eigenvalue  $\lambda_{1,p(x)}(\Omega_\varepsilon, \phi)$  of the  $p(x)$ -Laplacian is defined as the least number  $\lambda$  for which the Dirichlet problem has a nontrivial solution  $u \in W_0^{1,p(x)}(\Omega_\varepsilon)$ . It can be characterized by

$$\lambda_{1,p(x)}(\Omega_\varepsilon, \phi) = \inf_{u \in W_0^{1,p(x)}(\Omega_\varepsilon) \setminus \{0\}} \frac{\int_{\Omega_\varepsilon} |\nabla u|^{p(x)} dv_\phi}{\int_{\Omega_\varepsilon} |u|^{p(x)} dv_\phi},$$

where  $u$  runs over  $W_0^{1,p(x)}(\Omega_\varepsilon)$  and  $dv_\phi$  denotes the volume element of  $M$ . It turns out  $\lambda_{1,p(x)}(\Omega_\varepsilon, \phi) > 0$ . When  $B_\varepsilon = \emptyset$ , that is,  $\Omega_\varepsilon = M$ ;  $u$  runs over  $W^{1,p(x)}(\Omega_\varepsilon)$  and  $\int_M |u|^{p(x)-2} u dv_\phi = 0$ . We can easily see that  $\lambda_{1,p(x)}(M) = 0$ . The corresponding eigenfunctions are constant functions. In the case of  $p = 2$ , many people have studied the asymptotic expansion of the eigenvalues  $\lambda_{k,2}(\varepsilon)$  ( $k = 1, 2, \dots$ ) for the 2-Laplacian of a manifold  $M \setminus B_\varepsilon$  with the Dirichlet condition on the tubular neighborhood  $B_\varepsilon$ . Chavel and Feldman in [4] showed that  $\lambda_{k,2}(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  tends to zero under the condition  $\text{codim}(M^*) \geq 2$ . Eigenvalue problems for quasilinear operators of  $p$ -Laplace type have received considerable attention in the last years (see [1]).

In the present paper, we establish conditions ensuring a stability property of the first eigenvalue of the associated problem which deals with the  $p(x)$ -Laplacian. As far as we are concerned, this is the first paper that discusses this subject. We prove that the first eigenvalues of the  $p(x)$ -Laplacian on Riemannian manifolds  $\Omega_\varepsilon$  converges to zero as  $\varepsilon$  tends to zero. In precisely we show  $\lambda_{1,p(x)}(\Omega_\varepsilon, \phi) \rightarrow \lambda_{1,p(x)}(M, \phi) = 0$  as  $\varepsilon \rightarrow 0$ .

Moreover, the investigation of the Riemannian manifolds in the variable exponent Lebesgue space  $L^{p(x)}$  is firstly dealt with in our study.

## 2. Main results and proofs

**Lemma 1.** *Suppose that the codimension  $m - k \geq p^+$ . Given any  $f \in W^{1,p(x)}(M)$ , there exists a function  $f_\varepsilon \in W_0^{1,p(x)}(\Omega_\varepsilon)$  such that*

$$\lim_{\varepsilon \rightarrow 0} \|f_\varepsilon - f\|_{1,p(x)} = \lim_{\varepsilon \rightarrow 0} |f_\varepsilon - f|_{p(x)} + \lim_{\varepsilon \rightarrow 0} |\nabla(f_\varepsilon - f)|_{p(x)} = 0$$

for sufficiently small  $\varepsilon > 0$ .

**Proof.** By Proposition 1, we can write

$$\varrho_{1,p(x)}(f_\varepsilon - f) = \varrho_{p(x)}(f_\varepsilon - f) + \varrho_{p(x)} \nabla(f_\varepsilon - f).$$

Define a function  $\omega_\varepsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  as

$$\omega_\varepsilon(r) = \begin{cases} 1, & \varepsilon \leq r; \\ \left(\frac{r}{\varepsilon}\right)^\varepsilon, & \exp(-\varepsilon^{-2}) \leq r \leq \varepsilon; \\ \left(\frac{\exp(-\varepsilon^{-2})}{\varepsilon}\right)^\varepsilon \left(\frac{2r}{\exp(-\varepsilon^{-2})}\right), & 2^{-1} \exp(-\varepsilon^{-2}) \leq r \leq \exp(-\varepsilon^{-2}) \\ 0, & r \leq \exp(-\varepsilon^{-2}). \end{cases}$$

