Turkish Journal of Mathematics

Volume 34 | Number 4 Article 9

1-1-2010

# Order-isomorphism and a projection's diagram of C(X)

AHMED S. AL-RAWASHDEH

SULTAN M. AL-SULEIMAN

Turk J Math
34 (2010) , 523 – 536.
© TÜBİTAK
doi:10.3906/mat-0812-9

# Order-isomorphism and a projection's diagram of $C(X)$

*Ahmed S. Al-Rawashdeh and Sultan M. Al-Suleiman*

**Abstract**

A mapping between projections of $C^*$-algebras preserving the orthogonality, is called an orthoisomorphism. We define the order-isomorphism mapping on $C^*$-algebras, and using Dye's result, we prove in the case of commutative unital $C^*$-algebras that the concepts; order-isomorphism and the orthoisomorphism coincide. Also, we define the equipotence relation on the projections of $C(X)$; indeed, new concepts of finiteness are introduced. The classes of projections are represented by constructing a special diagram, we study the relation between the diagram and the topological space $X$. We prove that an order-isomorphism, which preserves the equipotence of projections, induces a diagram-isomorphism; also if two diagrams are isomorphic, then the $C^*$-algebras are isomorphic.

**Key Words:** Commutative $C^*$-algebras; projections order-isomorphism; infinite projections; clopen subsets.

## 1. Introduction

The algebra of continuous complex-valued functions on a compact Hausdorff space $X$, denoted by $C(X)$, is a commutative unital $C^*$-algebra. Let us recall the following main theorem, known as the Gelfand-Naimark Theorem.

**Theorem 1.1** *[8] Every commutative, unital $C^*$-algebra $A$ is isometrically $*$-isomorphic to $C(\mathcal{A})$, where $\mathcal{A}$ is the compact Hausdorff space of characters of $A$.*

Let $\mathcal{P}(A)$ denote the set of projections of $A$. A projection $p \in \mathcal{P}(A)$ is said to be minimal if $0$ is the only proper subprojection of $p$. For the case of $C(X)$, the projections are the characteristic functions on the clopen subsets of $X$ (see [4] § IX.3).

Let $A$ and $B$ be two unital $C^*$-algebras. A projection orthoisomorphism mapping was defined by H. Dye in [5] as a bijection $\theta$ between the projections of $A$ and $B$ which preserves the orthogonality; that is, for any projections $p$ and $q$ of $A$, $pq = 0$ if and only if $\theta(p)\theta(q) = 0$. Also, he proved the following lemma.

**Lemma 1.2** *[5] Any projection orthoisomorphism $\theta$ between $C^*$-algebras $A$ and $B$ preserves the following: $0$, $I$, the orthocomplement $I - p$ of $p$, and the order.*

2000 *AMS Mathematics Subject Classification:* 46L05; 46L99.

For commutative, unital $C^*$-algebras, A. Al-Rawashdeh and W. Shatanawi in [1] discussed the orthoisomorphism property (orthogonality-preserving) for certain bijection $\theta$ on the projections of $C(X)$, in other words, discussing the Boolean isomorphism of the Boolean algebra of clopen subsets of $X$.

As every projection orthoisomorphism preserves the order, then a natural question arises here: is every bijection between the projections of two $C^*$-algebras which preserves the order an orthoisomorphism? In the first part of this paper, we study this question where we define the projection order-isomorphism as a bijection between the projections of $C^*$-algebras which preserves the order and we prove the following theorem.

**Theorem 1.3** *Let $A$ and $B$ be two commutative unital $C^*$-algebras, and let $\theta$ be a one-to-one mapping from $\mathcal{P}(A)$ onto $\mathcal{P}(B)$. Then $\theta$ is orthoisomorphism if and only if $\theta$ is order-isomorphism.*

Recall that in a $C^*$-algebra $A$, two projections $p$ and $q$ are equivalent if there exists a partial isometry $w$ such that $ww^* = p$ and $w^*w = q$. A projection $p$ is called infinite if $p$ is equivalent to one of its proper subprojection, otherwise it is called a finite projection. If $A$ is a commutative $C^*$-algebra, then two projections are equivalent if and only if they are equal; this implies that every projection is finite. In the second part of this paper, we consider a compact Hausdorff subspace $X$ of $\mathbb{R}$ with the Lebesgue measure $\mu$. We define the equivalence relation, that two projections in $\mathcal{P}(C(X))$ are called equipotent if the measure of their supports are equal. Afterwards, we establish new concepts of projections which seems to be similar to the concept of finiteness in general $C^*$-algebras but not affected by commutativity, this concept of projections will be related to the Lebesgue measure and shall be denoted by $\mu$-finite and $\mu$-infinite.

Furthermore, we construct a special type of diagram that describes the class of projections of the $C^*$-algebras $C(X)$; these diagrams give information and descriptions about projections, in particular about $\mu$-infinite projections.

We define isomorphism between the diagrams and we prove (under some conditions) that if there is an order-isomorphism between $C^*$-algebras $C(X)$ and $C(Y)$, then the corresponding diagrams are isomorphic. Also, if two diagrams are isomorphic, then the corresponding $C^*$-algebras are isomorphic. Mainly we prove that following theorems.

**Theorem 1.4** *Let $X$ and $Y$ be two compact spaces in $(\mathbb{R}, \tau_u)$. If there exists an order-isomorphism between $\mathcal{P}(C(X))$ and $\mathcal{P}(C(Y))$, which preserves equipotence of projections, then $\mathfrak{D}(X)$ is isomorphic to $\mathfrak{D}(Y)$.*

**Theorem 1.5** *Let $X$ and $Y$ be two locally connected compact spaces in $(\mathbb{R}, \tau_u)$. If $\mathfrak{D}(X)$ is isomorphic to $\mathfrak{D}(Y)$, then $C(X)$ and $C(Y)$ are isomorphic as $C^*$-algebras.*

Let us now recall the following main result that will be used throughout the paper.

**Theorem 1.6** *[3] Let $X$ and $Y$ be two compact topological spaces. Then $X$ is homeomorphic to $Y$ if and only if $C(X)$ is isomorphic to $C(Y)$ as $C^*$-algebras.*

## 2. Order-Orthogonality Preserving

Let $A$ and $B$ be two unital $C^*$-algebras. We study the mappings between the set of projections $\mathcal{P}(A)$ and $\mathcal{P}(B)$ which preserve the order.

