[Turkish Journal of Mathematics](https://journals.tubitak.gov.tr/math)

Volume 34 | Number 1 Article 2

1-1-2010

Sufficient conditions for univalence obtained by using second order linear strong differential subordinations

GEORGIA IRINA OROS

Turk J Math
34 (2010) , 13 – 20.

doi:10.3906/mat-0810-6

Sufficient conditions for univalence obtained by using second order linear strong differential subordinations

Georgia Irina Oros

Abstract

The concept of differential subordination was introduced in [3] by S.S. Miller and P.T. Mocanu and the concept of strong differential subordination was introduced in [1], [2] by J.A. Antonino and S. Romaguera. In [5] we have studied the strong differential subordinations in the general case and in [6] we have studied the first order linear strong differential subordinations. In this paper we study the second order linear strong differential subordinations. Our results may be applied to deduce sufficient conditions for univalence in the unit disc, such as starlikeness, convexity, alpha-convexity, close-to-convexity respectively.

Key Words: Analytic function, differential subordination, strong differential subordination, linear strong differential subordinations, second order linear strong differential subordinations.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of analytic functions in U. For a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H};\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U\}.$$

Let A be the class of functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad z \in U,$$

which are analytic in the unit disk.

In addition, we need the classes of convex, alpha-convex, close-to-convex and starlike (univalent) functions given respectively by

$$K = \left\{ f \in A;\ \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0,\ z \in U \right\},$$

$$M_\alpha = \left\{ f \in A,\ \frac{f(z)f'(z)}{z} \neq 0, \right.$$

$$\left. \mathrm{Re}\ (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in U \right\}$$

2000 *AMS Mathematics Subject Classification:* 30C45, 34A30.

$$C = \{f \in A,\ \text{Re}\ f'(z) > 0,\ z \in U\},$$

and

$$S^* = \{f \in A,\ \text{Re}\ zf'(z)/f(z) > 0\}.$$

In order to prove our main results we use the following definitions and lemmas.

Definition 1 [1], *[2] Let $H(z,\xi)$ be analytic in $U \times \overline{U}$ and let $f(z)$ analytic and univalent in U. The function $H(z,\xi)$ is strongly subordinate to $f(z)$, written $H(z,\xi) \prec\prec f(z)$ if for each $\xi \in \overline{U}$, the function of z, $H(z,\xi)$ is subordinate to $f(z)$.*

Remark 1 (i) Since $f(z)$ is analytic and univalent, Definition 1 is equivalent to

$$H(0,\xi) = f(0) \text{ and } H(U \times \overline{U}) \subset f(U).$$

(ii) If $H(z,\xi) \equiv H(z)$, then the strong subordination becomes the usual notion of subordination.

Definition 2 [3], *[4, Definition 2.26 p.21] We denote by Q the set of functions q that are analytic and injective in $\overline{U} \setminus E(q)$, where*

$$E(q) = \left\{\zeta \in \partial U;\ \lim_{z \to \zeta} q(z) = \infty\right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Lemma A [4, Lemma 2.2.d, p.24] *Let $q \in Q(a)$, with $q(0) = a$ and $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ be analytic in U, with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,*

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, and

(iii) $\text{Re}\ \dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\text{Re}\ \left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right].$

Definition 3 [*5, Definition 4*] *Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r,s,t;z,\xi) \notin \Omega \tag{A}$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$,

$$\text{Re}\ \frac{t}{s} + 1 \geq m\text{Re}\ \left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right], \quad z \in U,\ \zeta \in \partial U \setminus E(q)$$

and $m \geq n$.

Remark 2 For the function $q(z) = Mz$, $M > 0$, $z \in U$, the condition of admissibility (A) becomes

$$\psi(Me^{i\theta}, Ke^{i\theta}, L; z, \xi) \notin \Omega \tag{A'}$$

whenever $K \geq nM$, Re $[Le^{-i\theta}] \geq (n-1)K$, $z \in U$, $\xi \in \overline{U}$ and $\theta \in \mathbb{R}$.