Then we have

$$\left|\omega'_\varepsilon(r)\right|^{p(r)} = \begin{cases} 0, & \varepsilon \leq r \\ \left(\frac{r}{\varepsilon}\right)^{\varepsilon p(r)-p(r)}, & \exp(-\varepsilon^{-2}) \leq r \leq \varepsilon \\ \left(\frac{\exp(-\varepsilon^{-2})}{\varepsilon}\right)^{\varepsilon p(r)} \left(\frac{2}{\exp(-\varepsilon^{-2})}\right)^{p(r)}, & 2^{-1} \exp(-\varepsilon^{-2}) \leq r \leq \exp(-\varepsilon^{-2}) \\ 0, & r \leq \exp(-\varepsilon^{-2}). \end{cases}$$

Define  $f_\varepsilon = \omega_\varepsilon f \in W_0^{1,p(x)}(\Omega_\varepsilon)$  and  $r(x) = d(M^*, x)$ . Then we have

$$\varrho_{p(x)}(f_\varepsilon - f) = \int_M |(f_\varepsilon - f)|^{p(x)} dv_\phi,$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_\varepsilon^r |f_\varepsilon - f|^{p(x)} dv_\phi = \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^r |(\omega_\varepsilon - 1)f|^{p(x)} dv_\phi \rightarrow 0,$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\exp(-\varepsilon^{-2})}^\varepsilon \left| \left( \left( \frac{r}{\varepsilon} \right)^\varepsilon - 1 \right) \cdot f \right|^{p(x)} dv_\phi \rightarrow 0, \quad \left( \lim_{\varepsilon \rightarrow 0} \varepsilon^\varepsilon \rightarrow 1 \right),$$

$$\lim_{\varepsilon \rightarrow 0} \int_{2^{-1} \exp(-\varepsilon^{-2})}^{\exp(-\varepsilon^{-2})} \left| \left( \left( \frac{\exp(-\varepsilon^{-2})}{\varepsilon} \right)^\varepsilon \left( \frac{2r}{\exp(-\varepsilon^{-2})} \right) - 1 \right) \cdot f \right|^{p(x)} dv_\phi \rightarrow 0,$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_r^{\exp(-\varepsilon^{-2})} |f_\varepsilon - f|^{p(\cdot)} dv_\phi \rightarrow 0;$$

so we have

$$\lim_{\varepsilon \rightarrow 0} \int_M |f_\varepsilon - f|^{p(x)} dv_\phi = 0.$$

Hence, by Proposition 1 we can write  $|f_\varepsilon - f|_{p(x)} = 0$ .

Let's consider

$$\varrho_{p(x)} \nabla (f_\varepsilon - f) = \int_M |\nabla (f_\varepsilon - f)|^{p(x)} dv_\phi = \int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |\nabla (f_\varepsilon - f)|^{p(x)} dv_\phi + \int_{B_{2^{-1} \exp(-\varepsilon^{-2})}} |\nabla f|^{p(x)} dv_\phi,$$

where  $T_{\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} = B_\varepsilon \setminus B_{2^{-1} \exp(-\varepsilon^{-2})}$ . Since  $f \in W^{1,p(x)}(M)$ ,

$$\int_{B_{2^{-1} \exp(-\varepsilon^{-2})}} |\nabla f|^{p(x)} dv_\phi \rightarrow 0 \text{ as } 2^{-1} \exp(-\varepsilon^{-2}) \rightarrow 0,$$

and

$$\int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |\nabla (f_\varepsilon - f)|^{p(x)} dv_\phi \leq 2^{p^+-1} \left( \int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |\nabla f_\varepsilon|^{p(x)} dv_\phi + \int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |\nabla f|^{p(x)} dv_\phi \right).$$

The first term on the right hand side tends to 0 as  $\varepsilon \rightarrow 0$  because of  $f \in W^{1,p(x)}(M)$ . By the definition of  $f_\varepsilon$ , we have

$$\begin{aligned} \int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |\nabla f_\varepsilon|^{p(x)} dv_\phi &\leq \int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |(\nabla \omega_\varepsilon) f + \omega_\varepsilon (\nabla f)|^{p(x)} dv_\phi \\ &\leq 2^{p^+-1} \left( \int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |f|^{p(x)} |\nabla \omega_\varepsilon|^{p(x)} dv_\phi + \int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |\nabla f|^{p(x)} |\omega_\varepsilon|^{p(x)} dv_\phi \right). \end{aligned}$$

