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DJAVVAT KHADJIEV

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Djavvat Khadjiev

Abstract

The system of generators of the differential field of all G-invariant differential rational functions of a vector field in the n-dimensional Euclidean space R^n is described for groups $G = M(n)$ and $G = SM(n)$, where $M(n)$ is the group of all isometries of R^n and $SM(n)$ is the group of all euclidean motions of R^n. Using these results, vector field analogues of the first part of the Bonnet theorem for groups Aff(n), $M(n), SM(n)$ in R^n are obtained, where Aff(n) is the group of all affine transformations of R^n. These analogues are given in terms of the first fundamental form and Christoffel symbols of a vector field.

Key Words: Vector field; Christoffel symbol; Bonnet theorem; Differential invariant.

1. Introduction

Let $M(n)$ be the group of all isometries of the n-dimensional Euclidean space R^n, $O(n)$ be the group of all orthogonal transformations of R^n and $SM(n)$ be the subgroup of $M(n)$ generated by rotations and translations of R^n.

According to the Bonnet theorem (see [3], p. 49; [13], p. 19), if U and W are regular hypersurfaces in R^{n+1} such that $I(U) = I(W), II(U) = II(W)$, where I and II are the first and the second fundamental forms of a hypersurface, then there exists $F \in SM(n+1)$ such that $W = FU$ (the first part of the Bonnet theorem). The following vector field analogue of the Bonnet theorem in R^3 is given in ([2], pp. 69–71):

Let us be given for $G \subseteq R^3$ the functions A_{ik} and B_{ik} of Cartesian coordinates, where $A_{1k} = -A_{2k}, i = 1, 2; k = 1, 2, 3$. Suppose that the functions satisfy the following system:

$$\frac{\partial A_{ik}}{\partial x_l} - \frac{\partial A_{il}}{\partial x_k} + B_{il}B_{jk} - B_{ik}B_{jl} = 0,$$

$$\frac{\partial B_{ik}}{\partial x_l} - \frac{\partial B_{il}}{\partial x_k} + A_{ik}B_{jl} - A_{il}B_{jk} = 0.$$

Then there are the orthonormal vector fields $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}$ in G such that

$$\frac{\partial \boldsymbol{a}_i}{\partial x_k} = A_{ik}\boldsymbol{a}_j + B_{ik}\boldsymbol{n}, i \neq j,$$

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$$\frac{\partial \boldsymbol{n}}{\partial x_k} = -B_{1k}\boldsymbol{a}_1 - B_{2k}\boldsymbol{a}_2.$$

The vector fields $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}$ *are defined uniquely up to their choice at one point.*

We note that the part *"The vector fields* $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}$ *are defined uniquely up to their choice at one point."* of this theorem is not clear. Indeed, the functions $A_{1k} = (\frac{\partial \mathbf{a}_1}{\partial x_k}, \mathbf{a}_2), A_{2k} = (\frac{\partial \mathbf{a}_2}{\partial x_k}, \mathbf{a}_1), B_{ik} = -(\frac{\partial \mathbf{n}}{\partial x_k}, \mathbf{a}_i)$ are $O(3)$-invariant, but they are not invariant with respect to parallel translations in R^3. Hence the clear form of the part of this theorem is as follows: *"The vector fields* $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{n}$ *are defined uniquely up to an orthogonal transformation of* R^3*."* This means that the system of functions A_{ik}, B_{ik} is the complete system of joint $O(3)$-invariants of orthonormal vector fields $\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}$.

In the present paper, we give vector field analogues of the first part of the Bonnet theorem for groups $\mathrm{Aff}(n)$, $M(n), SM(n)$ in R^n, where $\mathrm{Aff}(n)$ is the group of all affine transformations of R^n. First we describe systems of generators of the differential field of all H-invariant differential rational functions of a vector field in R^n for groups $H = M(n)$ and $H = SM(n)$. Using these results, we prove vector field analogues of the first part of the Bonnet theorem for mentioned groups. These analogues are given in terms of the first fundamental form and Christoffel symbols of a vector field.

Let G be a group and $\alpha(G)$ be an action of G on the set of all smooth vector fields in R^n. Investigations of the problem of $\alpha(G)$-equivalence of vector fields, $\alpha(G)$-invariant vector fields and $\alpha(G)$-invariants of vector fields have important role in many areas of mathematics and mathematical physics.

Let ρ be a linear representation of a group G in R^n and x be a smooth vector field in R^n. Consider the action $\rho^*(g)(x(a)) = \rho(g)x(\rho(g^{-1})a)$ of G on the set of all smooth vector fields in R^n. The problem of describing of the general form of all $\rho^*(G)$-invariant (that is, equivariant) polynomial vector fields for a compact Lie groups G has been studied intensively in bifurcation theory [4, 6, 9]. By using the Theorems of Schwartz and Poe'naru ([9], Theorem XII4.3 and Theorem XII5.2), this problem reduces to an algebraic problem in invariant theory. The problem of $\rho^*(G)$-equivalence of smooth vector fields and complete systems of $\rho^*(G)$-invariants of polynomial vector fields are investigated in the theory of differential equations [15-17, 8]. Invariants of vector fields are studied also in differential geometry [1, 2, 7].

The present paper is organized as follows. In section 2, we give some known definitions and propositions, which we use in the next sections.

In section 3, we describe the system of generators of the differential field of all G-invariant differential rational functions of a vector field for groups $G = M(n)$ and $G = SM(n)$ (Theorems 1 and 2).

In section 4, using results of the section 3, we prove that: 1. the set of all Christoffel symbols of the second kind of a vector field is a complete system of $\mathrm{Aff}(n)$-invariants on the set of all regular vector fields (Theorem 3); 2. the set of all coefficients of the first fundamental form of a vector field is a complete system of $M(n)$-invariants on the set of all regular vector fields (Theorem 4); 3. the set of all Christoffel symbols of the first and second kinds of a vector field is a complete system of $M(n)$-invariants (Theorem 5). A similar result has been obtained for the group $SM(n)$ (Theorem 6). Theorems 3-6 are vector field analogues of the first part of the Bonnet theorem for groups $G = \mathrm{Aff}(n)$, $M(n), SM(n)$.

In this paper, we use methods of invariant theory. A similar approach to the theory of curves was used in the book [11] and papers [5, 12, 14].

2. Complete systems of invariants

Let A be a set, G be a group and α be an action of G on A. Elements $a,b\in A$ is called G-equivalent if there exists $q\in G$ such that $b=\alpha(q,a)$. In this case, we write $a\overset{G}{\sim}b$. Let K be a set. A function $h:A\to K$ is called G-invariant if $a,b\in A, a\overset{G}{\sim}b$ implies $h(a)=h(b)$. Denote by $\mathrm{Map}(A,K)^G$ the set of all G-invariant functions $h:A\to K$.

Definition 1 ([15], p.11) *A system $\{f_1,f_2,\dots,f_m\}$, where $f_i\in Map(A,K)^G$, will be called a complete system of G-invariants of the action α if $a,b\in A, f_i(a)=f_i(b)$ for all $i\in\{1,2,\dots,m\}$ imply $a\overset{G}{\sim}b$.*

Let $P=\{f_1,f_2,\dots,f_m\}\subset\mathrm{Map}(A,K)^G$. Denote by $\mathrm{Map}(A,K;P)$ the set of all $h:A\to K$ such that h is a function of the system P.

Proposition 1 *Let $P=\{f_1,f_2,\dots,f_m\}$ be a complete system of G-invariant functions on A. Then Map $(A,K)^G=$ $Map(A,K;P)$.*

Proof. Proof is given in ([15], p. 11, Theorem 1.1). □

Definition 2 ([15], p. 11) *A complete system $P=\{f_1,f_2,\dots,f_m\}$ of G-invariant functions will be called a minimal complete system if $P\setminus\{f_i\}$ is not complete for any $i\in\{1,2,\dots,m\}$.*

Proposition 2 *Let $P=\{f_1,f_2,\dots,f_m\}$ be a complete system, where $f_i\in Map(A,K)^G$. Then P is a minimal complete system if and only if $f_j\notin Map(A,K;P\setminus\{f_j\})$ for all $j=1,2,\cdots,m$.*

Proof. $\Rightarrow$. Assume that P is a minimal complete system and $f_j\in\mathrm{Map}(A,K;P\setminus\{f_j\})$ for some $j=k$. Since P is a minimal complete system, the subsystem $P\setminus\{f_k\}$ is not a complete system. Hence there exist $a,b\in A$ such that $f_i(a)=f_i(b)$ for all $i\in\{1,2,\dots,m\}\setminus\{k\}$ but a is not G-equivalent to b. Using $f_k\in\mathrm{Map}(A,K;P\setminus\{f_k\})$ and equalities $f_i(a)=f_i(b)$ for all $i\in\{1,2,\dots,m\}\setminus\{k\}$, we obtain that $f_k(a)=f_k(b)$. Then $f_i(a)=f_i(b)$ for all $i\in\{1,2,\dots,m\}$. Since P is a complete system, we obtain $a\overset{G}{\sim}b$. It is a contradiction. Therefore $f_j\notin\mathrm{Map}(A,K;P\setminus\{f_j\})$ for all $j=1,2,\cdots,m$.

