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Warped product semi-slant submanifolds in Kenmotsu manifolds

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Abstract

In this paper, we research the existence or non-existence of warped product semi-slant submanifolds in Kenmotsu manifolds. Consequently, we see that there are no proper warped product semi-slant submanifolds in Kenmotsu manifolds such that totally geodesic and totally umbilical submanifolds of warped product are proper semi-slant and invariant (or anti-invariant), respectively.

Key Words: Warped product, Slant submanifold and Kenmotsu manifold.

1. Introduction

In [2], the notion of warped product manifolds was introduced by Bishop and O'Neill in 1969 and it was studied by many mathematicians and physicists. These manifolds are generalization of Riemannian product manifolds.

Also, the notion of slant submanifolds in a complex manifold was defined and studied by B-Y. Chen as a natural generalization of both invariant and anti-invariant submanifolds. Examples of slant submanifolds of complex Euclidean space C^2 and C^4 were given by B-Y. Chen[7]. Moreover, A. Lotta has defined and studied of slant immersions of a Riemannian manifold into an almost contact metric manifold and proved some properties of such immersions [12].

In [3, 4], Authors studied slant immersions in K-contact and Sasakian manifolds. They introduced many interesting examples of slant submanifolds in almost contact metric manifolds and Sasakian manifolds. They characterized slant submanifolds by means of the covariant derivative of the square of the tangent projection T over the submanifold of almost contact structure of a K-contact manifold.

In [9], Authors studied slant submanifolds of a Kenmotsu manifold and gave a necessary and sufficient condition for a 3-dimensional submanifold of a 5-dimensional Kenmotsu manifold to be minimal proper slant submanifold.

Recently, we have studied warped product semi-slant submanifolds in Locally Riemannian manifolds and characterized warped product semi-slant submanifolds in locally Riemannian product manifolds [1].

The geometry of warped product semi-slant submanifolds are very important subject in geometry because every structure on a manifold may not admit warped product semi-slant submanifolds [see Theorem 4.1 and Theorem 4.2].

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2. Preliminaries

In this section we review basic formulas and definitions for almost contact metric manifolds and their submanifolds, which we shall use later.

Let M be a $2m+1$-dimensional almost contact metric manifold with structure tensor (φ, ξ, η, g), where φ is a $(1,1)$-type tensor, ξ is a vector field and η is a 1-form on M such that

$$\begin{aligned} &\varphi^2 X = -X + \eta(X)\xi, \quad \varphi(\xi) = 1, \quad \eta(\varphi X) = 0 \\ &g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{1}$$

for any vector fields X and Y on M. An almost contact metric manifold is called Kenmotsu manifold if

$$(\bar{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \bar{\nabla}_X \xi = X - \eta(X)\xi, \tag{2}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on M [9].

Now, let N be an immersed submanifold in M. We denote the induced metric on N by g. $TN^{\perp}$ is the set of all vector fields normal to N in M. Also, we denote by ∇ the Levi-Civita connection on N. The Gauss and Weingarten formulas are, respectively, given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{3}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{4}$$

for any vector fields X and Y tangent to N and V normal to N, where $\nabla^{\perp}$ is the connection in the normal bundle, h is the second fundamental form of N and A_V is the Weingarten endomorphism associated with V. The second fundamental form h and the shape operator A are related by

$$g(h(X,Y), V) = g(A_V X, Y) \tag{5}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$ [13].

3. Warped product manifolds

B.Y. Chen studied warped product CR-submanifolds in Kaehler manifolds and introduced the notion of CR-warped product [5, 6]. After that I. Hasegawa and I. Mihai studied contact CR-warped product submanifolds in Sasakian manifolds [10]. In this paper we studied warped product semi-slant submanifolds of Kenmotsu manifolds which is more general.

The study of warped product manifolds was initiated by R.L. Bishop and B. O'Neill [2]. They defined these as follows.