Since  $f \in W^{1,p(x)}(M)$ , the term  $\int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |\nabla f|^{p(x)} |\omega_\varepsilon|^{p(x)} dv_\phi \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Next we shall show

$$\int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |f|^{p(x)} |\nabla \omega_\varepsilon|^{p(x)} dv_\phi \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

From  $\dim M^* = k$  and  $m - k \geq p^+$ , that is, the codimension of  $M^*$  and  $M$  is greater than  $p^+$  or equal to  $p^+$ . We consider the fact

$$p^- \leq \inf_{x \in (\varepsilon, \exp(-\varepsilon^{-2}))} p(x) \text{ and } \sup_{x \in (\varepsilon, \exp(-\varepsilon^{-2}))} p(x) \leq p^+,$$

with which we can estimate

$$\begin{aligned} \int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |f|^{p(x)} |\nabla \omega_\varepsilon|^{p(x)} dv_\phi &\leq C^{p^+} \int_{T_\varepsilon, 2^{-1} \exp(-\varepsilon^{-2})} |\nabla \omega_\varepsilon|^{p(x)} dv_\phi \\ &\leq 2^{p^+-1} C^{p^+} \left( C_1 \int_{\exp(-\varepsilon^{-2})}^\varepsilon \left(\frac{r}{\varepsilon}\right)^{\varepsilon p(r) - p(r)} r^{m-k-1} dr + C_2 \int_{2^{-1} \exp(-\varepsilon^{-2})}^{\exp(-\varepsilon^{-2})} \left(\frac{\exp(-\varepsilon^{-2})}{\varepsilon}\right)^{\varepsilon p(r)} \right) \end{aligned}$$

$$\begin{aligned}
 & \times \left( \frac{2}{\exp(-\varepsilon^{-2})} \right)^{p(r)} r^{m-k-1} dr \leq C_1(p^+) \int_{\exp(-\varepsilon^{-2})}^{\varepsilon} \left( \frac{r}{\varepsilon} \right)^{\varepsilon p^- - p^+} r^{m-k-1} dr \\
 & + \frac{C_2(p^+)}{\varepsilon^{\varepsilon p^+}} \exp(-(\varepsilon p^+ - p^+)/\varepsilon^2) \int_{2^{-1} \exp(-\varepsilon^{-2})}^{\exp(-\varepsilon^{-2})} r^{m-k-1} dr = \frac{C_1(p^+)}{\varepsilon^{\varepsilon p^- - p^+}} \left[ \frac{r^{\varepsilon p^- - p^+ + m - k}}{\varepsilon p^- - p^+ + m - k} \right]_{\exp(-\varepsilon^{-2})}^{\varepsilon} \\
 & + \frac{C_2(p^+)}{\varepsilon^{\varepsilon p^+}} \exp(-(\varepsilon p^+ - p^+)/\varepsilon^2) \left[ \frac{r^{m-k}}{m-k} \right]_{2^{-1} \exp(-\varepsilon^{-2})}^{\exp(-\varepsilon^{-2})} \\
 & \leq \frac{C_1(p^+)}{p^- \varepsilon^{\varepsilon p^- - p^+ + 1}} \left\{ \varepsilon^{\varepsilon p^- - p^+ + m - k} - \exp(-(\varepsilon p^- - p^+ + m - k)/\varepsilon^2) \right\} \\
 & \quad + \frac{C_2(p^+, m, k)}{\varepsilon^{\varepsilon p^+}} \left\{ \exp(-\varepsilon^{-2}(\varepsilon p^+ - p^+ + m - k)) \right\} \\
 & = \frac{C_1(p^+)}{p^-} \varepsilon^{m-k-1} - \frac{C_1(p^+) \varepsilon^{p^+ - 1}}{p^- \varepsilon^{\varepsilon p^-}} \exp(-(\varepsilon p^- - p^+ + m - k)/\varepsilon^2) \\
 & \quad + \frac{C_2(p^+, m, k)}{\varepsilon^{\varepsilon p^+}} \exp(-(\varepsilon p^+ - p^- + m - k)/\varepsilon^2).
 \end{aligned}$$