$\Longleftarrow$. Assume that $f_j\notin\mathrm{Map}(A,K;P\setminus\{f_j\})$ for all $j=1,2,\cdots,m$ and P is not a minimal complete system. Then there exists $k\in\{1,2,\dots,m\}$ such that $P\setminus\{f_k\})$ is a complete system. Then, by Proposition 1, $f_k\in\mathrm{Map}(A,K;P\setminus\{f_k\})$. It is a contradiction. Hence P is a minimal complete system. □

3. Generating systems of differential fields of all G-invariant differential rational functions of a vector field

In the sequel, n is a natural number such that $n>1$. Let J be an open subset of R^n.

Definition 3 *A C^∞-mapping $x:J\to R^n$ is called a vector field in R^n.*

Let $GL(n)$ be the group of all non-degenerate real $n \times n$- matrices. Put $Aff(n)=\{F : R^n \to R^n \mid Fx = gx + b,\ g \in GL(n),\ b \in R^n\}$, where gx is the multiplication of a matrix g and a column vector $x \in R^n$. Let $O(n)$ be the group of all orthogonal real $n \times n$-matrices. Then $M(n)=\{F : R^n \to R^n \mid Fx = gx+b,\ g \in O(n),\ b \in R^n\}$ and $SM(n) = \{F \in M(n) : det g = 1\}$.

Let $x(u)$ be a vector field in R^n. Then $Fx(u)$ is also a J-vector field in R^n for all $F \in M(n)$. Let G be a subgroup of $Aff(n)$.

Definition 4 J-vector fields $x(u)$ and $y(u)$ in R^n is called G-equivalent if there exists $F \in G$ such that $y(u) = Fx(u)$ for all $u \in J$. In this case, it will be denoted by $x \overset{G}{\sim} y$.

Denote by N_0 the set of all non-negative integers. Let $x(u) = x(u_1, u_2, \dots, u_n)$ be a vector field in R^n. For $m_i \in N_0, 1 \le i \le n$, we put

$$x^{(0,0,\dots,0)} = x, \quad x^{(m_1,m_2,\dots,m_n)} = \frac{\partial^{m_1+m_2+\cdots+m_n} x}{\partial u_1^{m_1} u_2^{m_2} \cdots \partial u_n^{m_n}}.$$

Definition 5 *(See [10, 11].) A polynomial $q(x, x^{(1,0,0,\dots,0)}, x^{(0,1,0,\dots,0)}, \dots, x^{(m_1,m_2,m_3,\dots,m_n)})$ of x and a finite number of partial derivatives $x^{(1,0,0,\dots,0)}, x^{(0,1,0,\dots,0)}, \dots, x^{(m_1,m_2,m_3,\dots,m_n)}$ of x with coefficients from R is called a differential polynomial of x.*

Denote such polynomials by $q\{x\}$. The set of all differential polynomials of x will be denoted by $R\{x\}$. It is a differential R-algebra (see [10]) with respect to the differentiations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}$. This differential R-algebra is also an integral domain. The quotient field of it will be denoted by $R < x >$. It is a differential field (see [10]) with respect to the differentiations $\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n}$. An element h of $R < x >$ will be called a differential rational function of x and denoted by $h < x >$.

Let $x(u_1, u_2, \dots, u_n), y(u_1, u_2, \dots, u_n), \dots, z(u_1, u_2, \dots, u_n)$ be a finite number of vector fields in R^n. A differential polynomial and a differential rational function of vector fields $x, y, \dots, z$ are defined similarly. They will be denoted by $p\{x, y, \dots, z\}$ and $p < x, y, \dots, z >$, respectively. The differential field of all differential rational functions of $x, y, \dots, z$ will be denoted by $R < x, y, \dots, z >$.

Definition 6 *A differential rational function $h < x, y, \dots, z >$ is called G-invariant if $h < gx, gy, \dots, gz >= h < x, y, \dots, z >$ for all $g \in G$.*

The set of all G-invariant differential rational functions of $x, y, \dots, z$ will be denoted by $R < x, y, \dots, z >^G$. It is a differential subfield of the differential field $R < x, y, \dots, z >$.

Definition 7 *A subset S of $R < x, y, \dots z >^G$ is called a generating system of $R < x, y, \dots z >^G$ if the smallest differential subfield of it containing S is $R < x, y, \dots z >^G$.*

Let $(x, y) = \sum_{i=1}^{n} x_i y_i$ be the scalar product of vectors $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in R^n. So $(x^{(m_1,m_2,\dots,m_n)}, x^{(p_1,p_2,\dots,p_n)})$ is the scalar product of vectors $x^{(m_1,m_2,\dots,m_n)}$, $x^{(p_1,p_2,\dots,p_n)}$ in R^n.

Theorem 1 . *The system*

$$\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right), 1 \le i \le j \le n, \tag{1}$$

is a generating system of the differential field $R < x >^{M(n)}$.

Proof. First we prove several lemmas. Let $R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >$ be the differential field of all differential rational functions of $\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x$ and $O(n)$ is the group of all $n \times n$-orthogonal real matrices.

Lemma 1 $R < x >^{M(n)} = R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{O(n)}$.

Proof. Let $q < x >= q(x, \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x, \ldots, x^{(m_1,m_2,m_3,\ldots,m_n)}) \in R < x >^{M(n)}$. Then it is invariant with respect to translations in R^n. This implies that

$$q < x >= q(\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x, \ldots, x^{(m_1,m_2,m_3,\ldots,m_n)}) = q < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x > .$$

It is also $O(n)$-invariant. Hence it is an $O(n)$-invariant differential rational function of $\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x$. Conversely, assume that q is a $O(n)$-invariant differential rational function of $\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x$. Then it is invariant with respect to translations in R^n. Hence it is $M(n)$-invariant. □

Lemma 2 *Let* $f \in R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{O(n)}$. *Then there exist* $O(n)$*-invariant differential polynomials* f_1, f_2 *such that* $f = f_1/f_2$.

Proof. Proof is similar to the proof in ([11], p. 106). □

Lemma 3 *The system of all elements* $(x^{(m_1,m_2,\ldots,m_n)}, x^{(p_1,p_2,\ldots,p_n)})$, *where* $m_1 + m_2 + \cdots + m_n \ge 1, p_1 + p_2 + \cdots + p_n \ge 1, m_i \in N_0, p_i \in N_0$, *is a generating system of* $R < x >^{M(n)}$ *as a field.*

Proof. Let $R[x^{(m_1,m_2,\ldots,m_n)}, m_i \in N_0]^{O(n)}$ be the R-algebra of all $O(n)$-invariant polynomials of the system $\left\{x^{(m_1,m_2,\ldots,m_n)}\right\}$, where $m_i \in N_0, m_1 + m_2 + \cdots + m_n \ge 1$. It is obvious that $R[x^{(m_1,m_2,\ldots,m_n)}, m_i \in N_0]^{O(n)} = R\left\{\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x\right\}^{O(n)}$. According to the First Main Theorem for $O(n)$ (see [19], p. 53), the system $\left\{(x^{(m_1,m_2,\ldots,m_n)}, x^{(p_1,p_2,\ldots,p_n)})\right\}$, where $m_1 + m_2 + \cdots + m_n \ge 1, p_1 + p_2 + \cdots + p_n \ge 1, m_i \in N_0, p_i \in N_0$, is a generating system of the R-algebra $R[x^{(m_1,m_2,\ldots,m_n)}, m_i \in N_0]^{O(n)} = R\left\{\frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x\right\}^{O(n)}$. Using Lemmas 1 and 2, we obtain that the system $\left\{(x^{(m_1,m_2,\ldots,m_n)}, x^{(p_1,p_2,\ldots,p_n)})\right\}$, where $m_1+m_2+\cdots+m_n \ge 1, p_1+p_2 + \cdots + p_n \ge 1, m_i \in N_0, p_i \in N_0$, is a generating system of $R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{O(n)} = R < x >^{M(n)}$ as a field. □

Denote by $\Delta = \Delta_x$ the determinant $\det \left\| (\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) \right\|_{i,j=1,2,\ldots,n}$. Let V be the system Eq.(1). Denote by $R\{V\}$ the differential R-subalgebra of $R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{O(n)}$ generated by elements of V.

