[Turkish Journal of Mathematics](https://dctubitak.researchcommons.org/math)

Volume 33 | Number 4 Article 4

1-1-2009

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Turk J Math
33 (2009) , 341 – 350.

doi:10.3906/mat-0711-23

Oscillation of nonlinear neutral delay differential equations of second-order with positive and negative coefficients

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Abstract

Some oscillation criteria for the following second-order neutral differential equation

$$\big[x(t) \pm r(t)f(x(t-\gamma))\big]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t)$$

where $t \geq t_0$, $\gamma, \alpha, \beta \in \mathbb{R}^+$ with $\alpha \geq \beta$, $r \in C^2([t_0,\infty),\mathbb{R}^+)$, $p,q \in C([t_0,\infty),\mathbb{R}^+)$ and $f,g \in C(\mathbb{R},\mathbb{R})$, $s \in C([t_0,\infty),\mathbb{R})$ have been obtained. Our results are not restricted with boundedness of solutions.

Key word and phrases: Delay differential equations, neutral, nonlinear, oscillation, second-order.

1. Introduction

In this paper, we consider the oscillation of the second-order nonlinear neutral delay differential equations of the form

$$\big[x(t) + r(t)f(x(t-\gamma))\big]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t), \tag{1}$$

$$\big[x(t) - r(t)f(x(t-\gamma))\big]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t), \tag{2}$$

where $t \geq t_0$, $\gamma \geq 0$, $\alpha \geq \beta \geq 0$, $r \in C^2([t_0,\infty),\mathbb{R}^+)$ and $p,q \in C([t_0,\infty),\mathbb{R}^+)$. Furthermore, we suppose that the following are satisfied:

(H1) $\liminf_{t\to\infty} h(t) > 0$, where $h(t) := p(t) - q(t-\alpha+\beta)$ for $t \geq t_0$.

(H2) $f \in C(\mathbb{R},\mathbb{R})$ is nondecreasing with $f(u)/u > 0$ for $u \neq 0$ and there exists positive constant M such that

$$0 < \frac{f(u)}{u} \leq M, \quad u \neq 0$$

holds.

2000 *AMS Mathematics Subject Classification:* 34C10, 34C15, 34K40.

(H3) $g \in C(\mathbb{R}, \mathbb{R})$ with $g(u)/u > 0$ for $u \neq 0$ and there exists positive constants N_1 and N_2 such that

$$N_1 \leq \frac{g(u)}{u} \leq N_2, \quad u \neq 0$$

holds.

(H4) $s \in C([t_0, \infty), \mathbb{R})$ and there exists a function $S \in C^2([t_0, \infty), \mathbb{R})$ such that $S'' = s$ and $S(t) \to 0$ as $t \to \infty$.

Note that if $S^* \in C^2([t_0, \infty), \mathbb{R})$ is a function satisfying $S^{*\prime\prime} = s$ and $L := \lim_{t\to\infty} S^*(t)$ exists and is finite, then $S := S^* - L$ holds (H4).

For the case f and g are identity functions, we obtain better results than those in [3]. Also in this case our results weaken assumptions on the coefficients. For the first-order case, see the results in [4]. Our results improve results in the literature. We refer readers to [1, 2, 5, 6, 7] for further results.

We restrict our attention only to those solutions x that are not eventually trivial. By a *solution*, we mean a function x identically satisfying the equation and $[x(t) - r(t)f(x(t-\gamma))] \in C^2([t_0, \infty), \mathbb{R})$ for all $t \geq t_0$. A solution is called *nonoscillatory* if it is eventually of single sign; otherwise, the solution is called *oscillatory*. Throughout the paper, we let $\kappa := \max\{\gamma, \alpha\}$.

2. Oscillatory behavior of solutions of homogenous equations

We start this section by giving the following sufficient condition on (1).