The right hand side tends to 0 by using

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \varepsilon^{m-k-1} &= 0 \text{ (because of } m - k \geq p^+ - 1 > 0), \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^{p^+ - 1} \exp(-(\varepsilon p^- - p^+ + m - k)/\varepsilon^2) = 0 \\
 \text{(because of } \lim_{\varepsilon \rightarrow 0} \varepsilon^\varepsilon &= 1), \quad \lim_{\varepsilon \rightarrow 0} \frac{\exp(-(\varepsilon p^- - p^+ + m - k)/\varepsilon^2)}{\varepsilon^{\varepsilon p^-}} = 0.
 \end{aligned}$$

This completes the proof of Lemma 1. □

**Theorem 2.** *Let  $(M, \phi)$  be a compact Riemannian manifold with  $\dim M = m$ , and  $M^*$  a closed submanifold in  $M$  with  $\dim M^* = k$ . Suppose that the codimension  $m - k \geq p^+$  of  $M^*$  in  $M$  is greater than or equal to  $p^+$ . Let  $\lambda_{1,p(x)}(\Omega_\varepsilon, \phi)$  be the first eigenvalue of the  $p(x)$ -Laplacian on  $\Omega_\varepsilon$ . We have*

$$\lim_{\varepsilon \rightarrow 0} \lambda_{1,p(x)}(\Omega_\varepsilon, \phi) = \lambda_{1,p(x)}(M, \phi) = 0.$$

The corresponding eigenfunctions  $\phi_\varepsilon$  in  $\Omega_\varepsilon$  converge to a constant function  $\phi_1$  in  $L^{p(x)}(M)$ .

**Proof.** Take the eigenfunction  $f$  for  $\lambda_{1,p(x)}(M, \phi)$  such that  $\int_M |f|^{p(x)} dv_\phi = 1$ . Then

$$\lambda_{1,p(x)}(M) = \int_M |\nabla f|^{p(x)} dv_\phi.$$

From Lemma 1, there exists  $f_\varepsilon \in W_0^{1,p(x)}(\Omega_\varepsilon)$  such that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |f_\varepsilon - f|^{p(x)} dv_\phi \rightarrow 0 \text{ and } \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla(f_\varepsilon - f)|^{p(x)} dv_\phi \rightarrow 0.$$

Thus

$$\lambda_{1,p(x)}(\Omega_\varepsilon, \phi) \leq \frac{\int_{\Omega_\varepsilon} |\nabla f_\varepsilon|^{p(x)} dv_\phi}{\int_{\Omega_\varepsilon} |f_\varepsilon|^{p(x)} dv_\phi} \rightarrow \frac{\int_M |\nabla f|^{p(x)} dv_\phi}{\int_M |f|^{p(x)} dv_\phi} = \lambda_{1,p(x)}(M, \phi) \text{ } (\varepsilon \rightarrow 0). \tag{1}$$

Now let  $\phi_\varepsilon$  be the first eigenvalue for the  $p(x)$ -Laplacian on  $\Omega_\varepsilon$ . By formula (1),  $\phi_\varepsilon$  is uniformly bounded.  $\phi_\varepsilon$  has a strongly convergence limit  $\phi_1 \in L^{p(x)}(M)$  in  $L^{p(x)}(M)$ . By formula (1), and the compactness of  $\phi_\varepsilon$ , we have  $\int_M |\phi_1|^{p(x)} dv_\phi = 1$  and

$$\lambda_{1,p(x)}(M, \phi) \leq \int_M |\nabla \phi_1|^{p(x)} dv_\phi \leq \lim_{\varepsilon \rightarrow 0} \int_M |\nabla \phi_\varepsilon|^{p(x)} dv_\phi \leq \lambda_{1,p(x)}(M, \phi).$$

This shows that  $\lambda_{1,p(x)}(\Omega_\varepsilon, \phi) \rightarrow \lambda_{1,p(x)}(M, \phi)$  as  $\varepsilon \rightarrow 0$ , and the limit function  $\phi_1 \in L^{p(x)}(M)$  is the first eigenfunction for  $p(x)$ -Laplacian on  $M$ .

This completes the proof of Theorem. □

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