Lemma 4 $\Delta \in R\{V\}$.

Proof. By the definition of V, $(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) \in V$ for all $1 \leq i, j \leq n$. Hence $\Delta \in R\{V\}$. □

Denote by $R\left\{V, \Delta^{-1}\right\}$ the differential R-subalgebra of $R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{O(n)}$ generated by elements of the system V and the function Δ^{-1}. According to Lemmas 1 and 3, for a proof of our theorem, it is enough to prove that $(x^{(m_1,m_2,\ldots,m_n)}, x^{(p_1,p_2,\ldots,p_n)}) \in R\left\{V, \Delta^{-1}\right\}$ for all $m_i, p_i \in N_0$ such that $m_1 + m_2 + \cdots + m_n \geq 1$ and $p_1 + p_2 + \cdots + p_n \geq 1$.

Denote by $\mathrm{Gr}(y_1, \ldots, y_m; z_1, \ldots, z_m)$ the matrix $||(y_i, z_j)||_{i,j=1,2,\ldots,m}$ of vectors $y_1, \ldots, y_m,\ z_1, \ldots, z_m$ in R^n. Let $\det\mathrm{Gr}(y_1, y_2, \ldots, y_m; z_1, z_2, \ldots, z_m)$ be the determinant of the matrix $\mathrm{Gr}(y_1, y_2, \ldots, y_m; z_1, z_2, \ldots, z_m)$. The following is known.

Lemma 5 *The equality*

$$detGr(y_1, y_2, \ldots, y_{n+1}; z_1, z_2, \ldots, z_{n+1}) = det|| < y_i, z_j > ||_{i,j=1,2,\ldots,n+1} = 0$$

holds for all vectors $y_1, y_2, \ldots, y_{n+1}, z_1, z_2, \ldots, z_{n+1}$ *in* R^n.

Proof. A proof is given in ([11], p. 106–107; [19], p. 75). □

Lemma 6 *Let* $x^{(b_1,b_2,\ldots,b_n)}$ *and* $x^{(c_1,c_2,\ldots,c_n)}$ *be elements such that* $1 \leq b_1 + b_2 + \cdots + b_n, 1 \leq c_1 + c_2 + \cdots + c_n$, $(x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_i}x) \in R\left\{V, \Delta^{-1}\right\}$ *and* $(x^{(c_1,c_2,\ldots,c_n)}, \frac{\partial}{\partial u_i}x) \in R\left\{V, \Delta^{-1}\right\}$ *for all* $1 \leq i \leq n$. *Then* $(x^{(b_1,b_2,\ldots,b_n)}, x^{(c_1,c_2,\ldots,c_n)}) \in R\left\{V, \Delta^{-1}\right\}$.

Proof. Applying Lemma 5 to vectors

$$y_1 = z_1 = \frac{\partial}{\partial u_1}x, y_2 = z_2 = \frac{\partial}{\partial u_2}x, \cdots, y_n = z_n = \frac{\partial}{\partial u_n}x, y_{n+1} = x^{(b_1,b_2,\ldots,b_n)}, z_{n+1} = x^{(c_1,c_2,\ldots,c_n)},$$

we obtain the equality $\det A = 0$, where

$$A = \left\|(y_i, z_j)\right\|_{i,j=1,2,\ldots,n+1}.$$

Let D_{n+1j} be the cofactor of the element (y_{n+1}, z_j) of the matrix A for $j = 1, 2, \ldots, n+1$. The equality $\det A = 0$ implies the equality

$$(y_{n+1}, z_1)D_{n+11} + (y_{n+1}, z_2)D_{n+12} + \cdots + (y_{n+1}, z_n)D_{n+1n} + (y_{n+1}, z_{n+1})D_{n+1n+1} = 0. \tag{2}$$

Since $\Delta = D_{n+1n+1}$, Eq. (2) implies the equality

$$(y_{n+1}, z_{n+1}) = (x^{(b_1,b_2,\ldots,b_n)}, x^{(c_1,c_2,\ldots,c_n)}) = -\frac{(y_{n+1}, z_1)D_{n+11} + (y_{n+1}, z_2)D_{n+12} + \cdots + (y_{n+1}, z_n)D_{n+1n}}{\Delta}. \tag{3}$$

In Eq. (3), by the supposition of the lemma, $(y_{n+1}, z_j) = (x^{(b_1,b_2,\dots,b_n)}, \frac{\partial}{\partial u_j}x) \in R\left\{V, \Delta^{-1}\right\}$ for every $j : 1 \le j \le n$. We prove that $D_{n+1s} \in R\left\{V, \Delta^{-1}\right\}$ for every $s : 1 \le s \le n$. We have $D_{n+1s} = (-1)^{n+1+s}$ $\det\mathrm{Gr}(y_1, y_2, \dots, y_n; z_1, z_2, \ \dots, z_{s-1}, z_{s+1}, \dots, z_n, z_{n+1}))$. By the definition of V, we obtain $(y_i, z_j) \in V \subset R\{V\}$ for all $i, j : 1 \le i, j \le n$. By the supposition of our lemma, we have $(y_i, z_{n+1}) = (\frac{\partial}{\partial u_i}x, x^{(c_1,c_2,\dots,c_n)}) \in R\left\{V, \Delta^{-1}\right\}$ for every $i : 1 \le i \le n$. Hence $D_{n+1s} \in R\left\{V, \Delta^{-1}\right\}$ for every $s : 1 \le s \le n$ and Eq.(3) implies $(y_{n+1}, z_{n+1}) \in R\left\{V, \Delta^{-1}\right\}$. $\square$

Lemma 7 $(\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x) \in R\left\{V, \Delta^{-1}\right\}$ *for all* $i, j, l \in \{1, 2, \dots, n\}$.

Proof. We have

$$\frac{\partial}{\partial u_j}(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_i}x) = 2(\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_i}x)$$

for all $i, j \in \{1, 2, \dots, n\}$. This equality implies $(\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_i}x) \in R\left\{V, \Delta^{-1}\right\}$ for all $i, j \in \{1, 2, \dots, n\}$. Using $(\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_i}x) \in R\left\{V, \Delta^{-1}\right\}$ and the equality

$$\frac{\partial}{\partial u_i}(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) = (\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) + (\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x),$$

we obtain $(\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) \in R\left\{V, \Delta^{-1}\right\}$ for all $i, j \in \{1, 2, \dots, n\}$. Assume that $i \ne j, i \ne l, j \ne l$. We have

$$\left.\begin{aligned}
\frac{\partial}{\partial u_j}(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_l}x) &= (\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x) + (\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}\frac{\partial}{\partial u_l}x),\\
\frac{\partial}{\partial u_i}(\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x) &= (\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x) + (\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_l}x),\\
\frac{\partial}{\partial u_l}(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) &= (\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_l}x, \frac{\partial}{\partial u_j}x) + (\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}\frac{\partial}{\partial u_l}x).
\end{aligned}\right\} \tag{4}$$

Put $\frac{\partial}{\partial u_j}(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_l}x) = b_1, \frac{\partial}{\partial u_i}(\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x) = b_2, \frac{\partial}{\partial u_l}(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) = b_3, (\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x) = w_1$, $(\frac{\partial}{\partial u_j}\frac{\partial}{\partial u_l}x, \frac{\partial}{\partial u_i}x) = w_2, (\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_l}x, \frac{\partial}{\partial u_j}x) = w_3$. Then system Eq. (4) has the form: $w_1 + w_2 = b_1, w_1 + w_3 = b_2, w_2 + w_3 = b_3$. We consider this system as a system of equations with respect to w_1, w_2, w_3. This system has the unique solution (w_1, w_2, w_3), where $w_1 = \frac{1}{2}(b_1 + b_2 - b_3) \in R\left\{V, \Delta^{-1}\right\}, w_2 = \frac{1}{2}(b_1 + b_3 - b_2) \in R\left\{V, \Delta^{-1}\right\}, w_3 = \frac{1}{2}(b_2 + b_3 - b_1) \in R\left\{V, \Delta^{-1}\right\}$. $\square$

Lemma 8 $(x^{(b_1,b_2,\dots,b_n)}, \frac{\partial}{\partial u_i}x) \in R\left\{V, \Delta^{-1}\right\}$ *for all* $1 \le i \le n$ *and* $b_1, b_2, \dots, b_n \in N_0$ *such that* $1 \le b_1 + b_2 + \dots + b_n$.

