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A note on the lévy constant for continued fractions

Ting Zhong

Abstract

In this note, we study the lévy constant of continued fraction expansions. We show that for all $x \in [0, 1)$, the upper lévy constant of x is finite except a set with Hausdorff dimension one-half.

Key Words: Continued fractions, lévy constant, Hausdorff dimension.

1. Introduction

It is well known that every irrational number $x \in [0, 1)$ has a unique standard continued fraction expansion of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}}},$$

where each partial quotient $a_n(x) \in \mathbb{N}$ is uniquely defined by the number x .

For any $n \geq 1$ and $a_1, \dots, a_n \in \mathbb{N}$, define a *CF-interval* of rank n as

$$I(a_1, a_2, \dots, a_n) = \{x \in [0, 1) : a_k(x) = a_k, 1 \leq k \leq n\}.$$

Therefore, (see [5], section 12), $I(a_1, \dots, a_n)$ is the interval with endpoints $\frac{p_n}{q_n}$ and $\frac{p_n + p_{n-1}}{q_n + q_{n-1}}$, where p_n and q_n are defined by following recurrence relations

$$\begin{aligned} p_{-1} &= 1; p_0 = 0; p_n = a_n p_{n-1} + p_{n-2}, \quad n \geq 1. \\ q_{-1} &= 0; q_0 = 1; q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 1. \end{aligned} \tag{1}$$

Thus, the length of $I(a_1, a_2, \dots, a_n)$ is

$$|I(a_1, a_2, \dots, a_n)| = \left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| = \frac{1}{q_n(q_n + q_{n-1})}. \tag{2}$$

For an irrational number $x \in [0, 1)$, we call

$$\beta^*(x) = \limsup_{n \rightarrow \infty} \frac{\log q_n(x)}{n} \quad \text{and} \quad \beta_*(x) = \liminf_{n \rightarrow \infty} \frac{\log q_n(x)}{n},$$

the upper lévy constant and lower lévy constant of x , respectively. If $\beta^*(x) = \beta_*(x)$, we say the lévy constant of x exists and denote the common value by $\beta(x)$. A famous result of P. Lévy [6] says that for almost all x , the lévy constant exists and

$$\beta(x) = \frac{\pi^2}{12 \log 2} \approx 1.18657.$$

$\beta^*(x)$ and $\beta_*(x)$ describe the exponential growth rates of $q_n(x)$ in n . Faiver [2] showed that every quadratic number has a lévy constant. It is easy to see that for any irrational number $x \in [0, 1)$, one has $\beta_*(x) \geq \log \frac{\sqrt{5}+1}{2}$, then Faiver [3] also established that for all $\lambda \geq \log \frac{\sqrt{5}+1}{2}$, there exists an $x \in I$ such that $\beta(x) = \lambda$ by employing an ergodic theorem. Later, Baxa [1] showed the following more general result by elementary means.

Theorem 1.1 *For any $\log \frac{\sqrt{5}+1}{2} \leq \lambda_* \leq \lambda^* < \infty$, there exist uncountably many $x \in [0, 1)$ such that $\beta_*(x) = \lambda_*$ and $\beta^*(x) = \lambda^*$.*

In 2006, Wu [7] improved Baxa's result by showing the following theorem.

Theorem 1.2 *For any $\log \frac{\sqrt{5}+1}{2} \leq \lambda_* \leq \lambda^* < \infty$, let*

$$E(\lambda_*, \lambda^*) = \{x \in [0, 1) : \beta_*(x) = \lambda_*, \beta^*(x) = \lambda^*\}.$$

Then

$$\dim_H E(\lambda_*, \lambda^*) \geq \frac{\lambda_* - \log \frac{\sqrt{5}+1}{2}}{\lambda^*}.$$

In this note, we consider the set of $x \in [0, 1)$ whose upper lévy constant is infinite and obtain

Theorem 1.3 *Let*

$$E^\infty = \left\{x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\log q_n(x)}{n} = \infty\right\}.$$

Then

$$\dim_H E^\infty = \frac{1}{2}.$$

Here and in what follows, \dim_H denotes the Hausdorff dimension of a subset of $[0, 1)$, and $|\cdot|$ denotes the diameter. We sketch, very briefly, the definition and some basic properties of Hasdorff dimension. If $E \subset R$ and $\delta > 0$, define for each $s \geq 0$,

$$H^s(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{n=1}^{\infty} |I_n|^s : E \subset \bigcup_{n=1}^{\infty} I_n, |I_n| \leq \delta, n = 1, 2, \dots \right\},$$

$$\dim_H E = \inf \{s \geq 0 : H^s(E) = 0\} = \sup \{s \geq 0 : H^s(E) = \infty\}.$$

The following two facts are basic in calculating Hasdorff dimension of various sets.