**Definition 2.1** *A projection order-isomorphism (simply, order-isomorphism) between two unital $C^*$-algebras $A$ and $B$ is a one-to-one mapping $\theta$ from $\mathcal{P}(A)$ onto $\mathcal{P}(B)$ which preserves the order. That is, $p \leq q$ if and only if $\theta(p) \leq \theta(q)$.*

Let us prove the following result concerning the order-isomorphism mappings.

**Lemma 2.2** *Let $A$ and $B$ be two $C^*$-algebras, and let $\theta : \mathcal{P}(A) \to \mathcal{P}(B)$ be an order-isomorphism. Then $\theta(0) = 0$.*

**Proof.** As $\theta$ preserves the order and $0 \leq p$, for any $p \in \mathcal{P}(A)$, then $\theta(0) \leq \theta(p)$, for all $p \in \mathcal{P}(A)$ which implies that $\theta(0) \leq q$, for all $q \in \mathcal{P}(B)$ and hence $\theta(0) = 0$. □

In the following theorem we show that, in commutative unital $C^*$-algebras, the order-isomorphism and the orthoisomorphism coincide.

**Theorem 2.3** *Let $A$ and $B$ be two commutative unital $C^*$-algebras, and let $\theta$ be a one-to-one mapping from $\mathcal{P}(A)$ onto $\mathcal{P}(B)$. Then $\theta$ is an orthoisomorphism if and only if $\theta$ is an order-isomorphism.*

**Proof.** If $\theta$ is orthoisomorphism, then by Lemma 1.2, $\theta$ is order-isomorphism. To prove the other direction, suppose that $p, q \in \mathcal{P}(A)$ with $pq = 0$. As B is commutative, $\theta(p)\theta(q) \in \mathcal{P}(B)$ and hence there exists $r \in \mathcal{P}(A)$ such that $\theta(r) = \theta(p)\theta(q)$. Since $\theta(r)\theta(p) = \theta(p)\theta(r) = \theta(r)$ and $\theta(r)\theta(q) = \theta(q)\theta(r) = \theta(r)$, then $\theta(r) \leq \theta(p)$ and $\theta(r) \leq \theta(q)$. As $\theta$ preserves the order, then $r \leq p$ and $r \leq q$. Thus, $r(pq) = (pq)r = r$ and hence $r \leq pq$. Therefore by Lemma 2.2, $\theta(r) = 0$ and hence $\theta(p)\theta(q) = 0$. □

In the following theorem we establish some properties of order-isomorphisms in the case of commutative $C^*$-algebras.

**Theorem 2.4** *Let $A$ and $B$ be two commutative unital $C^*$-algebras, and let $\theta : \mathcal{P}(A) \to \mathcal{P}(B)$ be an order-isomorphism. Then*

1. *if $pq = 0$, then $\theta(p + q) = \theta(p) + \theta(q)$, for any $p, q \in \mathcal{P}(A)$,*
2. *$\theta(pq) = \theta(p)\theta(q)$, for any $p, q \in \mathcal{P}(A)$,*
3. *if $p \leq q$, then $\theta(q - p) = \theta(q) - \theta(p)$, for any $p, q \in \mathcal{P}(A)$,*
4. *$\theta(p\Delta q) = \theta(p)\Delta\theta(q)$, for any $p, q \in \mathcal{P}(A)$.*

**Proof.**

1. If $pq = 0$, then $p + q \in \mathcal{P}(A)$, $\theta(p) \leq \theta(p + q)$, $\theta(q) \leq \theta(p + q)$ and by Theorem 2.3, $\theta(p)\theta(q) = 0$. Therefore we have $\theta(p) + \theta(q) \in \mathcal{P}(B)$ with $\theta(p) + \theta(q) \leq \theta(p + q)$. Let $r$ be a projection such that $\theta(r) = \theta(p + q) - (\theta(p) + \theta(q))$. Then $\theta(r) \leq \theta(p + q)$, $\theta(r)\theta(p) = 0$, and $\theta(r)\theta(q) = 0$. Thus, $r \leq p + q$

and by Theorem 2.3, $rp = 0$ and $rq = 0$ which is true only if $r = 0$. Therefore by Lemma 2.2, $\theta(r) = 0$ and hence $\theta(p+q) = \theta(p) + \theta(q)$.

2. As $pq \leq p$ and $pq \leq q$, then $\theta(pq) \leq \theta(p)$, $\theta(pq) \leq \theta(q)$ and hence $\theta(pq) \leq \theta(p)\theta(q)$. Let $r$ be a projection such that $\theta(r) = \theta(p)\theta(q) - \theta(pq)$. Then $\theta(r) \leq \theta(p)$ and $\theta(r) \leq \theta(q)$, so $r \leq p$, $r \leq q$ and thus $r \leq pq$. On the other hand, since $\theta(r)\theta(pq) = 0$, then by Theorem 2.3, we have $r(pq) = 0$. Thus $r = 0$ and by Lemma 2.2, $\theta(r) = 0$. Hence $\theta(pq) = \theta(p)\theta(q)$.

3. If $p \leq q$, then $q - p \in \mathcal{P}(A)$ with $q = p + (q-p)$. As $p(q-p) = 0$, then by (1) we have, $\theta(q) = \theta(p) + \theta(q-p)$ and hence $\theta(q-p) = \theta(q) - \theta(p)$.

4. As $p\Delta q = p + q - 2pq = (p - pq) + (q - pq)$ with $(p-pq)(q-pq) = 0$, then by (1), $\theta(p\Delta q) = \theta(p - pq) + \theta(q - pq)$. Also, since $p = (p - pq) + pq$ and $(p - pq)pq = 0$, then again by (1), $\theta(p) = \theta(p - pq) + \theta(pq)$, therefore $\theta(p - pq) = \theta(p) - \theta(pq)$. Similarly, we have $\theta(q - pq) = \theta(q) - \theta(pq)$. So,

$$\begin{aligned} \theta(p\Delta q) &= (\theta(p) - \theta(pq)) + (\theta(q) - \theta(pq)) \\ &= \theta(p) + \theta(q) - 2\theta(pq) \\ &= \theta(p) + \theta(q) - 2\theta(p)\theta(q) \quad \text{(by (2))} \\ &= \theta(p)\Delta\theta(q). \end{aligned}$$

□

Notice that in a commutative $C^*$-algebra $A$, the set of projections is a commutative ring under the operations: symmetric difference as addition and the usual multiplication.