Proof. We prove this assertion by induction on $p = b_1 + b_2 + \dots + b_n$. Let $p = 1$. By the definition of V, we have $(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x) \in V \subset R\left\{V, \Delta^{-1}\right\}$ for all $1 \le i, j \le n$. Hence the assertion holds for $p = 1$.

Assume that the assertion holds for $p > 1$ that is assume that $(x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_j}x) \in R\left\{V, \Delta^{-1}\right\}$ for all $1 \leq j \leq n$ and $b_1, b_2, \ldots, b_n \in N_0$ such that $b_1 + b_2 + \cdots + b_n = p$. By Lemma 7, we have $(\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x) \in R\left\{V, \Delta^{-1}\right\}$ for all $i, j, l \in \{1, 2, \ldots, n\}$. Using Lemma 6 to $(x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_j}x)$ and $(\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x, \frac{\partial}{\partial u_l}x)$, we obtain that $(x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x)) \in R\left\{V, \Delta^{-1}\right\}$ for all $1 \leq i, j \leq n$ and $b_s \in N_0$ such that $b_1 + b_2 + \cdots + b_n = p$. Since $(x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_j}x) \in R\left\{V, \Delta^{-1}\right\}$ by the supposition of our induction and $(x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x)) \in R\left\{V, \Delta^{-1}\right\}$, the following equality

$$\frac{\partial}{\partial u_i}(x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_j}x) = (\frac{\partial}{\partial u_i}x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_j}x) + (x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x).$$

implies that $(\frac{\partial}{\partial u_i}x^{(b_1,b_2,\ldots,b_n)}, \frac{\partial}{\partial u_j}x) \in R\left\{V, \Delta^{-1}\right\}$ for all $i, j : 1 \leq i, j \leq n$. This means that the assertion is proved for $p + 1$. □

Lemma 9 $(x^{(b_1,b_2,\ldots,b_n)}, x^{(c_1,c_2,\ldots,c_n)}) \in R\left\{V, \Delta^{-1}\right\}$ *for all* $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in N_0$ *such that* $1 \leq b_1 + b_2 + \cdots + b_n, 1 \leq c_1 + c_2 + \cdots + c_n$.

Proof. Using Lemmas 8 and 6, we obtain $(x^{(b_1,b_2,\ldots,b_n)}, x^{(c_1,c_2,\ldots,c_n)}) \in R\left\{V, \Delta^{-1}\right\}$. □

We complete the proof of our theorem. By Lemma 4, $\Delta \in R\{V\}$. Since $R < V >$ is a field, we obtain $\Delta^{-1} \in R < V >$. Hence $R\left\{V, \Delta^{-1}\right\} \subset R < V >$. By Lemma 9, $(x^{(b_1,b_2,\ldots,b_n)}, x^{(c_1,c_2,\ldots,c_n)}) \in R\left\{V, \Delta^{-1}\right\} \subset R < V > \subset R < x >^{M(n)}$ for all $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in N_0$ such that $1 \leq b_1 + b_2 + \cdots + b_n, 1 \leq c_1 + c_2 + \cdots + c_n$. By Lemma 3, the system of all elements $(x^{(m_1,m_2,\ldots,m_n)}, x^{(p_1,p_2,\ldots,p_n)})$, where $m_1 + m_2 + \cdots + m_n \geq 1, p_1 + p_2 + \cdots + p_n \geq 1, m_i \in N_0, p_i \in N_0$, is a generating system of $R < x >^{M(n)}$ as a field. Hence $R < V >= R < x >^{M(n)}$. The theorem is completed.

Let $a_m \in R^n$, $m = 1, \ldots, n$, $a_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$. The determinant $\det \|a_{ij}\|_{i,j=1,2,\ldots,n}$ will be denoted by $[a_1 a_2 \cdots a_n]$. So $\left[x^{(m_{11},m_{12},\cdots,m_{1n})} x^{(m_{21},m_{22},\cdots,m_{2n})} \ldots x^{(m_{n1},m_{n2},\cdots,m_{nn})}\right]$ is the determinant of the vectors $x^{(m_{i1},m_{i2},\cdots,m_{in})}$ in R^n, $i = 1, 2, \ldots, n$. Let $SM(n)$ be the subgroup of $M(n)$ generated by rotations and translations of R^n.

Theorem 2 *The system*

$$\left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x\right], \left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right), 1 \leq i \leq j \leq n, i + j < 2n, \tag{5}$$

is a generating system of the differential field $R < x >^{SM(n)}$.

Proof. First we prove several lemmas. Let $SO(n)$ be the subgroup of $O(n)$ generated by rotations of R^n.

Lemma 10 $R < x >^{SM(n)} = R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{SO(n)}$.

Proof. A proof is similar to the proof of Lemma 1. $\square$

Lemma 11 *Let* $f \in R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{SO(n)}$. *Then there exist* $SO(n)$*-invariant differential polynomials* f_1, f_2 *such that* $f = f_1/f_2$.

Proof. A proof is similar to the proof in ([11], p. 106). $\square$

Lemma 12 *The system of all elements*

$$\left[x^{(m_{11},m_{12},\cdots,m_{1n})}x^{(m_{21},m_{22},\cdots,m_{2n})}\cdots x^{(m_{n1},m_{n2},\cdots,m_{nn})}\right], (x^{(p_1,p_2,\ldots,p_n)}, x^{(q_1,q_2,\cdots,q_n)}), \tag{6}$$

where $m_{i1}+m_{i2}+\cdots+m_{in} \geq 1, p_1+p_2+\cdots+p_n \geq 1, q_1+q_2+\cdots+q_n \geq 1$, *is a generating system of* $R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{SO(n)}$ *as a field.*

Proof. Let $R[x^{(m_1,m_2,\ldots,m_n)}, m_1+m_2+\cdots+m_n \geq 1]^{SO(n)}$ be the R-algebra of all $SO(n)$-invariant polynomials of the system $x^{(m_1,m_2,\ldots,m_n)}$, where $m_1+m_2+\cdots+m_n \geq 1$. According to the First Main Theorem for $SO(n)$ (see [19], p. 53), the system Eq. (6) is a generating system of $R[x^{(m_1,m_2,\ldots,m_n)}, m_1+m_2+\cdots+m_n \geq 1]^{SO(n)}$. Lemma 11 implies that the system Eq. (6) is a generating system of $R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{SO(n)}$ as a field. $\square$

Denote by Z the system Eq. (5) of differential polynomials. Let $R\{Z\}$ be the differential R-subalgebra of $R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >^{SO(n)}$ generated by elements of the system Z.

Let $\delta = \delta_x$ be the determinant of the matrix $\mathrm{Gr}(y_1, y_2, \ldots, y_{n-1}; z_1, z_2, \ldots, z_{n-1})$, where $y_1 = z_1 = \frac{\partial}{\partial u_1}x, y_2 = z_2 = \frac{\partial}{\partial u_2}x, \cdots, y_{n-1} = z_{n-1} = \frac{\partial}{\partial u_{n-1}}x$.

Lemma 13 $(y_i, z_j) \in R\{Z\}$ *for all* $1 \leq i, j \leq n-1$, $\delta \in R\{Z\}$ *and* $\delta^{-1} \in R < Z >$.

Proof. Elements (y_i, z_j) of the determinant δ are functions $\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right)$, where $1 \leq i, j \leq n-1$. By the definition of Z, $\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right) \in Z \subset R\{Z\}$ for all $1 \leq i, j \leq n-1$. Hence $\delta \in R\{Z\}$ and $\delta^{-1} \in R < Z >$. $\square$

Let Δ be the function in the proof of Theorem 1.

Lemma 14 $\Delta \in R\{Z\}$ *and* $\Delta^{-1} \in R < Z >$.