Definition 3.1 *Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and f be a positive definite differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where*

$$g = g_1 + f^2 g_2.$$

More explicitly, if vector fields X and Y tangent to $N_1 \times_f N_2$ at (x, y), then

$$g(X, Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + f^2(x)g_2(\pi_{2*}X, \pi_{2*}Y),$$

where $\pi_i (i = 1, 2)$ are the canonical projections of $N_1 \times N_2$ onto N_1 and N_2, respectively, and $*$ stands for derivative map

Let $N = N_1 \times_f N_2$ be warped product manifold. This means that N_1 and N_2 are totally geodesic and totally umbilical submanifolds of N, respectively.

For warped product manifolds, we have the following proposition [2].

Proposition 3.1 *Let $N = N_1 \times_f N_2$ be a warped product manifold. Then*
1) $\nabla_X Y \in \Gamma(TN_1)$ is the lift of $\nabla_X Y$ on N_1
2) $\nabla_U X = \nabla_X U = X(\ln f)U$
3) $\nabla_U V = \nabla'_U V - g(U, V)\nabla \ln f$
for any $X, Y \in \Gamma(TN_1)$ and $U, V \in \Gamma(TN_2)$, where ∇ and ∇' denote the Levi-Civita connections on N and N_2, respectively.

Throughout this paper, let us suppose that M be a Kenmotsu manifold and $N_1 \times_f N_2$ be a warped product semi-slant submanifolds of a Kenmotsu manifold M. Such submanifolds are always tangent to the structure vector field ξ. If the manifolds N_θ and N_T (resp. $N_\perp$) are slant and invariant(resp. anti-invariant) submanifolds of a Kenmotsu manifold M, then their warped product semi-slant submanifolds may be given by one of the following forms:
1) $N_\theta \times_f N_T$
2) $N_\theta \times_f N_\perp$
3) $N_T \times_f N_\theta$
4) $N_\perp \times_f N_\theta$.

In this paper we are concerned with cases 1) and 2) and are left with the last two cases. cases 3) and 4) are out of the scope of this paper because they contain other search fields under some conditions.

Now let N be an immersed submanifold of Kenmotsu manifold M and we denote the orthogonal complementary of $\varphi(TN)$ in TN by ν. Then we have the direct sum

$$TN^\perp = \varphi(TN) \oplus \nu. \tag{6}$$

We can easily see that ν is an invariant subbundle with respect to φ. Furthermore, for any nonzero vector Z tangent to N, we put

$$\varphi Z = tZ + nZ, \tag{7}$$

where tZ and nZ denote the tangential and normal components of φZ, respectively.

The submanifold N is said to be invariant (resp. anti invariant) if n (resp. t) is identically zero.

Furthermore, for submanifolds tangent to the structure vector field ξ, there is another class of submanifolds which is called slant submanifold. For each nonzero vector Z tangent to N at x, the angle $\theta(x)$,

$0 \leq \theta(x) \leq \pi/2$, between φZ and T_xN is called the slant angle. If the slant angle is constant, then the submanifold is also called the slant submanifold. Invariant and anti-invariant submanifolds are particular slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold is said to be proper slant if it is neither invariant nor anti-invariant submanifold [8].

In the same way, for any vector V normal to N, we put

$$\varphi V = BV + CV, \tag{8}$$

where BV and CV denote the tangential and normal components of φV, respectively. If t and n are the endomorphism defined by (7), then

$$g(tZ, W) + g(Z, tW) = 0 \tag{9}$$

$$g(nZ, W) + g(Z, nW) = 0, \tag{10}$$

for any $Z, W \in \Gamma(TN)$. On the other hand, making use of Gauss and Weingarten formulas with (2), (7) and (8), we have

$$(\nabla_Z n)W = Ch(Z, W) - h(Z, tW) - \eta(W)nZ \tag{11}$$

$$(\nabla_Z t)W = A_{nW}Z + Bh(Z, W) + g(\varphi Z, W)\xi - \eta(W)tZ \tag{12}$$

for any $Z, W \in \Gamma(TN)$, where the covariant derivatives of n and t are, respectively, defined by

$$(\nabla_Z n)W = \nabla^{\perp}_Z nW - n(\nabla_Z W) \quad \text{and} \quad (\nabla_Z t)W = \nabla_Z tW - t(\nabla_Z W). \tag{13}$$

We recall the following results from [9] for later use.