Theorem 1 *Assume that (H1)–(H3) hold and* $r \in C([t_0, \infty), \mathbb{R}^+)$ *is bounded. If*

$$\int_{t_0}^{\infty} \int_{u-\alpha+\beta}^{u} q(v)dvdu < \infty \tag{3}$$

holds, then every solution of (1) *is oscillatory.*

Proof. Suppose that x is an eventually positive solution of (1). The case where x is eventually negative is similar and is omitted. Let $t_1 \geq t_0$ such that $x(t-\kappa) > 0$ for $t \geq t_1$. Then, considering (3) there exists $t_2 \geq t_1$ such that

$$\int_{t_2}^{\infty} \int_{u-\alpha+\beta}^{u} q(v)dvdu \leq \frac{1}{2N_2} \tag{4}$$

holds. Now, we set

$$w(t) := x(t) + r(t)f(x(t-\gamma)) \geq 0 \tag{5}$$

and

$$z(t) := w(t) - \int_{t_2}^{t} \int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu \tag{6}$$

for $t \geq t_2$. Then, we have

$$\begin{aligned} z''(t) &= w''(t) - q(t)g(x(t-\beta)) + q(t-\alpha+\beta)g(x(t-\alpha)) \\ &= -p(t)g(x(t-\alpha)) + q(t-\alpha+\beta)g(x(t-\alpha)) \\ &= -h(t)g(x(t-\alpha)) \leq 0 \end{aligned} \tag{7}$$

for all $t \geq t_2$. Hence, $z'(t)$ and $z(t)$ is strictly monotonic and constant of sign for all $t \geq t_3$, where $t_3 \geq t_2$ is sufficiently large. To prove $z'(t) > 0$ holds for all $t \geq t_2$, we assume contrary that $z'(t) < 0$ holds for all $t \geq t_2$. In the present case, we see that

$$\lim_{t\to\infty} z(t) = -\infty. \tag{8}$$

We also claim that x is bounded. For contrary assume x is unbounded. Thus, there is $t_4 \geq t_3$ such that

$$z(t_4) < 0, \quad x(t_4) = \max\{x(t) : t \in [t_3, t_4]\}. \tag{9}$$

Then, considering (H3), (4) and (9), we obtain

$$\begin{aligned} 0 > z(t_4) &= w(t_4) - \int_{t_2}^{t_4} \int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu, \\ &\geq x(t_4) - N_2 \int_{t_2}^{t_4} \int_{u-\alpha+\beta}^{u} q(v)x(v-\beta)dvdu, \\ &\geq x(t_4)(1 - N_2 \int_{t_2}^{t_4} \int_{u-\alpha+\beta}^{u} q(v)dvdu) \geq \frac{1}{2}x(t_4) \geq 0. \end{aligned}$$

This contradiction shows that x must be bounded. There is a positive constant K such that $x(t) \leq K$ holds for all $t \geq t_0$. Accordingly, we see that

$$z(t) \geq -KN_2 \int_{t_2}^{\infty} \int_{u-\alpha+\beta}^{u} q(v)dvdu \geq -\frac{K}{2} > -\infty$$

holds, which contradicts with (8) and proves that $z'(t) > 0$ holds for all $t \geq t_2$. By (H1), there exists $t_3 \geq t_2$ and $\varepsilon > 0$ such that $h(t) \geq \varepsilon$ holds for all $t \geq t_3$. Integrating (7) from t_3 to ∞, we get

$$\infty > z'(t_3) \geq \varepsilon \int_{t_3}^{\infty} g(x(u-\alpha))du \geq \varepsilon N_1 \int_{t_3}^{\infty} x(u-\alpha)du,$$

which implies $x \in L^1([t_0, \infty))$. Since r is bounded and (H2) holds, we see from (5) that $w \in L^1([t_2, \infty))$. Hence,

$$\liminf_{t\to\infty} w(t) = 0 \tag{10}$$

is true. On the other hand, we see from (6) that

$$w'(t) = z'(t) + \int_{t-\alpha+\beta}^{t} q(u)g(x(u-\beta))du > 0 \tag{11}$$

holds for all $t \geq t_3$. Note that w defined in (4) is positive and increasing by (11), hence (10) is impossible. This is a contradiction. Thus, every solution is oscillatory. □

Theorem 2 *Assume that (H1)–(H2) hold and* $r \in C([t_0,\infty),\mathbb{R}^+)$ *satisfies*

$$\limsup_{t\to\infty} r(t) < \frac{1}{M}. \tag{12}$$

If (3) *holds, then every solution of* (2) *is oscillatory or tending to zero as* t *tends to infinity.*