Proof. Since $\Delta = \left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x\right]^2$, we have $\Delta \in R\{Z\}$. Hence $\Delta^{-1} \in R < Z >$. $\square$

Denote by $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ the differential R-subalgebra of $R < \frac{\partial}{\partial u_1}x, \frac{\partial}{\partial u_2}x, \ldots, \frac{\partial}{\partial u_n}x >$ generated by Z and functions δ^{-1}, Δ^{-1}. By Lemmas 11 and 12, for a proof of our theorem, it is enough to prove that $(x^{(p_1,\ldots,p_n)}, x^{(q_1,\cdots,q_n)})$ and $\left[x^{(m_{11},\cdots,m_{1n})}\cdots x^{(m_{n1},\cdots,m_{nn})}\right]$ elements of $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $m_{ij}, p_i, q_i \in N_0$ such that $m_{i1}+m_{i2}+\cdots+m_{in} \geq 1, p_1+p_2+\cdots+p_n \geq 1, q_1+q_2+\cdots+q_n \geq 1$.

In the sequel, we need the following lemma.

Lemma 15 *The equality*

$$[y_1 \dots y_n][z_1 \dots z_n] = det||(y_i, z_j)||_{i,j=1,2,\dots,n}$$

holds for all vectors $y_1, \dots, y_n, z_1, \dots, z_n$ *in* R^n.

Proof. A proof of the this lemma is given in ([11], p.72; [19], p. 53). □

Let V be the system in the proof of Theorem 1.

Lemma 16 $\left(\frac{\partial}{\partial u_n}x, \frac{\partial}{\partial u_n}x\right) \in R\left\{Z, \delta^{-1}\right\}$ *and* $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$.

Proof. Using Lemma 15 to vectors $y_1 = z_1 = \frac{\partial}{\partial u_1}x, y_2 = z_2 = \frac{\partial}{\partial u_2}x, \cdots, y_n = z_n = \frac{\partial}{\partial u_n}x$, we obtain

$$\left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x\right]^2 = det\left\|(y_i, z_j)\right\|_{i,j=1,2,\dots n} = \Delta. \tag{7}$$

Denote by D_{ni} the cofactor of the element (y_n, z_j) of the matrix $A = \left\|(y_i, z_j)\right\|_{i,j=1,2,\dots n}$ for $i = 1, 2, \dots, n$. Then we obtain the equality

$$\Delta = (y_n, z_1)D_{n1} + (y_n, z_2)D_{n2} + \cdots + (y_n, z_{n-1})D_{nn-1} + (y_n, z_n)D_{nn}. \tag{8}$$

Since $\delta = D_{nn} \neq 0$, equalities Eq. (7) and Eq. (8) imply

$$(y_n, z_n) = \left(\frac{\partial}{\partial u_n}x, \frac{\partial}{\partial u_n}x\right) = \Delta\delta^{-1} - (y_n, z_1)D_{n1}\delta^{-1} - (y_n, z_2)D_{n2}\delta^{-1} - \cdots - (y_n, z_{n-1})D_{nn-1}\delta^{-1}. \tag{9}$$

In Eq. (9), $(y_n, z_j) \in Z \subset R\{Z\}$ for every $j : 1 \leq j \leq n-1$ by the definition of Z. We prove that $D_{ns} \in R\{Z\}$ for every $1 \leq s \leq n-1$. We have $D_{ns} = (-1)^{n+s}$ $\det\mathrm{Gr}(y_1, y_2, \dots, y_{n-1}; z_1, z_2, \ \dots, z_{s-1}, z_{s+1}, \dots, z_n)$. Elements of D_{ns} have the following forms (y_i, z_j), where $i, j < n$, and (y_i, z_n), where $i < n$. By the definition of Z, $(y_i, z_j) \in Z \subset R\{Z\}$ for all $i, j < n$ and $(y_i, z_n) = (y_n, z_i) \in Z \subset R\{Z\}$ for all $i < n$. Hence $D_{ns} \in R\{Z\}$ for every $1 \leq s \leq n-1$ and Eq. (9) implies $(y_n, z_n) \in R\left\{Z, \delta^{-1}\right\}$. Using $\left(\frac{\partial}{\partial u_n}x, \frac{\partial}{\partial u_n}x\right) \in R\left\{Z, \delta^{-1}\right\}$ and $V \subset Z \cup \{(y_n, z_n)\}$, we obtain $V \subset R\left\{Z, \delta^{-1}\right\}$. Hence $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. □

Lemma 17 $(x^{(p_1,p_2,\cdots,p_n)}, x^{(r_1,r_2,\cdots,r_n)}) \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ *for all* $p_i, r_i \in N_0$ *such that* $1 \leq p_1 + p_2 + \cdots + p_n$ *and* $1 \leq r_1 + r_2 + \cdots + r_n$.

Proof. By Lemma 16, $R\left\{V, \Delta^{-1}\right\} \subseteq R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. Using $R\left\{V, \Delta^{-1}\right\} \subseteq R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ and Lemma 9, we obtain $(x^{(p_1,p_2,\cdots,p_n)}, x^{(r_1,r_2,\cdots,r_n)}) \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $p_i, r_i \in N_0$ such that $1 \leq p_1 + p_2 + \cdots + p_n$ and $1 \leq r_1 + r_2 + \cdots + r_n$. □

Lemma 18 $\left[x^{(m_{11},m_{12},\cdots,m_{1n})}x^{(m_{21},m_{22},\cdots,m_{2n})} \dots x^{(m_{n1},m_{2},\cdots,m_{nn})}\right] \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ *for all* $m_{ij} \in N_0$ *such that* $m_{i1} + m_{i2} + \cdots + m_{in} \geq 1, i = 1, 2, \dots, n$.

Proof. Using Lemma 15 to $y_1 = \frac{\partial}{\partial u_1}x, y_2 = \frac{\partial}{\partial u_2}x, \cdots, y_n = \frac{\partial}{\partial u_n}x; z_1 = x^{(m_{11},m_{12},\cdots,m_{1n})}, z_2 = x^{(m_{21},m_{22},\cdots,m_{2n})} \ldots, z_n = x^{(m_{n1},m_{n2},\cdots,m_{nn})}$, we obtain that

$$[y_1 \ldots y_n][z_1 \ldots z_n] = det||(y_i, z_j)||_{i,j=1,2,\ldots,n}. \tag{10}$$

Since $\Delta = [y_1 \ldots y_n]^2$, Eq. (10) implies

$$[z_1 \ldots z_n] = \Delta^{-1}[y_1 \ldots y_n]det||(y_i, z_j)||_{i,j=1,2,\ldots,n}.$$

By Lemma 17, $(y_i, z_j) = (\frac{\partial}{\partial u_i}x, x^{(m_{j1},m_{j2},\cdots,m_{jn})}) \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $i, j = 1, 2, \ldots, n$. Since $[y_1 \ldots y_n] \in Z \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$, we obtain $[z_1 \ldots z_n] \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. □

We complete the proof of our theorem. By Lemmas 13 and 14, $\delta^{-1}, \Delta^{-1} \in R < Z >$. Hence $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\} \subset R < Z >$. By Lemma 17, $(x^{(b_1,b_2,\ldots,b_n)}, x^{(c_1,c_2,\ldots,c_n)}) \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\} \subset R < Z >$ for all $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in N_0$ such that $1 \leq b_1 + b_2 + \cdots + b_n, 1 \leq c_1 + c_2 + \cdots + c_n$. By Lemma 18, $\left[x^{(m_{11},m_{12},\cdots,m_{1n})}x^{(m_{21},m_{22},\cdots,m_{2n})} \ldots x^{(m_{n1},m_2,\cdots,m_{nn})}\right] \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\} \subset R < Z >$ for all $m_{ij} \in N_0$ such that $m_{i1}+m_{i2}+\cdots+m_{in} \geq 1, i = 1, 2, \ldots, n$. Hence Lemmas 10 and 12 imply that $R < Z >= R < x >^{SM(n)}$. The theorem is complete.

4. The conditions of G-equivalence of vector fields

In this section, J is a connected open subset of R^n.