For the function $q(z) = \dfrac{1+z}{1-z}$, $z \in U$, the condition of admissibility (A) becomes

$$\psi(\rho i, \sigma, \mu + \nu i; z, \xi) \notin \Omega \tag{A''}$$

whenever $\rho, \sigma, \mu, \nu \in \mathbb{R}$, $\sigma \leq -\dfrac{n}{2}[1+\rho^2]$, $\sigma + \mu \leq 0$, $z \in U$, $\xi \in \overline{U}$, and $n \geq 1$.

2. Main results

Definition 4 *A strong differential subordination of the form*

$$A(z,\xi)z^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi) \prec\prec h(z),$$

where $A, B, C, D : U \times \overline{U} \to \mathbb{C}$, $A(z,\xi)z^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)$ *is an analytic function of* z *for all* $\xi \in \overline{U}$ *and function* h *is analytic and univalent in* U, *is called second order linear strong differential subordination.*

Remark 3 (i) For $A(z,\xi) \equiv 0$ the second order linear strong differential subordination reduces to the first order linear strong differential subordination studied in [6].

(ii) For

$$A(z,\xi) = A(z), \quad B(z,\xi) = B(z),$$
$$C(z,\xi) = C(z), \quad D(z,\xi) = D(z)$$

the second order linear strong differential subordination reduces to the second order linear differential subordination studied in [4, Chapter 4].

Theorem 1 *Let* $A, B, C, D : U \times \overline{U} \to \mathbb{C}$, $A(z,\xi)z^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)$ *a function of* z, *analytic for all* $\xi \in \overline{U}$ *with*

$$A(z,\xi) = A > 0, \quad D(0,\xi) = 0, \quad \text{Re } B(z,\xi) \geq 0 \tag{1}$$

$$(n-1)nA + n\text{Re } B(z,\xi) + \text{Re } C(z,\xi) \geq 1 + \frac{|D(z,\xi)|}{M}, \quad z \in U, \ \xi \in \overline{U}.$$

If $p \in \mathcal{H}[0,n]$ *and the strong differential subordination*

$$Az^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi) \prec\prec Mz \tag{2}$$

holds, then

$$p(z) \prec Mz, \quad z \in U.$$

Proof. Let $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ denote the function

$$\psi(r, s, t; z, \xi) = At + B(z, \xi)s + C(z, \xi)r + D(z, \xi) \tag{3}$$

for $r = p(z)$, $s = zp'(z)$, $t = z^2 p''(z)$, $z \in U$, $\xi \in \overline{U}$.

Also, let $h(z) = Mz$, $z \in U$.

Then (2) becomes

$$\psi(r, s, t; z, \xi) \prec\prec h(z), \quad z \in U,\ \xi \in \overline{U}. \tag{4}$$

Since $h(U) = U(0, M)$, (4) implies

$$\psi(r, s, t; z, \xi) \in U(0, M), \quad z \in U,\ \xi \in \overline{U}. \tag{5}$$

Suppose that p is not subordinate to function h. Then, by Lemma A, we have that there exist $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$ and $\zeta_0 \in \partial U$ such that $p(z_0) = h(\zeta_0) = Me^{i\theta_0}$, $z_0 p'(z_0) = m\zeta_0 h'(\zeta_0) = Ke^{i\theta_0}$, $z_0^2 p''(z_0) = \zeta_0^2 h''(\zeta_0) = L$ with $K \geq nM$, Re $[Le^{-i\theta_0}] \geq (n-1)K$ where $z \in U$, $\theta_0 \in \mathbb{R}$.