Theorem 3.1 *Let N be submanifold of a Kenmotsu manifold M such that ξ is tangent to N. Then N is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$t^2 = \lambda(-I + \eta \otimes \xi). \tag{14}$$

Furthermore, if θ is the slant angle of N, then $\lambda = \cos^2\theta$.

Corollary 3.2 *Let N be a slant submanifold with slant angle θ of a Kenmotsu manifold M such that ξ is tangent to N. Then we have*

$$g(tZ, tW) = \cos^2\theta\{g(Z, W) - \eta(Z)\eta(W)\} \tag{15}$$

$$g(nZ, nW) = \sin^2\theta\{g(Z, W) - \eta(Z)\eta(W)\} \tag{16}$$

for any $Z, W \in \Gamma(TN)$.

In the following section, we shall investigate warped product semi-slant submanifolds in Kenmotsu manifolds.

4. Warped product semi-slant submanifolds of a Kenmotsu manifold

Theorem 4.1 *There do not exist proper warped product semi-slant submanifolds $N = N_\theta \times_f N_T$ in a Kenmotsu manifold M such that N_θ is a proper slant submanifold, N_T is an invariant submanifold of M and ξ is tangent to N.*

Proof. Let $N = N_\theta \times_f N_T$ be a proper warped product semi-slant submanifold of a Kenmotsu manifold M. For any $X, Y \in \Gamma(TN_\theta)$ and $U, V \in \Gamma(TN_T)$,

$$\begin{aligned}(\bar{\nabla}_X \varphi)U &= \bar{\nabla}_X \varphi U - \varphi(\bar{\nabla}_X U) \\ g(\varphi X, U)\xi - \eta(U)\varphi X &= h(X, tU) - Bh(X, U) - Ch(X, U).\end{aligned} \tag{17}$$

From the tangential and normal components of (17), respectively, we get

$$g(\varphi X, U) = 0 \tag{18}$$

$$\eta(U)tX = Bh(X, U) \tag{19}$$

and

$$\eta(U)nX = Ch(X, U) - h(X, tU). \tag{20}$$

On the other hand, by interchanging roles of U and X in (17), we conclude

$$tX \log(f)U = A_{nX}U + X \log(f)tU + Bh(X, U) - \eta(X)tU \tag{21}$$

and

$$\nabla^\perp_U nX = Ch(X, U) - h(U, tX). \tag{22}$$

From (21), we reach

$$\begin{aligned}tX \log(f)g(U, U) &= g(A_{nX}U, U) + g(Bh(X, U), U) \\ &= g(h(U, U), nX) + g(Bh(X, U), U) \\ &= g(h(U, U), nX) - g(h(X, U), \varphi U) \\ &= g(h(U, U), nX).\end{aligned} \tag{23}$$

On the other hand, since the ambient space M is a Kenmotsu manifold, by using Gauss formulae and (2), we get

$$h(Z, \xi) = 0 \tag{24}$$

for any $Z \in \Gamma(TN)$. By using (20) and (22), we get $nX = Ch(X, \xi) = 0$. Thus we have $tX \log(f)g(U, U) = 0$ which implies that $tX \log(f) = 0$, that is, the warping function f is constant on N_θ. The proof is complete. □

Theorem 4.2 *There do not exist proper warped product semi-slant submanifolds $N = N_\theta \times_f N_\perp$ in a Kenmotsu manifold M such that N_θ is a proper slant submanifold, $N_\perp$ is an anti-invariant submanifold of M and ξ is tangent to N.*