Definition 8 *A J-vector field x will be called regular if $\left[\frac{\partial}{\partial u_1}x(u)\frac{\partial}{\partial u_2}x(u) \ldots \frac{\partial}{\partial u_n}x(u)\right] \neq 0$ for all $u \in J$. The set of all regular vector fields in R^n will be denoted by $H_{reg}(n)$.*

Let $x(u)$ be a regular vector field in R^n. Put $D_i x(u) = \frac{\partial}{\partial u_i}x(u)$. Let $\left\{D^1x(u), \ldots, D^nx(u)\right\}$ be the biorthonormal system of the system $\{D_1x(u), \ldots, D_nx(u)\}$ that is $(D_ix(u), D^jx(u)) = \delta_i^j$ for all $u \in J$ and $i, j = 1, \ldots, n$. The following derivation formulas for vector fields are known ([18], p. 116):

$$\left\{\begin{array}{l} \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x(u) = \sum_{s=1}^n p_{ij}^s\{x\} D_s x(u), \\ i, j = 1, 2, \ldots, n; \end{array}\right. \tag{11}$$

$$\left\{\begin{array}{l} \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x(u) = \sum_{s=1}^n p_{ij,s}\{x\} D^s x(u), \\ i, j = 1, 2, \ldots, n. \end{array}\right. \tag{12}$$

The functions $p_{ij,s}\{x\}, p_{ij}^s\{x\}$ are called the Christoffel symbols of the first and second kinds of a vector field respectively.

Let $x(u)$ be a vector field. Put $g_{ij}(x) = \left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right)$. The form $I(x) = \sum_{i,j=1}^n g_{ij}(x)du_idu_j$ will be called the first fundamental form of a vector field $x(u)$. According to Lemma 7, we have $p_{ij,s}\{x\} = (\frac{\partial^2}{\partial u_i \partial u_j}x, \frac{\partial}{\partial u_s}x) = \frac{1}{2}(\frac{\partial}{\partial u_j}g_{si}(x) + \frac{\partial}{\partial u_i}g_{sj}(x) - \frac{\partial}{\partial u_s}g_{ij}(x))$ for all $i, j, s = 1, 2, \ldots, n$.

Theorem 3 *Let $x(u)$ and $y(u)$ be regular J-vector fields in R^n. Then the following conditions are equivalent:*

1. $p^s_{ij}\left\{x(u)\right\} = p^s_{ij}\left\{y(u)\right\}$ *for all* $u \in J$ *and* $i, j, s = 1, 2, \dots, n$;

2. $x \overset{Aff(n)}{\sim} y$.

Proof. First we prove the following lemma.

Lemma 19 $p^s_{ij}\left\{x\right\} \in R\left\{V, \Delta^{-1}\right\}$ *for all* $1 \le i, j, s \le n$.

Proof. Let i, j be fixed. We consider $\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}x(u) = \sum_{s=1}^{n} p^s_{ij}\left\{x\right\} D_s x(u)$ as a system of linear equations with respect to $p^1_{ij}\left\{x\right\}, \dots, p^n_{ij}\left\{x\right\}$. Since $[D_1x(u), D_2x(u), \dots, D_nx(u)]$ is the determinant of this system and $x(u)$ is a regular vector field, $[D_1x(u), D_2x(u), \dots, D_nx(u)] \ne 0$ for all $u \in J$. Hence the system has the following solutions:

$$p^s_{ij}\left\{x\right\} = \frac{\left[D_1x \dots D_{s-1}x\frac{\partial}{\partial u_i}(\frac{\partial}{\partial u_j}x)D_{s+1}x \dots D_nx\right]}{[D_1xD_2x \dots D_nx]},$$

where $s = 1, 2, \dots, n$. This equality implies

$$p^s_{ij}\left\{x\right\} = \frac{\left[D_1x \dots D_{s-1}x\frac{\partial}{\partial u_i}(\frac{\partial}{\partial u_j}x)D_{s+1}x \dots D_nx\right][D_1xD_2x \dots D_nx]}{[D_1xD_2x \dots D_nx]^2}. \tag{13}$$

Using Lemmas 15 and 9, we obtain that

$$\left[D_1x \dots D_{s-1}x\frac{\partial}{\partial u_i}(\frac{\partial}{\partial u_j}x)D_{s+1}x \dots D_nx\right][D_1xD_2x \dots D_nx] \in R\left\{V, \Delta^{-1}\right\}.$$

Since $\Delta_x = [D_1xD_2x \dots D_nx]^2$, we have $p^s_{ij}\left\{x\right\} \in R\left\{V, \Delta^{-1}\right\}$ for all $1 \le i, j, s \le n$. The lemma is proved.

$(2) \to (1)$. Let $x \overset{Aff(n)}{\sim} y$. Then Eq. (13) implies that the function $p^s_{ij}\left\{x\right\}$ is $\mathrm{Aff}(n)$-invariant. Hence $p^s_{ij}\left\{x(u)\right\} = p^s_{ij}\left\{y(u)\right\}$ for all $i, j, s = 1, 2, \dots, n$ and $u \in J$.

$(1) \to (2)$. Assume that the equality $p^s_{ij}\left\{x(u)\right\} = p^s_{ij}\left\{y(u)\right\}$ holds for all $i, j, s = 1, 2, \dots, n$ and $u \in J$. Put $A(x) = \|D_1xD_2x \dots D_nx\|$ and $\frac{\partial}{\partial u_i}A(x) = \left\|\frac{\partial}{\partial u_i}D_1x\frac{\partial}{\partial u_i}D_2x \dots \frac{\partial}{\partial u_i}D_nx\right\|$, where we consider D_ix as a column-vector. Eq. (11) implies $A(x)^{-1}\frac{\partial}{\partial u_i}A(x) = \left\|p^s_{ij}\left\{x\right\}\right\|_{s=1,\dots,n;j=1,\dots,n}$. Using $p^s_{ij}\left\{x(u)\right\} = p^s_{ij}\left\{y(u)\right\}$ for all $i, j, s = 1, 2, \dots, n$, we obtain $A(x(u))^{-1}\frac{\partial}{\partial u_i}A(x(u)) = A(y(u))^{-1}\frac{\partial}{\partial u_i}A(y(u))$ for all $i = 1, 2, \dots, n$ and $u \in J$.

Now we complete the proof of our theorem. The equality $A(x)^{-1}\frac{\partial}{\partial u_i}A(x) = A(y)^{-1}\frac{\partial}{\partial u_i}A(y)$ implies

$$\frac{\partial}{\partial u_i}(A(y)A(x)^{-1}) = (\frac{\partial}{\partial u_i}A(y))A(x)^{-1} + A(y)\frac{\partial}{\partial u_i}(A(x)^{-1}) = (\frac{\partial}{\partial u_i}A(y))A(x)^{-1} -$$

$$A(y)A(x)^{-1}(\frac{\partial}{\partial u_i}A(x))A(x)^{-1} = A(y)(A(y)^{-1}\frac{\partial}{\partial u_i}A(y) - A(x)^{-1}\frac{\partial}{\partial u_i}A(x))A(x)^{-1} = 0$$

for all $i = 1, 2, \ldots, n$ and $u \in J$. Using this equality for all $i = 1, 2, \ldots, n$, we obtain that the matrix $A(y(u))A(x(u))^{-1}$ is not depend on $u \in J$. Put $F = A(y)A(x)^{-1}$. According to $detA(x(u)) \neq 0$ and $detA(y(u)) \neq 0$ for all $u \in J$, we have $detF \neq 0$ and $A(y) = FA(x)$ for all $u \in J$. The equality $A(y(u)) = FA(x(u))$ implies $\frac{\partial}{\partial u_i} y(u) = F \frac{\partial}{\partial u_i} x(u)$ for all $i = 1, 2, \ldots, n$ and $u \in J$. Then there exists a constant vector $b \in R^n$ such that $y(u) = Fx(u) + b$ for all $u \in J$. Theorem 3 is complete. □

This theorem means the system $\left\{p_{ij}^s \{x(u)\}, i, j, s = 1, 2, \ldots, n\right\}$ of all Christoffel symbols of the second kind is a complete system of Aff(n)-invariants of a vector field on the set $H_{reg}(n)$. Below we prove that the system of all Christoffel symbols of the first and second kind is a complete system of $M(n)-$invariants of a vector field (Theorem 5).