**Theorem 2.5** *Let $A$ and $B$ be two commutative unital $C^*$-algebras, and let $\theta$ be a one-to-one mapping from $\mathcal{P}(A)$ onto $\mathcal{P}(B)$. Then the following are equivalent*

1. *$\theta$ is an orthoisomorphism,*

2. *$\theta$ is an order-isomorphism,*

3. *$\theta$ is a ring isomorphism.*

**Proof.** We proved in Theorem 2.3 that (1) and (2) are equivalent. Also, by Theorem 2.4(2 and 4), it is evident that (2) implies (3). So, it is sufficient to prove that (3) implies (1): Let $p, q \in \mathcal{P}(A)$. Then

$$\begin{aligned} pq = 0 \quad &\text{iff} \quad \theta(pq) = 0 \\ &\text{iff} \quad \theta(p)\theta(q) = 0. \end{aligned}$$

Therefore $\theta$ preserves orthogonality, and this completes the proof. □

## 3. A Projection's Diagram of $C(X)$

Let $X$ be a compact space in $(\mathbb{R}, \tau_u)$. We recall and establish some main results concerning components and clopen subsets of $X$.

**Theorem 3.1** *[2] Let $X$ be a compact space in $(\mathbb{R}, \tau_u)$. Then every component of $X$ has the form $[a, b]$, for some $a, b \in X$ with $a \leq b$.*

**Theorem 3.2** *[2] Let $X$ be a locally connected compact space in $(\mathbb{R}, \tau_u)$. Then*

1. *a subset $B$ of $X$ is a component if and only if $B$ is non-empty connected and clopen in $X$,*

2. *the number of clopen subsets of $X$ is finite,*

3. *for $a \in X$, $\{a\}$ is a component of $X$ if and only if $a$ is an isolated point.*

Let $X$ be a compact space in $(\mathbb{R}, \tau_u)$, and let $\mu$ denote the Lebesgue measure. Define the equipotence relation $\cong$ on $\mathcal{P}(C(X))$ as

$$p \cong q \text{ if and only if } \mu(\text{support}(p)) = \mu(\text{support}(q)).$$

It is easy to show that the previous relation is an equivalence relation. If $p \cong q$, then we say that $p$ is equipotent to $q$. The equivalence class of $p$ is denoted by $[p]$ (the notation $[(w_{[p]})]$ is also used, where $w_{[p]} = \mu(\text{support}(p))$ and called the weight of the class $[p]$), and the cardinal number of $[p]$ is denoted by $n_{[p]}$.

**Definition 3.3** *Let $X$ be a compact space in $(\mathbb{R}, \tau_u)$.*

1. *For two distinct classes $[p], [q] \in \mathcal{P}(C(X))/_{\cong}$, we say that $[p]$ is a subclass of $[q]$, we write $[p] \prec [q]$, if there exist $p_1 \in [p]$ and $q_1 \in [q]$ such that $p_1 < q_1$.*

2. *A subclass $[p]$ of $[q]$ is called maximal, denoted by $[p] \overset{max}{\prec} [q]$, if there is no $[r] \in \mathcal{P}(C(X))/_{\cong}$ such that $[p] \prec [r] \prec [q]$.*

3. *For $[p] \prec [q]$, we denote by $m_{[p],[q]}$ the total number of subprojections of elements in $[q]$ which are equipotent to $p$. This number is called the multiplicity of $[p]$ in $[q]$.*

4. *A class $[r] \in \mathcal{P}(C(X))/_{\cong}$ is said to be minimal if $[0]$ is the only subclass of $[r]$.*

In the following definition we introduce new concepts concerning projections of commutative $C^*$-algebra.

**Definition 3.4** *Let $X$ be a compact space in $(\mathbb{R}, \tau_u)$. A projection $p$ in the $C^*$-algebra $C(X)$ is said to be $\mu$-infinite projection if $p$ is equipotent to one of its proper subprojections. A projection that is not $\mu$-infinite is called $\mu$-finite projection. For $A \subseteq \mathcal{P}(C(X))$, we denote by $\mu F(A)$ the set of all $\mu$-finite projections in $A$.*

**Remark 3.5** *The concept of $\mu$-finite will be essentially used in the studying of the projection's diagram of $C(X)$.*

Let us now prove the following results.

**Theorem 3.6** *Let $\theta : \mathcal{P}(C(X)) \to \mathcal{P}(C(Y))$ be an order-isomorphism which preserves equipotence of projections. Then for any $p \in \mathcal{P}(C(X))$, $p$ is $\mu$-finite if and only if $\theta(p)$ is $\mu$-finite.*

**Proof.** Let $p$ be a $\mu$-finite projection of $C(X)$. Suppose that $\theta(p) \cong r$ with $r \leq \theta(p)$, for some $r \in \mathcal{P}(C(Y))$. As $\theta$ is a bijection, there exists a unique projection $q \in \mathcal{P}(C(X))$ such that $\theta(q) = r$. As $\theta$ preserves equipotence and order of projections, we have $p \cong q$ and $q \leq p$ which implies that $p = q$. Thus $\theta(p) = \theta(q)$, and hence $\theta(p)$ is $\mu$-finite. Since $\theta^{-1}$ is an order-isomorphism and preserves equipotence, the converse holds. □

**Lemma 3.7** *Let $X$ be a locally connected compact space in $(\mathbb{R}, \tau_u)$. Then*

1. *every non-zero projection of $C(X)$ can be written uniquely as a sum of minimal projections,*
2. *the number of $\mu$-finite projections of $C(X)$ in a class $[p]$ is $\frac{n_{[p]}}{n_{[0]}}$, for any $[p] \in \mathcal{P}(C(X))/_{\cong}$,*
3. *if $[p] \prec [q]$, then $m_{[p],[q]} = N\, n_{[0]}$, where $N$ is the number of subprojections of elements of $\mu F([q])$ in $\mu F([p])$.*