Using the conditions given by (1), we calculate

$$|\psi(p(z_0), z_0 p'(z_0), z_0^2 p''(z_0); z_0, \xi)| \tag{6}$$

$$= |AL + B(z_0, \xi)Ke^{i\theta_0} + C(z_0, \xi)Me^{i\theta_0} + D(z_0, \xi)|$$

$$= |ALe^{-i\theta_0} + KB(z_0, \xi) + MC(z_0, \xi) + D(z_0, \xi)e^{-i\theta_0}|$$

$$\geq |ALe^{-i\theta_0} + KB(z_0, \xi) + MC(z_0, \xi)| - |D(z_0, \xi)|$$

$$\geq \text{Re } [ALe^{-i\theta_0} + KB(z_0, \xi) + MC(z_0, \xi)] - |D(z_0, \xi)|$$

$$\geq A\text{Re } Le^{-i\theta_0} + K\text{Re } B(z_0, \xi) + M\text{Re } C(z_0, \xi) - |D(z_0, \xi)|$$

$$\geq A(n-1)K + K\text{Re } B(z_0, \xi) + M\text{Re } C(z_0, \xi) - |D(z_0, \xi)|$$

$$\geq A(n-1)nM + nM\text{Re } B(z_0, \xi) + M\text{Re } C(z_0, \xi) - |D(z_0, \xi)|$$

$$\geq M[A(n-1)n + n\text{Re } B(z_0, \xi) + \text{Re } C(z_0, \xi)] - |D(z_0, \xi)| \geq M, \quad \xi \in \overline{U}.$$

Since (6) contradicts (5), we have that the assumption made is false and hence, $p(z) \prec Mz$, $z \in U$. □

Example 1 *Let*

$$A(z, \xi) = 5, \quad B(z, \xi) = z + 2\xi + 4 - 3i,$$

$$C(z, \xi) = -2 + \xi + 3 - i, \quad D(z, \xi) = 2z, \quad n = 3, \quad M = 2.$$

Since $z \in U$, $\xi \in \overline{U}$, *we have*

$$\text{Re } B(z, \xi) \geq 0, \quad \text{Re } C(z, \xi) \geq 0,$$

$$|D(z, \xi)| \leq 2, \quad 3\text{Re } B(z, \xi) + \text{Re } C(z, \xi) \geq -28.$$

From Theorem 1, we obtain:

If:

$$[5z^2p''(z)+(z+2\xi+4-3i)zp'(z)+(-z+\xi+3-i)p(z)+2z]$$

is a function of z, *analytic for all* $\xi\in\overline{U}$ *and*

$$5z^2p''(z)+(z+2\xi+4-3i)zp'(z)+(-z+\xi+3-i)p(z)+2z\prec\prec 2z$$

then

$$p(z)\prec 2z,\quad z\in U.$$

Theorem 2 *Let* $A,B,C,D:U\times\overline{U}\to\mathbb{C}$, $A(z,\xi)z^2p''(z)+B(z,\xi)zp'(z)+C(z,\xi)p(z)+D(z,\xi)$ *a function of* z, *analytic for all* $\xi\in\overline{U}$ *with*

$$A(z,\xi)=A>0,\quad \text{Re } B(z,\xi)\ge A, \tag{7}$$

$$C(0,\xi)+D(0,\xi)=1,\quad \frac{n}{2}[\text{Re } B(z,\xi)-A]\ge \text{Re } D(z,\xi),$$

$$\text{Im } C(z,\xi)\le\sqrt{n[\text{Re } B(z,\xi)-A][n\text{Re } B(z,\xi)-nA-2\text{Re } D(z,\xi)]},$$

$z\in U,\ \xi\in\overline{U}$.

If $p\in\mathcal{H}[1,n]$ *and the following strong differential subordination holds*

$$Az^2p''(z)+B(z,\xi)zp'(z)+C(z,\xi)p(z)+D(z,\xi)\prec\prec\frac{1+z}{1-z}, \tag{8}$$

$z\in U,\ \xi\in\overline{U}$, *then*

$$p(z)\prec\frac{1+z}{1-z},\quad z\in U.$$

Proof. Let $\psi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ denote the function

$$\psi(r,s,t;z,\xi)=At+B(z,\xi)s+C(z,\xi)r+D(z,\xi) \tag{9}$$

for $r=p(z)$, $s=zp'(z)$, $t=z^2p''(z)$. Also, let $h(z)=\dfrac{1+z}{1-z}$.