Lemma 1.4 *Let $E \subset \mathbb{R}$ and let $s \geq 0$ be given. Suppose for each $\delta > 0$ there is a sequence of intervals $\{I_n\}$ such as $E \subset \bigcup I_n$, $|I_n| \leq \delta$ for all n , and $\sum_{n=1}^{\infty} |I_n|^s \leq 1$. Then $\dim_H E \leq s$.*

Lemma 1.5 *Let $E \subset \mathbb{R}$ be a Borel set and μ be a measure with $\mu(E) > 0$. If for any $x \in E$*

$$\liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq s.$$

where $B(x, r)$ denotes the open ball with center at x and radius r . Then $\dim_H E \geq s$.

Lemma 1 is obvious; for Lemma 2, see ([4], Proposition 2.3).

2. Proof of Theorem 1.3

In this section, we show Theorem 1.3 in detail and divide the proof into two parts: upper bound and lower bound.

I. Upper bound. $\dim_H E \leq \frac{1}{2}$.

Proof. By (1), we have

$$a_n q_{n-1} \leq q_n \leq 2a_n q_{n-1}.$$

Successive application of this inequality gives

$$a_1 a_2 \cdots a_n \leq q_n \leq 2^n a_1 a_2 \cdots a_n. \tag{3}$$

Thus we get the following alternative description of E^∞ :

$$E^\infty = \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\log a_1(x) + \log a_2(x) + \cdots + \log a_n(x)}{n} = \infty \right\}.$$

Let

$$E^{(m)} = \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\log a_1(x) + \log a_2(x) + \cdots + \log a_n(x)}{n} > m \right\}.$$

Then E^∞ can be written

$$E^\infty = \bigcap_{m=1}^{\infty} E^{(m)} = \lim_{n \rightarrow \infty} E^{(m)},$$

and for every $x = [a_1, a_2, a_3, \dots] \in E^{(m)}$, there exist infinitely many positive integers n_i such that

$$\frac{\log a_1(x) + \cdots + \log a_{n_i}(x)}{n_i} > m, \quad i = 1, 2, 3, \dots$$

So that, for any $\delta > 0$, the family of the *CF-intervals*

$$\mathcal{A}(m, \delta) = \left\{ I(a_1, a_2, \dots, a_{n_i}) : \frac{\log a_1(x) + \cdots + \log a_{n_i}(x)}{n_i} > m, |I(a_1, a_2, \dots, a_{n_i})| \leq \delta, n_i \in \mathbb{N} \right\}$$

is a δ -cover of $E^{(m)}$.

Note that, for any two *CF-intervals*, say I and I' , the following relation holds:

$$I \cap I' \neq \emptyset \implies I \subseteq I' \text{ or } I' \subseteq I.$$

In fact, if $I = I(a_1, \dots, a_n)$ and $I' = I(a'_1, \dots, a'_{n+k})$ ($k \geq 0$) have a common point $x = [x_1, x_2, \dots]$, then $a_1 = x_1 = a'_1, a_2 = x_2 = a'_2, \dots, a_n = x_n = a'_n$. It follows that $I' \subseteq I$.

We remove from $\mathcal{A}(m, \delta)$ all those the *CF-intervals* which are contained in other *CF-interval* in $\mathcal{A}(m, \delta)$, and denote the complement by $A(m, \delta)$. Then, $A(m, \delta)$ is a non-overlapping δ -cover of $E^{(m)}$.

Now we define a family of measures $\{\mu_t : t > 1\}$ as

$$\mu_t(I(a_1, a_2, \dots, a_n)) = e^{-np(t) - t \log \sum_{i=1}^n \log a_i} \tag{4}$$

where $p(t) = \log \zeta(t) = \log \sum_{n \geq 1} \frac{1}{n^t}, (t > 1)$.