**Proof.**

1. By Theorem 3.2(2), the number of projections of $C(X)$ is finite. Let $\{p_i\}_{i=1}^{k}$ be the set of minimal projections of $C(X)$. Clearly it is a set of pair-wise orthogonal projections. We claim that every non-zero projection in $\mathcal{P}(C(X))$ can be written as a sum of such projections and this representation is unique. Let $p \in \mathcal{P}(C(X))$. If $p$ is minimal, then the result holds. Otherwise, decompose $p$ as $p = q + (p-q)$, for some projection $q < p$. Apply the argument on $q$ and on $(p-q)$ then continue with this process. As the number of projections is finite, we shall attain that $p$ can be written as a sum of minimal projections. To prove the uniqueness, suppose that $p = \sum_{j=1}^{s} q_j$ and $p = \sum_{t=1}^{m} r_t$, where $q_j$ and $r_t$ are minimal projections for all $j = 1, 2, \ldots, s$ and for all $t = 1, 2, \ldots, m$, we may assume that $s \leq m$. As $\sum_{j=1}^{s} q_j = \sum_{t=1}^{m} r_t$, then for every $1 \leq n \leq s$, $q_n \sum_{j=1}^{s} q_j = q_n \sum_{t=1}^{m} r_t$, which implies that $q_n = q_n \sum_{t=1}^{m} r_t$. As $r_1, r_2, \ldots, r_m$ are pair-wise orthogonal projections and $q_n$ cannot be divided, then $q_n = r_{t_n}$, for some $1 \leq t_n \leq m$. To show that $s = m$, suppose on the contrary that $s < m$, then $\sum_{t=1}^{m} r_t - \sum_{j=1}^{s} q_j = \sum_{\substack{t \neq t_n \\ 1 \leq t \leq m}} r_t = 0$. As $0 < r_t \leq 1$, for every $1 \leq t \leq m$, we get a contradiction.
2. Let $[p] \in \mathcal{P}(C(X))/_{\cong}$. We claim that $[p] = \bigcup_{p_s \in \mu F([p])} (p_s + [0])$. To this end, let $r \in [p]$. Then by (1), $r = \sum_{j=1}^{n} r_j$, where $r_1, r_2, \ldots, r_n$ are minimal projections. Let $r_0 = \sum_{r_j \in [0]} r_j$ and $r_s = r - r_0$. Then $r_s \in \mu F([p])$ and $r = r_s + r_0$. So, $r \in r_s + [0]$ and hence $[p] \subseteq \bigcup_{p_s \in \mu F([p])} (p_s + [0])$. The other direction is obvious. Our next claim is that $(q+[0]) \cap (r+[0]) = \emptyset$, for any $q, r \in \mu F([p])$ with $q \neq r$. To prove this, let $q, r \in \mu F([p])$ with $q \neq r$, and suppose that $(q+[0]) \cap (r+[0]) \neq \emptyset$, then there exist $q_0, r_0 \in [0]$ such that $q + q_0 = r + r_0$. Clearly $\text{support}(q) \cap \text{support}(q_0) = \text{support}(r) \cap \text{support}(r_0) = \emptyset$. So, if $x \in \text{support}(q)$, then either $x \in \text{support}(r)$ or $x \in \text{support}(r_0)$. If $x \in \text{support}(r_0)$, then $q \cong (q - qr_0) < q$ which is impossible because $q$ is $\mu$-finite. Therefore, $x \in \text{support}(r)$ and hence $\text{support}(q) \subseteq \text{support}(r)$. By the same argument we get that $\text{support}(r) \subseteq \text{support}(q)$. Therefore, $\text{support}(q) = \text{support}(r)$ and thus $q = r$, which is a contradiction. Finally, since $|p_s + [0]| = |[0]| = n_{[0]}$, for any $p_s \in \mu F([p])$, then by the previous two claims $n_{[p]} = |\mu F([p])|\, n_{[0]}$, which implies that $|\mu F([p])| = \frac{n_{[p]}}{n_{[0]}}$ and this is the desired result.

3. For $[p] \prec [q]$. Let $\mathcal{M} = \{p' \in [p] \; : \; p' < q', \text{ for some } q' \in [q]\}$, and let $\mathcal{N} = \{p_s \in \mu F([p]) \; : \; p_s < q_s, \text{ for some } q_s \in \mu F([q])\}$, notice that $m_{[p],[q]} = |\mathcal{M}|$ and $N = |\mathcal{N}|$. We claim that $\mathcal{M} = \bigcup_{p_s \in \mathcal{N}} (p_s + [0])$. Indeed, let $p' \in \mathcal{M}$. Then $p' \in [p]$ with $p' < q'$, for some $q' \in [q]$. Now, $p'$ can be decomposed as $p' = p'_s + p'_0$ where $p'_s \in \mu F([p])$ and $p'_0 \in [0]$, also $q'$ can be decomposed as $q' = q'_s + q'_0$ where $q'_s \in \mu F([q])$ and $q'_0 \in [0]$. Therefore, $p'_s < q'_s$, and this implies that $p'_s \in \mathcal{N}$, hence $p' \in \bigcup_{p_s \in \mathcal{N}} (p_s + [0])$. For the converse, let $p' \in \bigcup_{p_s \in \mathcal{N}} (p_s + [0])$. Then $p' = p'_s + r_0$, for some $p'_s \in \mathcal{N}$ and $r_0 \in [0]$. As $p'_s \in \mathcal{N}$, then $p'_s \in \mu F([p])$ and there exists $q'_s \in \mu F([q])$ such that $p'_s < q'_s$. Let $q' = q'_s + r_0$. Then $p' < q'$ and hence $p' \in \mathcal{M}$ and this completes the proof of the claim. Finally, by the same argument used in (2), we have $|\mathcal{M}| = |\mathcal{N}| \, n_{[0]}$ and hence $m_{[p],[q]} = N \, n_{[0]}$.

This completes the proof. □

Let us introduce the following definition.