Corollary 1 *Let K be a set. Every Aff(n)-invariant function $h : H_{reg}(n) \to K$ is a function of the system $\left\{p_{ij}^s \{x(u)\}, i, j, s = 1, 2, \ldots, n\right\}$.*

Proof. A proof follows from Theorem 3 and Proposition 1. □

Let $x(u)$ and $y(u)$ be vector fields in R^n such that $x \overset{M(n)}{\sim} y$. Then $f\{x\} = f\{y\}$ for any $M(n)$-invariant differential polynomial $f\{x\}$. In particularly, $g_{ij}(x(u)) = g_{ij}(y(u))$ for all $u \in J$ and i, j such that $1 \leq i \leq j \leq n$. The converse theorem is true for regular vector fields:

Theorem 4 *Let $x(u)$ and $y(u)$ be regular vector fields in R^n such that*

$$g_{ij}(x(u)) = g_{ij}(y(u)) \tag{14}$$

for all $u \in J$ and i, j such that $1 \leq i \leq j \leq n$. Then $x \overset{M(n)}{\sim} y$.

Proof. Since $\Delta_x(u) = det \|g_{ij}(x)\|$, equalities Eq. (14) imply $\Delta_x(u) = \Delta_y(u)$ for all $u \in J$. Using $\Delta_x(u) \neq 0, \Delta_y(u) \neq 0$ for regular vector fields x and y, we get

$$\Delta_x(u)^{-1} = \Delta_y(u)^{-1} \tag{15}$$

for all $u \in J$. Let V be the system $\{g_{ij}(x), 1 \leq i \leq j \leq n\}$ and $f\{x\} \in R\left\{V, \Delta^{-1}\right\})$. Then Eq. (14) and Eq. (15) imply

$$f\{x(u)\} = f\{y(u)\} \tag{16}$$

for all $u \in J$.

By Lemma 19, $p_{ij}^s\{x\} \in R\left\{V, \Delta^{-1}\right\}$ for all $i, j, s = 1, 2, \ldots, n$. Then Eq. (16) implies $p_{ij}^s\{x(u)\} = p_{ij}^s\{y(u)\}$ for all $u \in J$ and $i, j, s = 1, 2, \ldots, n$. By Theorem 3, there exists $F \in GL(n)$ and $b \in R^n$ such that $y(u) = Fx(u) + b$ for all $u \in J$. We prove that $F \in O(n)$.

Let $A(x)^\top$ be the transpose matrix of the matrix $A(x)$ in Theorem 3. Using the equality $A(x)^\top A(x) = \|g_{ij}(x(u))\|_{i,j=1,2,\ldots,n}$ and Eq. (14), we obtain $A(x)^\top A(x) = A(y)^\top A(y)$. Since $x(u)$ is a regular vector field, we have $detA(x(u)) \neq 0$ for all $u \in J$. Hence equalities $A(x)^\top A(x) = A(y)^\top A(y)$ and $A(y) = FA(x)$ imply $F^\top F = E$, where E is the unit matrix. Thus $F \in O(n)$. □

Corollary 2 *Let K be a set. Every $M(n)$-invariant function $h : H_{reg}(n) \to K$ is a function of elements of V.*

Proof. A proof follows from Theorem 4 and Proposition 1. □

Theorem 5 *Let $x(u)$ and $y(u)$ be regular vector fields in R^n. Assume that the following conditions hold:*

1. *there exists $u_0 \in J$ such that $det(\frac{\partial}{\partial u_i}A(x(u_0))) \neq 0$ for some $i = 1, 2, \ldots, n$;*
2. *$p^s_{ij}\{x(u)\} = p^s_{ij}\{y(u)\}$ for all $u \in J$ and $i, j, s = 1, 2, \ldots, n$;*
3. *$p_{ij,s}\{x(u)\} = p_{ij,s}\{y(u)\}$ for all $u \in J$ and $i, j, s = 1, 2, \ldots, n$.*

Then $x \overset{M(n)}{\sim} y$.

Proof. As in Theorem 3, equalities $p^s_{ij}\{x\} = p^s_{ij}\{y\}$ for all $i, j, s = 1, 2, \ldots, n$ imply the existence of $F \in GL(n)$ and $b \in R^n$ such that $y(u) = Fx(u) + b$ for all $u \in J$. We prove that $F \in O(n)$. Equalities $p_{ij,s}\{x\} = (\frac{\partial^2}{\partial u_i \partial u_j}x, \frac{\partial}{\partial u_s}x)$ imply $\|p_{ij,s}\{x(u)\}\|_{s,j=1,2,\ldots,n} = A(x)^\top \frac{\partial}{\partial u_i}A(x(u))$. By the condition 3 of our theorem, we obtain $A(x)^\top \frac{\partial}{\partial u_i}A(x(u)) = A(y)^\top \frac{\partial}{\partial u_i}A(y(u))$ for all $i = 1, 2, \ldots, n$. Using $A(y(u)) = FA(x(u))$ and $detA^\top(x(u)) = detA(x(u)) \neq 0$ for all $u \in J$, we get $\frac{\partial}{\partial u_i}A(x(u)) = F^\top F \frac{\partial}{\partial u_i}A(x(u))$ for all $i = 1, 2, \ldots, n$. Since $det(\frac{\partial}{\partial u_i}A(x(u_0))) \neq 0$ for some u_0 and i, this equality implies $F^\top F = E$. Hence $F \in O(n)$. □

Proposition 3 *The system $V = \{g_{ij}(x), 1 \leq i, j \leq n\}$ is a minimal complete system of invariants of a vector field on $H_{reg}(n)$ for the group $M(n)$.*

Proof. Prove the subset $V \setminus \{g_{11}\}$ is not a complete system of invariants on $H_{reg}(n)$. Let $J = I^n$, where $I = (0, 1)$ is the open interval of R. Consider the following two J-vector fields in R^n: $x(u) = (u_1, u_2, \ldots, u_n), y(u) = (2u_1, u_2, \ldots, u_n)$. We have $g_{11}(x)(u) = 1, g_{11}(y)(u) = 4, g_{jj}(x)(u) = g_{jj}(y)(u) = 1$ for all $j \in \{2, \ldots, n\}$ and $g_{pq}(x)(u) = g_{pq}(y)(u) = 0$ for all $p, q \in \{1, 2, \ldots, n\}$ such that $p \neq q$. Since $g_{11}(x)$ and $g_{11}(y)$ are $M(n)$-invariants, $g_1(x) = 1$, $g_{11}(y) = 4$, we obtain that the vector fields x and y are not $M(n)$-equivalent. Hence the subsystem $V \setminus \{g_{11}(x)\}$ is not complete on $H_{reg}(n)$. Similarly, the subsystem $V \setminus \{g_{ii}(x)\}$ is not complete on $H_{reg}(n)$ for every $i \in \{2, \ldots, n\}$.

Prove the subset $V \setminus \{g_{12}\}$ is not a complete system of invariants on $H_{reg}(n)$: Consider the following two vector fields in R^n: $x(u) = (u_1, u_2, \ldots, u_n), y(u) = (\frac{1}{\sqrt{2}}u_1, \frac{1}{\sqrt{2}}u_1 + u_2, \ldots, u_n)$. We have $g_{jj}(x)(u) = g_{jj}(y)(u) = 1$ for all $j \in \{1, 2, \ldots, n\}$, $g_{pq}(x)(u) = g_{pq}(y)(u) = 0$ for all $p, q \in \{1, 2, \ldots, n\}$ such that $p \neq q, (p, q) \neq (1, 2)$. Since $g_{12}(x)$ and $g_{12}(y)$ are $M(n)$-invariants, $g_{12}(x) = 0$, $g_{12}(y) = \frac{1}{\sqrt{2}}$, we obtain that the vector fields x and y are not $M(n)$-equivalent. Hence the subsystem $V \setminus \{g_{12}(x)\}$ is not complete on $H_{reg}(n)$. Similarly, the subsystem $V \setminus \{g_{ij}(x)\}$ is not complete on $H_{reg}(n)$ for every $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$. □

Theorem 6 *Assume that $x(u)$ and $y(u)$ are regular vector fields in R^n such that*

$$\left.\begin{array}{r}\left(\frac{\partial}{\partial u_i}x(u), \frac{\partial}{\partial u_j}x(u)\right) = \left(\frac{\partial}{\partial u_i}y(u), \frac{\partial}{\partial u_j}y(u)\right), \\ \left[\frac{\partial}{\partial u_1}x(u) \quad \frac{\partial}{\partial u_2}x(u) \cdots \frac{\partial}{\partial u_n}x(u)\right] = \left[\frac{\partial}{\partial u_1}y(u) \quad \frac{\partial}{\partial u_2}y(u) \cdots \frac{\partial}{\partial u_n}y(u)\right]\end{array}\right\} \tag{17}$$

for all $u \in J$ and all $1 \leq i \leq j \leq n, i+j < 2n$. Then $x \overset{SM(n)}{\sim} y$.