Proof. Suppose that x is a nonoscillatory solution of (2), then we have to show that $\lim_{t\to\infty} x(t) = 0$ is true. Without loss of generality, we suppose that x is an eventually positive solution. There exists $t_1 \geq t_0$ such that $x(t-\kappa) > 0$ holds for all $t \geq t_1$. Considering (12), there exists $t_2 \geq t_1$ and $0 < \delta < 1/M$ such that

$$r(t) \leq \frac{1}{M} - \delta \tag{13}$$

for all $t \geq t_2$. And (3) ensures existence of $t_3 \geq t_2$ such that

$$\int_{t_3}^{\infty} \int_{u-\alpha+\beta}^{u} q(v)dvdu < \frac{\delta M}{2N_2}, \tag{14}$$

Now, we set

$$w(t) := x(t) - r(t)f(x(t-\gamma)) \tag{15}$$

and

$$z(t) := w(t) - \int_{t_3}^{t} \int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu \tag{16}$$

for $t \geq t_3$. Then, we have

$$\begin{aligned} z''(t) &= w''(t) - q(t)g(x(t-\beta)) + q(t-\alpha+\beta)g(x(t-\alpha)) \\ &= -p(t)g(x(t-\alpha)) + q(t-\alpha+\beta)g(x(t-\alpha)) \\ &= -h(t)g(x(t-\alpha)) \leq 0 \end{aligned} \tag{17}$$

for all $t \geq t_3$. Hence, $z'(t)$ and $z(t)$ is strictly monotonic and constant of sign for all $t \geq t_4$, where $t_4 \geq t_3$ is sufficiently large. To prove $z'(t) > 0$ for all $t \geq t_4$, we assume on the contrary that $z'(t) < 0$ holds all $t \geq t_4$. In the present case, since z' is negative and nonincreasing, we see that

$$\lim_{t\to\infty} z(t) = -\infty \tag{18}$$

holds. We also claim that x is bounded. Again on the contrary, assume that x is unbounded. Thus, there exists $t_5 \geq t_4$ such that

$$z(t_5) < 0, \quad x(t_5) = \max\{x(t) : t \in [t_4, t_5]\} \tag{19}$$

hold. Then, from (H2), (H3), (13), (14) and (19), we obtain

$$\begin{aligned} 0 > z(t_5) &= w(t_5) - \int_{t_3}^{t_5} \int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu \\ &\geq x(t_5)\Big(1 - Mr(t_5) - N_2 \int_{t_3}^{t_5} \int_{u-\alpha+\beta}^{u} q(v)dvdu\Big) \\ &\geq x(t_5)\Big(1 - M(\frac{1}{M} - \delta) - \frac{\delta M}{2}\Big) = \frac{\delta M}{2} x(t_5) \geq 0. \end{aligned}$$

This contradiction implies that x is bounded. There is a positive K such that $x(t) \leq K$ for all $t \geq t_0$. Accordingly, for all $t \geq t_4$, we obtain

$$z(t) \geq -\Big(KMr(t) + KN_2 \int_{t_3}^{t} \int_{u-\alpha+\beta}^{u} q(v)dvdu\Big) \geq -\frac{\delta KM}{2} > -\infty,$$

which contradicts with (18) and proves that $z'(t) > 0$ holds for $t \geq t_2$. By (H1), there exists $t_5 \geq t_4$ and $\varepsilon > 0$ such that $h(t) \geq \varepsilon$ holds for all $t \geq t_5$. Integrating (17) from t_5 to ∞, we get

$$\infty > z'(t_5) \geq z'(t_5) - z'(\infty) \geq \varepsilon \int_{t_5}^{\infty} g(x(u-\alpha))du \geq \varepsilon N_1 \int_{t_5}^{\infty} x(u-\alpha)du$$

which implies $\underline{L} = 0$ and $\overline{L} < \infty$ hold, where $\underline{L} := \liminf_{t\to\infty} x(t)$ and $\overline{L} := \limsup_{t\to\infty} x(t)$. On the other hand, we have from (15) and (14) that

$$w'(t) = z'(t) + \int_{t-\alpha+\beta}^{t} q(u)g(x(u-\beta))du \geq 0$$

holds for all $t \geq t_5$, which implies w is nondecreasing. Therefore, from (H2), (15) and $\overline{L} < \infty$, we see that $-\infty < L < \infty$ holds, where $L := \lim_{t\to\infty} w(t)$.