**Definition 3.8** *Consider the $C^*$-algebra $C(X)$. Represent each class $[p] \in \mathcal{P}(C(X))/_{\cong}$ by a rectangle which contains the weight of the class $[p]$ and contains a small sub-rectangle filled by the cardinal number of $[p]$. For $[p] \overset{max}{\prec} [q]$, draw an arrow from $[q]$ to $[p]$ merged with the multiplicity of $[p]$ in $[q]$. The sequence of these pictures shall be called the diagram of projection classes of $C(X)$ and denoted by $\mathfrak{D}(X)$.*

We show a variety of diagrams by the following examples.

**Example 3.9** *Consider the $C^*$-algebra $C(X_1)$, where $X_1 = ([0,1] \cup [2,4], \tau_u)$. The set of projections of $C(X_1)$ is $\mathcal{P}(C(X_1)) = \{0, \chi_{[0,1]}, \chi_{[2,4]}, 1\}$ and the set of equivalence classes $\mathcal{P}(C(X_1))/_{\cong} = \{[0], [\chi_{[0,1]}], [\chi_{[2,4]}], [1]\}$. Clearly $n_{[p]} = 1$, for every $[p] \in \mathcal{P}(C(X))/_{\cong}$, also*

$$[0] \overset{\max}{\prec} [\chi_{[0,1]}] \overset{\max}{\prec} [1], \text{ and}$$

$$[0] \overset{\max}{\prec} [\chi_{[2,4]}] \overset{\max}{\prec} [1].$$

*The weights of the classes $[0]$, $[\chi_{[0,1]}]$, $[\chi_{[2,4]}]$, and $[1]$ are $0, 1, 2,$ and $3$, respectively. Thus, we obtain the diagram $\mathfrak{D}(X_1)$ as shown in Figure 1.*

**Example 3.10** *For $X_2 = ([0, 0.5] \cup [1, 2] \cup [3, 4], \tau_u)$, the set of equivalence classes $\mathcal{P}(C(X_2))/_{\cong} = \{[(0)], [(0.5)], [(1)], [(1.5)], [(2)], [(2.5)]\}$, and the corresponding diagram is $\mathfrak{D}(X_2)$, as shown in Figure 2.*

**Example 3.11** *Let $X_3 = ([0, 0.5] \cup [1, 1.5] \cup [2, 3], \tau_u)$. Then the set of equivalence classes $\mathcal{P}(C(X_3))/_{\cong} = \{[(0)], [(0.5)], [(1)], [(1.5)], [(2)]\}$, and the corresponding diagram is $\mathfrak{D}(X_3)$, as shown in Figure 3.*

The diagram $\mathfrak{D}(X)$ gives a good description of the projection classes of $C(X)$, in the following theorem we show that different quantities of $C(X)$ can be derived from the diagram $\mathfrak{D}(X)$.

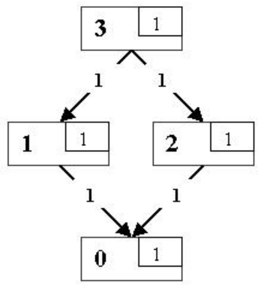


**Figure 1**. $\mathfrak{D}(X_1)$.

**Theorem 3.12** *Let $X$ be a locally connected compact space in $(\mathbb{R}, \tau_u)$, and let $\mathfrak{D}(X)$ be the corresponding diagram of $C(X)$. Then*

1. *the number of projections of $C(X)$ is $\sum_{[p]} n_{[p]}$;*
2. *the number of minimal projections of $C(X)$ is $\log_2 \sum_{[p]} n_{[p]}$;*
3. *the number of $\mu$-finite projections of $C(X)$ is $\frac{\sum_{[p]} n_{[p]}}{n_{[0]}}$;*
4. *the number of minimal $\mu$-finite projections of $C(X)$ is $\log_2 \frac{\sum_{[p]} n_{[p]}}{n_{[0]}}$;*
5. *the number of $\mu$-infinite projections of $C(X)$ is $\frac{n_{[0]}-1}{n_{[0]}} \sum_{[p]} n_{[p]}$;*
6. *the number of minimal projections in $[0]$ is $\log_2 n_{[0]}$.*

**Proof.**

1. As $\cong$ is an equivalence relation, the equivalence classes form a partition of the set of projections and hence $|\mathcal{P}(C(X))| = \sum_{[p]} n_{[p]}$.

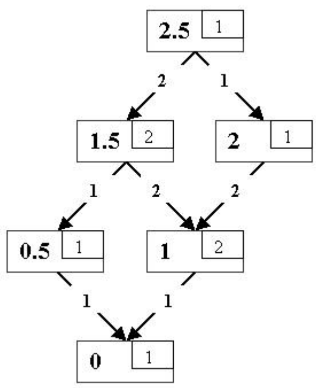


**Figure 2**. $\mathfrak{D}(X_2)$.

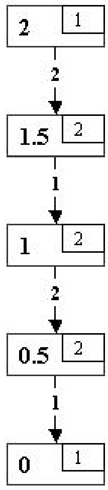


**Figure 3**. $\mathfrak{D}(X_3)$

2. Assume that $k$ is the number of minimal projections of $C(X)$. As the minimal projections are pair-wise orthogonal, then every sum of these projections is also a projection. Therefore, by Lemma 3.7(1),

$$\begin{aligned} |\mathcal{P}(C(X))| &= \binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \cdots + \binom{k}{k} \\ &= 2^k. \end{aligned}$$

Thus, $k = \log_2 |\mathcal{P}(C(X))|$ and by (1), $k = \log_2 \sum_{[p]} n_{[p]}$.

3. As the set $\mathcal{P}(C(X))/_{\cong}$ forms a partition of $\mathcal{P}(C(X))$, the set $\{\mu F([p]) : [p] \in \mathcal{P}(C(X))/_{\cong}\}$ also forms a partition of $\mu F(\mathcal{P}(C(X)))$. Therefore,

$$\begin{aligned} |\mu F(\mathcal{P}(C(X)))| &= \sum_{[p]} |\mu F([p])| \\ &= \sum_{[p]} \frac{n_{[p]}}{n_{[0]}} \quad \text{(Lemma 3.7(2))} \\ &= \frac{\sum_{[p]} n_{[p]}}{n_{[0]}}. \end{aligned}$$

4. Clearly every sum of pair-wise orthogonal $\mu$-finite projections is also $\mu$-finite projection. On the other hand, by Lemma 3.7(1) any $\mu$-finite projection can be written as a sum of minimal projections. It is easy to show that these projections are also $\mu$-finite. By applying the argument used to prove (2) and by (3) above, we get that the number of minimal $\mu$-finite projections is $\log_2 \frac{\sum_{[p]} n_{[p]}}{n_{[0]}}$.