By (2) and (3), we have for any $\epsilon > 0$,

$$\log |I(a_1, a_2, \dots, a_n)|^{\frac{\epsilon+t}{2}} \leq -(\epsilon+t) \log(a_1 a_2 \dots a_n) = -\epsilon \sum_{i=1}^n \log a_i - t \sum_{i=1}^n \log a_i. \tag{5}$$

By the definition of $A(m, \delta)$, for every $I = I(a_1, a_2, \dots, a_n) \in A(m, \delta)$ with $m \geq \frac{p(t)}{\epsilon}$, we have

$$-\epsilon \sum_{i=1}^n \log a_i \leq -\epsilon \cdot mn \leq -np(t). \tag{6}$$

Combining (4), (5) and (6), we get for any $\epsilon > 0$, and $I = I(a_1, a_2, \dots, a_n) \in A(m, \delta)$ with $m \geq \frac{p(t)}{\epsilon}$,

$$|I(a_1, a_2, \dots, a_n)|^{\frac{\epsilon+t}{2}} \leq e^{-np(t) - t \log \sum_{i=1}^n \log a_i} = \mu_t(I(a_1, a_2, \dots, a_n)).$$

Since $A(m, \delta)$ is a non-overlapping δ -cover of $E^{(m)}$, we sum the above inequality to have

$$\sum_{I \in A(m, \delta)} |I|^{\frac{\epsilon+t}{2}} \leq \sum_{I \in A(m, \delta)} \mu_t(I) = \mu_t\left(\bigcup_{I \in A(m, \delta)} I\right) \leq 1.$$

By Lemma 1.4, we get for any $t > 1, \epsilon > 0$ and $m \geq \frac{p(t)}{\epsilon}$,

$$\dim_H E^{(m)} \leq \frac{t + \epsilon}{2}.$$

Letting $\epsilon \rightarrow 0$ and since $t > 1$ is arbitrary, we obtain

$$\dim_H E^\infty \leq \frac{1}{2}.$$

□

II. **Lower bound.** $\dim_H E^\infty \geq \frac{1}{2}$.

Proof. Put

$$F = \left\{ x \in [0, 1) : 2^n \leq a_n(x) < 2^{n+1}, \text{ for all } n \geq 1 \right\}. \quad (7)$$

It is easy to check that $F \subset E^\infty$. So it is enough to prove $\dim_H F \geq \frac{1}{2}$. To give a precise view on the structure of F , we shall make use of a kind of symbolic space defined as follows.

$$\mathcal{D}_n = \left\{ (a_1, \dots, a_n) \in \mathbb{N}^n : 2^k \leq a_k < 2^{k+1}, \text{ for all } 1 \leq k \leq n \right\}.$$

For any $(a_1, \dots, a_n) \in \mathcal{D}_n$, call

$$J(a_1, \dots, a_n) = cl \left\{ x \in [0, 1) : a_k(x) = a_k, 1 \leq k \leq n \right\}$$

an admissible *CF-intervals* of rank n , where “*cl*” denotes the closure of a set in $[0, 1)$. It is observable that

$$F = \bigcap_{n=1}^{\infty} \bigcup_{(a_1, \dots, a_n) \in \mathcal{D}_n} J(a_1, \dots, a_n).$$

Let μ be a probability measure supported on F such that for every admissible intervals $J(a_1, \dots, a_n)$,

$$\mu(J(a_1, \dots, a_n)) = \frac{1}{\#\mathcal{D}_n} = \frac{1}{2^{1+2+\dots+n}}, \quad (8)$$

where $\#$ denotes the cardinality.

Now we estimate the μ -measure of arbitrary ball $B(x, r)$ with center $x \in F$ and radius r small enough. Choose $n \geq 1$ such that

$$|J(a_1, \dots, a_{n+1})| \leq r < |J(a_1, \dots, a_n)|.$$

Calculations show

$$|J(a_1, \dots, a_n)| \leq \sum_{1 \leq i \leq n} |J(a_1, \dots, a_{n-1}, a_n + i)|.$$

So that, from $a_n \geq 2$ and $r < |J(a_1, \dots, a_n)|$ we have

$$B(x, r) \subset J(a_1, \dots, a_{n-1}). \quad (9)$$

On the other hand, from (2), (3) and (7), We have

$$r \geq |J(a_1, \dots, a_{n+1})| > \frac{1}{2a_{n+1}^2} > \frac{1}{2^{2n+3}a_1^2a_2^2 \dots a_{n+1}^2} > \frac{1}{2^{2n+3+(n+1)(n+4)}}. \quad (10)$$

Combining (8), (9) and (10), we get

$$\liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq \liminf_{r \rightarrow 0} \frac{\frac{(n-1)n}{2}}{2n+3+(n+1)(n+4)} = \frac{1}{2}.$$

By using lemma 1.5, we obtain $\dim_H F \geq \frac{1}{2}$, which shows $\dim_H E^\infty \geq \frac{1}{2}$ since $F \subset E^\infty$. This completes the proof. \square

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