Now we investigate the following three possible ranges of L as follows:

(i) $0 < L < \infty$. Then, there exists a sufficiently large $t_6 \geq t_5$ such that $w(t) \geq L/2$ holds for all $t \geq t_6$. So, for all $t \geq t_6$, we obtain

$$w(t) = x(t) - r(t)f(x(t-\gamma)) \geq \frac{L}{2},$$

which implies $x(t) \geq L/2$ for all $t \geq t_6$. This contradicts with $\underline{L} = 0$.

(ii) $-\infty < L < 0$. Then, there exists a sufficiently large $t_6 \geq t_5$ such that $w(t) \leq L$ holds for all $t \geq t_6$. So, for all $t \geq t_6$, we see that

$$w(t) = x(t) - r(t)f(x(t-\gamma)) \leq L$$

holds, and together with (H2) and (13), we have

$$-L \leq r(t)f(x(t-\gamma)) \leq M(\frac{1}{M} - \delta)x(t-\gamma),$$

which simply implies $x(t-\gamma) > -L/(M(1/M - \delta))$ for all $t \geq t_6$. This contradicts the fact that $\underline{L} = 0$.

(iii) $L = 0$. Now, we claim that $\bar{L} = 0$. On the contrary, assume that $\overline{L} > 0$. Therefore, from (H2) and (13), we see that

$$w(t) \geq x(t) - \delta x(t-\gamma).$$

holds for all $t \geq t_6$, where $t \geq t_5$ is sufficiently large. Then, there is an increasing divergent sequence $\{u_n\}_{n=1}^{\infty}$ on $[t_7, \infty)$, where $t_7 \geq t_6 + \kappa$ such that $\overline{L} = \lim_{n\to\infty} x(u_n)$ and a sequence $\{v_n\}_{n=1}^{\infty}$ satisfying $x(v_n) = \max\{x(t) : u_n - \kappa \leq t \leq u_n\}$ for all $n \in \mathbb{N}$ Since, $x(v_n) \geq x(u_n)$ for all $n \in \mathbb{N}$, we have $\overline{L} = \lim_{n\to\infty} x(v_n)$ Therefore, from (H2) and (13), we obtain

$$w(u_n) \geq x(u_n) - \delta x(u_n - \gamma),$$

for all $n \in \mathbb{N}$, taking limit as $n \to \infty$, we see that

$$\begin{aligned} L = 0 &\geq \lim_{n\to\infty} [x(u_n) - \delta x(u_n - \gamma)] \\ &\geq \lim_{n\to\infty} x(u_n) - \delta \lim_{n\to\infty} x(v_n) \\ &= \overline{L}(1-\delta) \geq 0, \end{aligned}$$

which implies $\overline{L} = 0$. This contradicts to the assumption that x is not tending to zero as $t \to \infty$.

The proof is complete. □

3. Oscillatory behavior of solutions of forced equations

In this section, we shall consider (1) and (2) with forcing terms of the forms:

$$[x(t) + r(t)f(x(t-\gamma))]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t), \tag{20}$$

$$[x(t) - r(t)f(x(t-\gamma))]'' + p(t)g(x(t-\alpha)) - q(t)g(x(t-\beta)) = s(t) \tag{21}$$

for $t \geq t_0$.

Theorem 3 *Assume that (H1)–(H4) hold and $r \in C([t_0, \infty), \mathbb{R}^+)$ is bounded. If (3) holds, then every solution of (20) is oscillatory or tending to zero as $t \to \infty$.*

Proof. Suppose that x is an eventually positive solution of (20). Let $t_1 \geq t_0$ satisfy $x(t-\kappa) > 0$ for all $t \geq t_1$. There exists $t_2 \geq t_1$ such that (4) holds.