Proof. Let Z be the system Eq. (5) in Theorem 2 and $f\{x\} \in R\{Z\}$. Then Eq. (17) imply

$$f\{x(u)\} = f\{y(u)\} \tag{18}$$

for all $u \in J$. Let $\delta = \delta_x$ be the function in the proof of Theorem 2. By Lemma 13, $\delta_x \in R\{Z\}$. Hence Eq. (18) implies $\delta_x(u) = \delta_y(u)$ for all $u \in J$. The following lemma will help us to complete the proof. □

Lemma 20 *$\delta_x(u) \neq 0$ and $\delta_y(u) \neq 0$ for all $u \in J$.*

Proof. Since $x(u)$ is a regular vector field, we have $\left[\frac{\partial}{\partial u_1}x(u)\frac{\partial}{\partial u_2}x(u)\cdots\frac{\partial}{\partial u_n}x(u)\right] \neq 0$ for all $u \in J$. Hence vectors $\frac{\partial}{\partial u_1}x(u), \frac{\partial}{\partial u_2}x(u), \cdots, \frac{\partial}{\partial u_n}x(u)$ are linearly independent for all $u \in J$. Then vectors $\frac{\partial}{\partial u_1}x(u), \frac{\partial}{\partial u_2}x(u), \cdots, \frac{\partial}{\partial u_{n-1}}x(u)$ also are linearly independent. This implies $\det\left\|\left(\frac{\partial}{\partial u_i}x, \frac{\partial}{\partial u_j}x\right)\right\|_{i,j=1,2,\ldots,n-1} = \delta_x(u) \neq 0$ for all $u \in J$. Similarly, $\delta_y(u) \neq 0$ for all $u \in J$. □

The equality $\delta_x = \delta_y$ for all $u \in J$ and Lemma 20 imply $\delta_x^{-1} = \delta_y^{-1}$ for all $u \in J$. Let $f\{x\} \in R\left\{Z, \delta^{-1}\right\})$. Then the equality $\delta_x^{-1} = \delta_y^{-1}$, Eq. (17) and Eq. (18) imply

$$f\{x(u)\} = f\{y(u)\} \tag{19}$$

for all $u \in J$. Using Eq. (19) and Lemma 16, we obtain the equality $\left(\frac{\partial}{\partial u_n}x(u), \frac{\partial}{\partial u_n}x(u)\right) = \left(\frac{\partial}{\partial u_n}y(u), \frac{\partial}{\partial u_n}y(u)\right)$. This equality and Eq. (17) imply Eq. (14). Hence by theorem 4 there exist $F \in O(n)$ and $b \in R^n$ such that $y(u) = Fx(u) + b$. Using this equality and the equality $\left[\frac{\partial}{\partial u_1}x \cdots \frac{\partial}{\partial u_n}x\right] = \left[\frac{\partial}{\partial u_1}y \cdots \frac{\partial}{\partial u_n}y\right]$ in Eq.(17), we get $\left[\frac{\partial}{\partial u_1}x \cdots \frac{\partial}{\partial u_n}x\right] = \det F\left[\frac{\partial}{\partial u_1}x \cdots \frac{\partial}{\partial u_n}x\right]$. Since $\Delta_x(u) = \left[\frac{\partial}{\partial u_1}x \cdots \frac{\partial}{\partial u_n}x\right]^2 \neq 0$ for all $u \in J$, we obtain $det F = 1$. This means that $x \overset{SM(n)}{\sim} y$. Theorem 6 is proved.

Proposition 4 *The system Z is a minimal complete system of invariants of a vector field on $H_{reg}(n)$ for the group $SM(n)$.*

Proof. Prove that the subsystem $Z \setminus \left\{ \left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x \right] \right\}$ is not complete on $H_{reg}(n)$. Let $J = I^n$, where $I = (0,1)$ is the open interval of R. Consider the following vector fields in R^n: $x(u) = (u_1, u_2, \dots, u_n), y(u) = (-u_1, u_2, \dots, u_n)$. Then $g_{jj}(x)(u) = g_{jj}(y)(u) = 1$ for all $j \in \{1, 2, \dots, n-1\}$ and $g_{pq}(x)(u) = g_{pq}(y)(u) = 0$ for all $p, q \in \{1, 2, \dots, n\}$ such that $p \neq q$. We have $\left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x \right] = 1$ and $\left[\frac{\partial}{\partial u_1}y \quad \frac{\partial}{\partial u_2}y \cdots \frac{\partial}{\partial u_n}y \right] = -1$. Since the function $\left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x \right]$ is $SM(n)$-invariant, $\left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x \right] \neq \left[\frac{\partial}{\partial u_1}y \quad \frac{\partial}{\partial u_2}y \cdots \frac{\partial}{\partial u_n}y \right]$, we obtain that the vector fields x and y are not $SM(n)$-equivalent. Hence the subsystem $Z \setminus \left\{ \left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x \right] \right\}$ is not complete on $H_{reg}(n)$.

Prove that the subsystem $Z \setminus \{g_{11}\}$ is not complete on $H_{reg}(n)$: Consider the following two vector fields in R^n: $x(u) = (u_1, u_2, \dots, u_n), y(u) = (2u_1, u_2, \dots, \frac{1}{2}u_n)$. We have $g_{11}(x)(u) = 1, g_{11}(y)(u) = 4, g_{jj}(x)(u) = g_{jj}(y)(u) = 1$ for all $j \in \{2, \dots, n-1\}$, $\left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x \right] = \left[\frac{\partial}{\partial u_1}y \quad \frac{\partial}{\partial u_2}y \cdots \frac{\partial}{\partial u_n}y \right] = 1$ and $g_{pq}(x)(u) = g_{pq}(y)(u) = 0$ for all $p, q \in \{1, 2, \dots, n\}$ such that $p \neq q$. Since $g_{11}(x)$ and $g_{11}(y)$ are $SM(n)$-invariants, $g_{11}(x) = 1$, $g_{11}(y) = 4$, we obtain that the vector fields x and y are not $SM(n)$-equivalent. Hence the subsystem $Z \setminus \{g_{11}(x)\}$ is not complete on $H_{reg}(n)$. Similarly, the subsystem $Z \setminus \{g_{ii}(x)\}$ is not complete on $H_{reg}(n)$ for every $i \in \{2, \dots, n-1\}$.

Prove that the subsystem $Z \setminus \{g_{12}\}$ is not complete on $H_{reg}(n)$. Consider the following two vector fields in R^n: $x(u) = (u_1, u_2, \dots, u_n), y(u) = (\frac{1}{\sqrt{2}}u_1, \frac{1}{\sqrt{2}}u_1 + u_2, \dots, \sqrt{2}u_n)$. We have $g_{jj}(x)(u) = g_{jj}(y)(u) = 1$ for all $j \in \{1, 2, \dots, n-1\}$, $\left[\frac{\partial}{\partial u_1}x \quad \frac{\partial}{\partial u_2}x \cdots \frac{\partial}{\partial u_n}x \right] = \left[\frac{\partial}{\partial u_1}y \quad \frac{\partial}{\partial u_2}y \cdots \frac{\partial}{\partial u_n}y \right] = 1$ and $g_{pq}(x)(u) = g_{pq}(y)(u) = 0$ for all $p, q \in \{1, 2, \dots, n\}$ such that $p \neq q, (p,q) \neq (1,2)$. Since $g_{12}(x)$ and $g_{12}(y)$ are $SM(n)$-invariants, $g_{12}(x) = 0$, $g_{12}(y) = \frac{1}{\sqrt{2}}$, we obtain that the vector fields x and y are not $SM(n)$-equivalent. Hence the subsystem $Z \setminus \{g_{12}(x)\}$ is not complete on $H_{reg}(n)$. Similarly, the subsystem $Z \setminus \{g_{ij}(x)\}$ is not complete on $H_{reg}(n)$ for every $i, j \in \{1, 2, \dots, n\}$ such that $i \neq j$. □

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Djavvat KHADJIEV
Department of Mathematics,
Karadeniz Technical University,
61080, Trabzon, TURKEY
e-mail: khdjavvat@gmail.com

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