5. As $\mathcal{P}(C(X)) = (\mu F(\mathcal{P}(C(X)))) \cup (\mu F(\mathcal{P}(C(X))))^c$,

$$
\begin{aligned}
|(\mu F(\mathcal{P}(C(X))))^c| &= |\mathcal{P}(C(X))| - |\mu F(\mathcal{P}(C(X)))| \\
&= \sum_{[p]} n_{[p]} - \frac{\sum_{[p]} n_{[p]}}{n_{[0]}} \qquad \text{(by (1) and (3))} \\
&= \frac{n_{[0]} - 1}{n_{[0]}} \sum_{[p]} n_{[p]}.
\end{aligned}
$$

6. Assume that $l$ is the number of minimal projections in $[0]$. By Lemma 3.7(1), any projection in $[0]$ can be written uniquely as a sum of minimal projections, clearly these projections are equipotent to $0$. On the other hand, every sum of distinct minimal projections in $[0]$ is a projection in $[0]$. Therefore

$$
\begin{aligned}
|[0]| &= \binom{l}{0} + \binom{l}{1} + \binom{l}{2} + \cdots + \binom{l}{l} \\
&= 2^l.
\end{aligned}
$$

Thus, $l = \log_2 |[0]| = \log_2 n_{[0]}$.

This completes the proof. □

As a corollary we obtain a link between the diagrams and the topological invariants of $X$.

**Corollary 3.13** *Let $X$ be a locally connected compact space in $(\mathbb{R}, \tau_u)$, and let $\mathfrak{D}(X)$ be the corresponding diagram of $C(X)$. Then*

1. *the number of clopen subsets of $X$ is $\sum_{[p]} n_{[p]}$;*
2. *the number of components of $X$ is $\log_2 \sum_{[p]} n_{[p]}$;*
3. *the number of components of $X$ of non-zero measure is $\log_2 \frac{\sum_{[p]} n_{[p]}}{n_{[0]}}$;*
4. *the number of isolated points of $X$ is $\log_2 n_{[0]}$.*

**Proof.**

1. Obvious.
2. We claim that $B$ is a component of $X$ if and only if $\chi_B$ is a minimal projection.

$$
\begin{aligned}
B \text{ is a component of } X \quad &\text{iff} \quad B \text{ is non-empty connected and clopen subset of } X \\
&\text{iff} \quad \chi_B \text{ is non-zero and has no proper subprojection} \\
&\text{iff} \quad \chi_B \text{ is minimal.}
\end{aligned}
$$

So by Theorem 3.12(2), the number of components of $X$ is $\log_2 \sum_{[p]} n_{[p]}$.

3. By the argument used in (2), $B$ is a component of non-zero measure if and only if $\chi_B$ is a minimal projection in $\mu F(\mathcal{P}(C(X)))$, hence by Theorem 3.12(4), the number of these components is $\log_2 \frac{\sum_{[p]} n_{[p]}}{n_{[0]}}$.

4. By the argument used in (2), a component $B$ of $X$ has the form $\{a\}$ if and only if $\chi_B$ is a minimal projection in $[0]$, hence by Theorem 3.12(6) and Theorem 3.2(3), the number of isolated points of $X$ is $\log_2 n_{[0]}$.

This completes the proof. □

## 4. Diagram Isomorphisms

Let us introduce the concept of diagram isomorphism as follows.

**Definition 4.1** *Let $X$ and $Y$ be two compact spaces in $(\mathbb{R}, \tau_u)$. A diagram isomorphism between $\mathfrak{D}(X)$ and $\mathfrak{D}(Y)$ is a one-to-one mapping $\varphi$ from $\mathcal{P}(C(X))/_{\cong}$ onto $\mathcal{P}(C(Y))/_{\cong}$ such that:*

1. $n_{[p]} = n_{\varphi([p])}$, *for any* $[p] \in \mathcal{P}(C(X))/_{\cong}$,

2. $[p] \overset{\max}{\prec} [q]$ *if and only if* $\varphi([p]) \overset{\max}{\prec} \varphi([q])$, *for any* $[p], [q] \in \mathcal{P}(C(X))/_{\cong}$,

3. $m_{[p],[q]} = m_{\varphi([p]),\varphi([q])}$, *for any* $[p], [q] \in \mathcal{P}(C(X))/_{\cong}$ *with* $[p] \overset{\max}{\prec} [q]$.

*That is, $\varphi$ preserves cardinal number of classes, maximal subclasses and preserves the multiplicity between maximal classes. If there is an isomorphism between $\mathfrak{D}(X)$ and $\mathfrak{D}(Y)$, then we say that $\mathfrak{D}(X)$ and $\mathfrak{D}(Y)$ are isomorphic and write $\mathfrak{D}(X) \simeq \mathfrak{D}(Y)$.*

The following example discusses two different $C^*$-algebras whose diagrams are isomorphic.

**Example 4.2** *Let $X = ([-20, -14] \cup [-10, -9] \cup [-6, -4], \tau_u)$ and $Y = ([1, 3.5] \cup [4.5, 5] \cup [6.5, 7.5], \tau_u)$ be two spaces such that their corresponding diagrams $\mathfrak{D}(X)$ and $\mathfrak{D}(Y)$ are as shown in Figure 4,*

then $\mathfrak{D}(X)$ is isomorphic to $\mathfrak{D}(Y)$. Indeed, define $\varphi : \mathcal{P}(C(X))/_{\cong} \rightarrow \mathcal{P}(C(Y))/_{\cong}$ by: $\varphi = \{([(9)], [(5)]), ([(8)], [(4.5)]), ([(7)], [(4)]), ([(6)], [(3.5)]), ([(3)], [(1.5)]), ([(2)], [(0.5)]), ([(1)], [(1)]), ([(0)], [(0)])\}$. It is easy to check that $\varphi$ is a diagram isomorphism, hence the example is explained.