Let w and z as in (5) and (6) respectively. And if we define

$$W(t) := w(t) - S(t) \text{ and } Z(t) := z(t) - S(t), \tag{22}$$

from (20), we obtain

$$Z''(t) \leq -h(t)g(x(t-\alpha)) \leq 0, \quad t \geq t_1. \tag{23}$$

This shows that Z' is an eventually nonincreasing function. We claim that Z' can not be eventually negative function. Suppose the contrary, i.e. $Z(t) < 0$ for all $t \geq t_3$, for some $t_3 \geq t_2$. Then, we have $\lim_{t\to\infty} Z(t) = -\infty$. We can come to the conclusion that x is bounded from above. As a matter of fact, if x is unbounded from above, there exists an increasing divergent sequence $\{s_n\}_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} Z(s_n) = -\infty \text{ and } x(s_n) = \max\{x(t) : t_3 \leq t \leq s_n\} \tag{24}$$

for all $n \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} x(s_n) = \infty$ holds. Then, from (4) and (24), we have

$$\begin{aligned}
Z(s_n) &= x(s_n) + r(s_n)f(x(s_n-\gamma)) - \int_{t_2}^{s_n}\int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu - S(s_n) \\
&\geq x(s_n) - N_2\int_{t_2}^{s_n}\int_{u-\alpha+\beta}^{u} q(v)x(v-\beta)dvdu - S(s_n) \\
&\geq \frac{1}{2}x(s_n) - S(s_n),
\end{aligned}$$

and taking the limit as $n \to \infty$, leads the way to the contradiction $\lim_{t\to\infty} Z(t) = \infty$. Since x is bounded from above, there exists a constant $K > 0$ such that $x(t) \leq K$ holds for all $t \geq t_0$. Hence, from (22), we have

$$Z(t) \geq -KN_2\int_{t_0}^{t}\int_{u-\alpha+\beta}^{u} q(v)dvdu + S(t)$$

for all $t \geq t_3$, which according to (4) yields the following:

$$\lim_{t\to\infty} Z(t) \geq -KN_2\int_{t_2}^{\infty}\int_{u-\alpha+\beta}^{u} q(v)dvdu \geq \frac{K}{2}.$$

This contradicts to the fact that $\lim_{t\to\infty} Z(t) = -\infty$.

Therefore, we conclude that Z is an eventually nondecreasing function. Integrating (23) from t_3 to ∞, we have that $x \in L^1([t_0,\infty))$ because of (H1), and accordingly from (5), this implies that $w \in L^1([t_2,\infty))$ holds since (H2) holds and r is bounded. From (22), we obtain that

$$W'(t) = Z'(t) + \int_{t-\alpha+\beta}^{t} q(u)g(x(u-\beta))du \geq 0$$

holds for all $t \geq t_3$, so that W is nondecreasing. Therefore, using the assumption (H4), we have

$$L := \lim_{t\to\infty} W(t) = \lim_{t\to\infty} w(t),$$

where $0 \leq L < \infty$.

(i) If $0 < L < \infty$. Then, there exists a sufficiently large $t_4 \geq t_3$ such that $w(t) > L/2$ for all $t \geq t_4$. Hence, $w \notin L^1([t_2, \infty))$, and this yields to a contradiction.

(ii) If $L = 0$ is true, then since $x(t) \leq w(t)$ holds for all $t \geq t_2$, we have that $\lim_{t\to\infty} x(t) = 0$.

The proof is therefore completed. □

Theorem 4 *Assume that (H1)–(H4) hold and* $r \in C([t_0,\infty), \mathbb{R}^+)$ *satisfies* (12). *If* (3) *holds, then every solution of* (21) *is oscillatory or tending to zero as* $t \to \infty$.

Proof. Suppose that x is a nonoscillatory solution of (21), which is not tending to zero as $t \to \infty$. Without loss of generality, we suppose that x is eventually positive that is $x(t-\kappa) > 0$ holds for all $t \geq t_1$, where $t_1 \geq t_0$. Then, there exists $t_2 \geq t_1$ such that (13) holds. We have $t_3 \geq t_2$ such that (14) holds because of (3). If we now define W and Z by considering (22) with w and z are defined as in (15) and (16) respectively, from (21), we obtain

$$Z''(t) \leq -h(t)g(x(t-\alpha)) \leq 0, \quad t \geq t_1. \tag{25}$$

for all $t \geq t_3$. We claim that

$$Z'(t) \geq 0, \quad t \geq t_4 \tag{26}$$

holds for some $t_4 \geq t_3$. If this is not a case, then $Z'(t) < 0$ holds for all $t \geq t_4$, implies $\lim_{t\to\infty} Z(t) = -\infty$. On the other hand, x must be bounded from above. Otherwise, there exists an increasing divergent sequence $\{s_n\}_{n=1}^{\infty}$ satisfying (24). Clearly, we have $\lim_{n\to\infty} x(s_n) = \infty$ and $\lim_{n\to\infty} S(s_n) = 0$ from (H4). Since, we have that