Then (8) becomes

$$\psi(r,s,t;z,\xi)\prec\prec h(z),\quad z\in U,\ \xi\in\overline{U}. \tag{10}$$

Since $h(U)=\{w\in\mathbb{C};\ \text{Re } w>0\}$, (10) becomes

$$\text{Re } \psi(r,s,t;z,\xi)>0,\quad z\in U,\ \xi\in\overline{U}. \tag{11}$$

Suppose p is not subordinate to $h(z)=\dfrac{1+z}{1-z}$, $z\in U$. Then, from Lemma A, we have that there exist $z_0\in U$, $z_0=r_0e^{i\theta_0}$, $\theta_0\in\mathbb{R}$ and $\zeta_0\in\partial U$, such that $p(z_0)=h(\zeta_0)=\rho i$, $\rho\in\mathbb{R}$, $z_0p'(z_0)=m\zeta_0h'(\zeta_0)=\sigma$, $\sigma\in\mathbb{R}$, $z_0^2p''(z_0)=\zeta_0^2h''(\zeta_0)=\mu+i\nu$, $\mu,\nu\in\mathbb{R}$ with $\sigma\le-\dfrac{n}{2}(1+\rho^2)$ and $\sigma+\mu\le 0$, $m\ge n\ge 1$.

Using the conditions given by (7), we calculate

$$\operatorname{Re}\,\psi(p(z_0), z_0p'(z_0), z_0^2p''(z_0); z_0, \xi) = \tag{12}$$

$$= \operatorname{Re}\,[A(\mu + i\nu) + B(z_0,\xi)\sigma + C(z_0,\xi)\rho i + D(z_0,\xi)]$$

$$= A\mu + \sigma\operatorname{Re}\,B(z_0,\xi) - \rho\operatorname{Im}\,C(z_0,\xi) + \operatorname{Re}\,D(z_0,\xi)$$

$$\le -\sigma A + \sigma\operatorname{Re}\,B(z_0,\xi) - \rho\operatorname{Im}\,C(z_0,\xi) + \operatorname{Re}\,D(z_0,\xi)$$

$$\le \sigma[\operatorname{Re}\,B(z_0,\xi) - A] - \rho\operatorname{Im}\,C(z_0,\xi) + \operatorname{Re}\,D(z_0,\xi)$$

$$\le -\frac{n}{2}(1+\rho^2)[\operatorname{Re}\,B(z_0,\xi) - A] - \rho\operatorname{Im}\,C(z_0,\xi) + \operatorname{Re}\,D(z_0,\xi)$$

$$\le -\rho^2\frac{n}{2}[\operatorname{Re}\,B(z_0,\xi) - A] - \rho\operatorname{Im}\,C(z_0,\xi)$$

$$-\left[\frac{n}{2}\operatorname{Re}\,B(z_0,\xi) - \frac{n}{2}A - \operatorname{Re}\,D(z_0,\xi)\right] \le 0$$

Since (12) contradicts (11), we have that the assumption made is false and hence,

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in U.$$

□

Example 2 *Let*

$$A(z,\xi) = 2, \quad B(z,\xi) = -3z + 2\xi + 10 - i,$$

$$C(z,\xi) = z + \xi - 2 - 4i, \quad D(z,\xi) = \frac{z}{2} - \xi + 3 + 4i, \quad n = 3.$$

Since $z \in U$, $\xi \in \overline{U}$, *we have*

$$\operatorname{Re}\,B(z,\xi) \ge 2, \quad \frac{3}{2}[\operatorname{Re}\,B(z,\xi) - 2] \ge \operatorname{Re}\,D(z,\xi),$$

$$C(0,\xi) + D(0,\xi) = 1, \quad -6 \le \operatorname{Im}\,C(z,\xi) \le -2.$$

From Theorem 2, we obtain:

If

$$[2z^2p''(z) + (-3z + 2\xi + 10 - i)zp'(z) + (z + \xi - 2 - 4i)p(z) + \frac{z}{2} - \xi + 3 + 4i]$$

is a function of z, *analytic for all* $\xi \in \overline{U}$ *and*

$$[2z^2p''(z) + (-3z + 2\xi + 10 - i)zp'(z) + (z + \xi - 2 - 4i)p(z)$$

$$+\frac{z}{2} - \xi + 3 + 4i] \prec\prec \frac{1+z}{1-z}, \quad z \in U,\ \xi \in \overline{U}$$

then

$$p(z) \prec \frac{1+z}{1-z}, \quad z \in U.$$

Remark 4 Theorem 2 can be rewritten as the following corollary:

Corollary 1 *Let* $A, B, C, D : U \times \overline{U} \to \mathbb{C}$,

$$Az^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)$$

a function of z, *analytic for all* $\xi \in \overline{U}$ *with*

$$A(z,\xi) = A > 0, \quad \text{Re } B(z,\xi) \geq A,$$

$$\frac{n}{2}[\text{Re } B(z,\xi) - A] \geq \text{Re } D(z,\xi),$$

$$\text{Im } C(z,\xi) \leq \sqrt{n[\text{Re } B(z,\xi) - A][n\text{Re } B(z,\xi) - nA - 2\text{Re } D(z,\xi)]},$$

$z \in U, \ \xi \in \overline{U}$.

If $p \in \mathcal{H}[1,n]$ *and satisfies the inequality*

$$\text{Re } [Az^2p''(z) + B(z,\xi)zp'(z) + C(z,\xi)p(z) + D(z,\xi)] > 0$$

then

$$\text{Re } p(z) > 0, \quad z \in U.$$

□

Remark 5 Note that the result contained in Theorem 2 can be applied to obtain sufficient conditions for univalence on the unit disc, such as starlikeness, convexity, alpha-convexity, close-to-convexity. Indeed, it suffices to consider

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(z) = 1 + \frac{zf''(z)}{f'(z)},$$

$$p(z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right],$$

and $p(z) = f'(z)$, $z \in U$ respectively.

Example 3 *Let*

$$A(z,\xi) = 1, \quad B(z,\xi) = -z - \xi + 6 - 5i,$$

$$C(z,\xi) = 2z - 3\xi - 5 - 8i, \quad D(z,\xi) = z + 3\xi + 6 + 8i, \quad n = 2.$$

Since $z \in U$, $\xi \in \overline{U}$, *we have*

$$\text{Re } B(z,\xi) \geq 4, \quad [\text{Re } B(z,\xi) - 2] \geq \text{Re } D(z,\xi), \quad C(0,\xi) + D(0,\xi) = 1,$$

$$-13 \leq \text{Im } C(z,\xi) \leq -3.$$

From Corollary 1, we obtain:

If

$$[z^2p''(z)+(-z-\xi+6-5i)zp'(z)+(2z-3\xi-5-8i)p(z)+z+3\xi+6+8i]$$

is a function of z*, analytic for all* $\xi\in\overline{U}$ *and*

$$\text{Re}\,[z^2p''(z)+(-z-\xi+6-5i)zp'(z)+(2z-3\xi-5-8i)p(z)$$

$$+z+3\xi+6+8i]>0,\quad z\in U,\ \xi\in\overline{U},$$

then

$$\text{Re}\,p(z)>0,\quad z\in U.$$

References

[1] Antonino, J.A., Romaguera, S.: Strong differential subordination to Briot-Bouquet differential equations. Journal of Differential Equations 114, 101-105 (1994).

[2] Antonino, J.A.: Strong differential subordination and applications to univalency conditions. J. Korean Math. Soc. 43, no.2, 311-322, (2006).

[3] Miller, S.S., Mocanu, P.T.: Differential subordinations and univalent functions. Michigan Math. J. 28, no.2, 157-172, (1981).

[4] Miller, S.S., Mocanu, P.T.: Differential subordinations. Theory and applications. Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2000.

[5] Oros, G.I., Oros, Gh.: Strong differential subordination, Turk. J. Math. (to appear).

[6] Oros, G.I., Oros, Gh.: First order linear strong differential subordinations. General Mathematics vol.15, Nr. 2-3, 98-107, (2007).

Georgia Irina OROS
Department of Mathematics
University of Oradea
Str. Universităţii, No.1
410087 Oradea, ROMANIA
e-mail: georgia_oros_ro@yahoo.co.uk

Received 07.10.2008