**Example 4.3** *The diagrams given in Figure 5 are not isomorphic.*

Let $X = (\{1, 2\}, \tau_u)$ and $Y = ([1, 2] \cup [3, 4], \tau_u)$. Then

$$\mathcal{P}(C(X)) = \{0, \chi_{\{1\}}, \chi_{\{2\}}, 1\} \text{ and } \mathcal{P}(C(Y)) = \{0, \chi_{[1,2]}, \chi_{[3,4]}, 1\}.$$

Define

$$\theta : \mathcal{P}(C(X)) \rightarrow \mathcal{P}(C(Y))$$

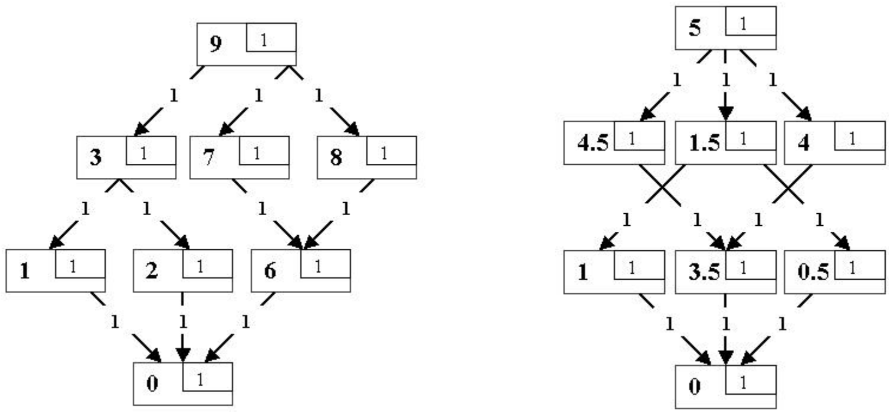


**Figure 4**. $\mathfrak{D}(X)$ $\mathfrak{D}(Y)$

by

$$\theta = \{(0,0)\,,\,(\chi_{\{1\}},\chi_{[1,2]})\,,\,(\chi_{\{2\}},\chi_{[3,4]})\,,\,(1,1)\}$$

then $\theta$ is an order-isomorphism but $\mathfrak{D}(X)$ is not isomorphic to $\mathfrak{D}(Y)$. Let us prove the following theorem.

**Theorem 4.4** *Let $X$ and $Y$ be two compact spaces in $(\mathbb{R},\tau_u)$. If there exists an order-isomorphism between $\mathcal{P}(C(X))$ and $\mathcal{P}(C(Y))$, which preserves equipotence of projections, then $\mathfrak{D}(X)$ is isomorphic to $\mathfrak{D}(Y)$.*

**Proof.** Let $\theta : \mathcal{P}(C(X)) \to \mathcal{P}(C(Y))$ be an order-isomorphism which preserves equipotence of projections. For any $[p],[q] \in \mathcal{P}(C(X))$, we have

$$\begin{aligned}[p]=[q] \quad &\text{iff} \quad p \cong q\\ &\text{iff} \quad \theta(p) \cong \theta(q)\\ &\text{iff} \quad [\theta(p)] = [\theta(q)],\end{aligned}$$

this allows us to define the map $\hat{\theta} : \mathcal{P}(C(X))/_{\cong} \to \mathcal{P}(C(Y))/_{\cong}$ by $\hat{\theta}([p]) = [\theta(p)]$. We claim that $\hat{\theta}$ is a diagram isomorphism. To this end, we first show that $\hat{\theta}$ is a bijection. By the above construction, $\hat{\theta}$ is well-defined and is an injection. let $[r'] \in \mathcal{P}(C(Y))/_{\cong}$. As $\theta$ is surjective, there exists $r \in \mathcal{P}(C(X))$ such that $r' = \theta(r)$. So, we have an element, namely $[r] \in \mathcal{P}(C(X))/_{\cong}$, such that $\hat{\theta}([r]) = [\theta(r)] = [r']$, hence $\hat{\theta}$ is a bijection.
Let $[p] \in \mathcal{P}(C(X))/_{\cong}$. As $\theta$ preserves the equipotence of projections and since $\theta$ is a bijection, we have

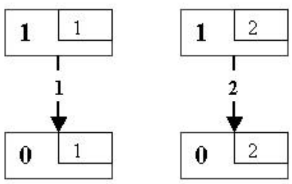


**Figure 5**. Non-isomorphic diagrams.

$n_{[p]} = n_{[\theta(p)]}$ and hence $n_{[p]} = n_{\hat{\theta}([p])}$, for any $[p] \in \mathcal{P}(C(X))/_{\cong}$.

To prove that $\hat{\theta}$ preserves subclasses, let $[p],[q] \in \mathcal{P}(C(X))/_{\cong}$ with $[p] \prec [q]$. Then there exist $p_1 \in [p]$ and $q_1 \in [q]$ such that $p_1 < q_1$. As $\theta$ preserves the order, $\theta(p_1) < \theta(q_1)$. Since $\theta$ preserves the equipotence of projections, then $p_1 \in [p]$ implies that $\theta(p_1) \in [\theta(p)]$ and hence $\theta(p) \in \hat{\theta}([p])$, also $q_1 \in [q]$ implies that $\theta(q_1) \in [\theta(q)]$ and thus $\theta(q_1) \in \hat{\theta}([q])$. Therefore, $\hat{\theta}([p]) \prec \hat{\theta}([q])$. The converse is true as $\theta^{-1}$ is also an order-isomorphism and preserves equipotence of projections.

Now we claim that $\hat{\theta}$ preserves maximal subclasses. To show this, let $[p],[q] \in \mathcal{P}(C(X))/_{\cong}$ and suppose that $[p] \overset{\max}{\prec} [q]$. By the previous claim, $[p] \prec [q]$ implies that $\hat{\theta}([p]) \prec \hat{\theta}([q])$. Let $[r'] \in \mathcal{P}(C(Y))/_{\cong}$ with $\hat{\theta}([p]) \prec [r'] \prec \hat{\theta}([q])$. As $\hat{\theta}$ is surjective, there exists $[r] \in \mathcal{P}(C(X))/_{\cong}$ such that $[r'] = \hat{\theta}([r])$. Then $\hat{\theta}([p]) \prec \hat{\theta}([r]) \prec \hat{\theta}([q])$, which implies that $[p] \prec [r] \prec [q]$, hence $[p]$ is not a maximal subclass of $[q]$ and this is a contradiction. The converse is true because $\hat{\theta}^{-1}$ is also surjective and preserves subclasses.