$$\begin{aligned} Z(s_n) &= x(s_n) - r(s_n)f(x(s_n-\gamma)) - \int_{t_3}^{s_n}\int_{u-\alpha+\beta}^{u} q(v)g(x(v-\beta))dvdu - S(s_n) \\ &\geq \frac{\delta M}{2}x(s_n) - S(s_n), \end{aligned}$$

by letting $n \to \infty$ and considering (H4), we get

$$\lim_{t\to\infty} Z(t) = \infty.$$

This is a contradiction. Therefore, x is bounded from above, there exists a constant $K > 0$ such that $x(t) \leq K$ holds for all $t \geq t_0$. Then, we have

$$Z(t) \geq -\frac{\delta K M}{2} + S(t), \quad t \geq t_4.$$

Taking the limit of the above inequality as $t \to \infty$, we have $\lim_{t\to\infty} Z(t) \geq -(\delta KM)/2$ as in the proof of Theorem 2. This is a contradiction. Consequently, we have that (26) holds.

From (H1), (H3) and (25), we obtain $x \in L^1([t_1,\infty))$. Hence, (H2), (5) and boundedness of r implies that $w \in L^1([t_1,\infty))$. On the other hand, we see from (22) that

$$W'(t) = Z'(t) + \int_{t-\alpha+\beta}^{t} q(u)g(x(u-\beta))du \geq 0, \quad t \geq t_4,$$

holds, so that $-\infty < L < \infty$ is true, where

$$L := \lim_{t\to\infty} W(t) = \lim_{t\to\infty} w(t).$$

Now, we investigate possible ranges of L as follows:

(i) $L \neq 0$. In this case, we obtain contradiction as obtained in Theorem 2.

(ii) $L = 0$. In this case, we see that $\lim_{t\to\infty} x(t) = 0$ holds as in Theorem 2.

Proof is done. □

Remark 1 *Letting f, g as the identity functions, we see that our results are still better than those in [3].*

4. An Application

Example 1 *Consider the forced neutral equation*

$$\begin{aligned}&\left[x(t) - \frac{1}{2}x(t-2\pi)\right]''\\ &+ (\mathrm{e}^{-t} + \frac{1}{2}(1 - \frac{1}{\sin^2(t)+2}))\frac{x(t-4\pi)([x(t-4\pi)]^2+2)}{[x(t-4\pi)]^2+1}\\ &- \mathrm{e}^{-t}\frac{x(t-2\pi)([x(t-2\pi)]^2+2)}{[x(t-2\pi)]^2+1} = 0\end{aligned} \tag{27}$$

for $t \geq 1$. For this equation, we have $r(t) = 1/2$, $\gamma = 2\pi$, $f(u) = u$, $p(t) = (\mathrm{e}^{-t} + (1 - 1/(\sin^2(t)+2))/2)$, $\alpha = 4\pi$, $q(t) = \mathrm{e}^{-t}$, $\beta = 2\pi$, $g(u) = u(u^2+2)/(u^2+1)$. In this case, we may let $M = 1$, and since we have $g(u)/u = 1 + 1/(u^2+1)$ for all $u \neq 0$, we may let $N_1 = 1$ and $N_2 = 2$. On the other hand, we have $\liminf_{t\to\infty} h(t) = 1/4 > 0$, where $h(t) = \mathrm{e}^{-t}(1-\mathrm{e}^{2\pi}) + (1 - 1/(\sin^2(t)+2))/2$, and $\int_1^\infty \int_{u-2\pi}^u \mathrm{e}^{-v} dv du = (\mathrm{e}^{2\pi}-1)/\mathrm{e} < \infty$. All the conditions of Theorem 4 are satisfied, thus every solution of (27) is oscillatory or convergent to zero as t tends to infinity. One can see by direct substation that $x(t) = \sin(t)$ is an oscillatory solution.

Acknowledgement

The authors are thankful to the referee for his/her valuable suggestions to improve the presentation of the paper.

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Received 26.11.2007