Proof. Let $N = N_\theta \times_f N_\perp$ be a proper warped product semi-slant submanifold of a Kenmotsu manifold M such that ξ is tangent to N. For any $X, Y \in \Gamma(TN_\theta)$ and $U, V \in \Gamma(TN_\perp)$ we have

$$\begin{aligned}(\bar{\nabla}_X\varphi)U &= \bar{\nabla}_X\varphi U - \varphi(\bar{\nabla}_X U)\\ g(\varphi X, U)\xi - \eta(U)\varphi X &= -A_{nU}X + \nabla^\perp_X nU - X\log(f)nU\\ &- \varphi h(X,U).\end{aligned} \tag{25}$$

Considering the tangential and normal components of (25), respectively, we get

$$\eta(U)tX = A_{nU}X + Bh(X,U) \tag{26}$$

and

$$\eta(U)nX = X\log(f)nU + Ch(X,U) - \nabla^\perp_X nU. \tag{27}$$

By interchanging roles of X and U in (25), we reach

$$\begin{aligned}g(\varphi U, X)\xi - \eta(X)\varphi U &= tX\log(f)U + h(U,tX) - A_{nX}U + \nabla^\perp_U nX\\ &- X\log(f)nU - Bh(X,U) - Ch(X,U).\end{aligned} \tag{28}$$

From the tangential and normal components of (28), respectively, we have

$$tX\log(f)U = A_{nX}U + Bh(X,U) \tag{29}$$

and

$$-\eta(X)nU = -X\log(f)nU + h(U,tX) - Ch(U,X) + \nabla^\perp_U nX. \tag{30}$$

From (29), we have

$$g(A_{nX}U, tY) + g(Bh(X,U), tY) = 0. \tag{31}$$

Since the ambient space M is a Kenmotsu manifold, ξ is tangent to N and by using (1), we obtain

$$\begin{aligned}g(Bh(X,U), tY) &= g(\varphi h(X,U), \varphi Y)\\ &= g(h(X,U), Y) - \eta(Y)\eta(h(X,U)) = 0,\end{aligned}$$

that is,

$$g(Bh(X,U), tY) = g(h(U,tY), nX) = 0. \tag{32}$$

Thus we have

$$g(h(U,tY), \varphi X) = 0, \tag{33}$$

for any $X, Y \in \Gamma(TN_\theta)$. Moreover, making use of (26) and (33), we get

$$\begin{aligned}\eta(U)g(tX,tY) &= g(h(X,tY), nU) + g(Bh(X,U), tY)\\ &= g(h(X,tY), \varphi U).\end{aligned} \tag{34}$$

By using the Gauss-Weingarten formulas and considering N_θ is totally geodesic in N, we arrive at

$$\begin{aligned} g(h(X,tY),\varphi U) &= g(\bar{\nabla}_{tY}X,\varphi U) = -g(\varphi(\bar{\nabla}_{tY}X),U) \\ &= -g(\bar{\nabla}_{tY}\varphi X - (\bar{\nabla}_{tY}\varphi)X,U) \\ &= -g(\bar{\nabla}_{tY}tX,U) - g(\bar{\nabla}_{tY}nX,U) + g(g(\varphi tY,X)\xi \\ &- \eta(X)\varphi tY,U) \\ &= g(A_{nX}tY,U) - \eta(U)g(tY,tX) \\ &= g(h(tY,U),nX) - \eta(U)g(tY,tX) \\ &= -\eta(U)g(tY,tX). \end{aligned} \tag{35}$$

Thus from (34) and (35), we conclude

$$\eta(U)g(tX,tY) = g(h(X,tY),\varphi U) = 0. \tag{36}$$

Here, if $\eta(U) = 0$, then by using (11) and (27), we have

$$X\log(f)nU = \eta(\nabla_X U) = g(\nabla_X U,\xi) = -g(\nabla_X\xi,U) = -g(X-\eta(X)\xi,U) = 0.$$

This is impossible. Because U is a nonzero vector field and $N_\perp \neq 0$. Thus $g(tX,tY) = \cos^2\theta\{g(X,Y) - \eta(X)\eta(Y)\} = 0$, which implies that the slant angle θ is either identically $\pi/2$ or the warping function f is constant on N_θ . The proof is complete. □

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