Finally, by the previous claim and since $\theta$ is a bijection which preserves the order, then for any $[p] \overset{\max}{\prec} [q]$ in $\mathcal{P}(C(X))/_{\cong}$, the number of subprojections of elements in $[q]$ whose equipotent to $p$ is equal to the number of subprojections of elements in $\hat{\theta}([q])$ whose equipotent to $\theta(p)$, hence $m_{[p],[q]} = m_{\hat{\theta}([p]),\hat{\theta}([q])}$, and this completes the proof. □

**Remark 4.5** *If $C(X)$ and $C(Y)$ are isomorphic as $C^*$-algebras, then the corresponding diagrams $\mathfrak{D}(X)$ and $\mathfrak{D}(Y)$ are not necessarily isomorphic in general. Indeed, take $X = ([1,2] \cup [3,4], \tau_u)$ and $Y = ([1,2] \cup [3,5], \tau_u)$.*

Now let us prove the following theorem.

**Theorem 4.6** *Let $X$ and $Y$ be two locally connected compact spaces in $(\mathbb{R}, \tau_u)$. If $\mathfrak{D}(X)$ is isomorphic to $\mathfrak{D}(Y)$, then $C(X)$ and $C(Y)$ are isomorphic as $C^*$-algebras.*

**Proof.** Suppose that $\mathfrak{D}(X)$ is isomorphic to $\mathfrak{D}(Y)$. By Corollary 3.13(3 and 4), we conclude that the number of components of $X$ of non-zero measure is equal to the number of components of $Y$ of non-zero measure, also the number of isolated points of $X$ is equal to the number of isolated points of Y. By Theorem 3.1, the components of non-zero measure are of the form $[a,b]$ with $a < b$, and the components of measure zero are of the form $\{a\}$, where $a$ is an isolated point of $X$. As the components form a partition, then $X$ and $Y$ can be written as a union of their components:

$$\begin{aligned} X &= \bigcup_{i=1}^{n} [a_i, b_i] \cup \bigcup_{j=1}^{m} \{x_j\},\ a_i < b_i, \text{ for all } i = 1,2,\ldots,n, \text{ and} \\ Y &= \bigcup_{i=1}^{n} [c_i, d_i] \cup \bigcup_{j=1}^{m} \{y_j\},\ c_i < d_i, \text{ for all } i = 1,2,\ldots,n. \end{aligned}$$

Define $f : X \to Y$ by

$$f(x) = \begin{cases} \frac{d_i - c_i}{b_i - a_i}(x - a_i) + c_i & ;\quad x \in [a_i, b_i], \text{ for some } i, \\ y_j & ;\quad x = x_j, \text{ for some } j. \end{cases}$$

Then $f$ is a homeomorphism and hence by Theorem 1.6, $C(X)$ and $C(Y)$ are isomorphic as $C^*$-algebras. □

**Corollary 4.7** *Let $X$ and $Y$ be two locally connected compact spaces in $(\mathbb{R}, \tau_u)$. If $\mathfrak{D}(X)$ is isomorphic to $\mathfrak{D}(Y)$, then there exists an order-isomorphism between $\mathcal{P}(C(X))$ and $\mathcal{P}(C(Y))$.*

**Proof.** As $\mathfrak{D}(X)$ is isomorphic to $\mathfrak{D}(Y)$, then by Theorem 4.7, $C(X)$ and $C(Y)$ are isomorphic as $C^*$-algebras. Let $\psi$ be an isomorphism from $C(X)$ onto $C(Y)$. Then the restriction $\psi|_{\mathcal{P}(C(X))}$ of $\psi$ on the projections of $C(X)$ is an orthoisomorphism from $\mathcal{P}(C(X))$ onto $\mathcal{P}(C(Y))$, so by Theorem 2.3, $\psi|_{\mathcal{P}(C(X))}$ is an order-isomorphism. □

Combining Theorems 4.4 and 4.7, we obtain the following corollary.

**Corollary 4.8** *Let $X$ and $Y$ be two locally connected compact spaces in $(\mathbb{R}, \tau_u)$. If there exists an order-isomorphism between $\mathcal{P}(C(X))$ and $\mathcal{P}(C(Y))$, which preserves the equipotence of the projections, then $C(X)$ is isomorphic to $C(Y)$ as $C^*$-algebras.*

## References


[1] Al-Rawashdeh, A., Shatanawi, W.: Induced Mappings on Boolean Algebras of Clopen Sets and on Projections of the $C^*$-Algebra $C(X)$, Turkish J. Math, 31, 1-13 (2007).

[2] Al-Suleiman, S.: $C^*$-Algebras of Continuous Complex-Valued Functions on a Compact Set, Master Thesis, Jordan University of Scirence and Technology (2007).

[3] Conway, J.: A Course in Functional Analysis, New York, Springer-Verlag (1985).

[4] Davidson, K.: $C^*$-Algebras by Example, Fields Institute Monographs, Amer. Math. Soc. Providencs RI (1996).

[5] Dye, H.: On the Geometry of Projections in Certain Operator Algebras, Ann. of Math., Second Series, 61:73-89 (1955).

[6] de la Harpe, P. and Jones, V.F.R.: An Introduction to $C^*$-Algebras, Université de Genève (1995).

[7] Pedersen, G.: $C^*$-Algebras and Their Automorphism Groups, London, Academic Press (1979).

[8] Pedersen, G.: Analysis Now, Corrected Second Printing 1994, Springer-Verlag, Berlin (1988).



Ahmed S. AL-RAWASHDEH
Department of Mathematical Sciences
United Arab Emirates University
B.O.Box 17551, Al-Ain, Abu Dhabi, UAE
e-mail: aalrawashdeh@uaeu.ac.ae
On Leave from Jordan University of Science and Technology
Department of Mathematics and Statistics
Irbid-JORDAN
e-mail: rahmed72@just.edu.jo


Received 18.12.2008


Sultan M. AL-SULEIMAN Department of Mathematics and Statistics
Jordan University of Science and Technology
P.O. Box3030 Irbid 22110, JORDAN
e-mail: msultan82@gmail.com